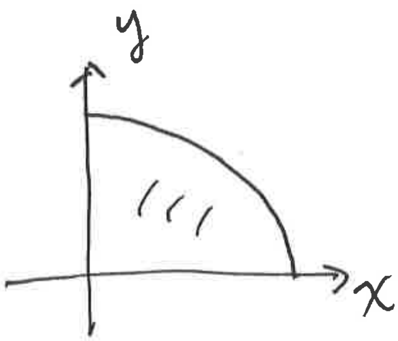
Math 1497 – Calc 2

So we have the standard integrals.

Consider the following problem. Find the area under $y = \sqrt{1-x^2}$ on $[0,1]$



$$A = \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$ (area is $\frac{1}{4}$ circle radius 1)

However, this is not one of my standard $\int$'s!

So we need more ways (techniques) to integrate more functions

1. Substitution

Consider $\int 2x(x^2+1)^5\,dx$

we could expand (Yikes) but we will try and turn this into something we know how to solve. So what part of

$$\int 2x(x^2+1)^5\,dx$$

is the most complicated - $(x^2+1)^5$

Try and let $u = x^2+1$

Now we convert the integral into a new one involving u

$$du = 2x\,dx$$

and we see our $\int$ becomes

$$\int u^5\,du$$

so $\frac{u^6}{6} + C = \frac{(x^2+1)^6}{6} + C$ $\underline{\underline{\text{Ans}}}$

We now consider a few examples illustrating this technique.

Ex 2 $\int e^{-2x}\,dx$

most complicated e^{-2x} Try $u=-2x$

the thing we first do to x.

$du = -2dx \Rightarrow dx = -\frac{du}{2}$

$\Rightarrow \int -e^{u}\frac{du}{2} = -\frac{1}{2}\int e^{u}du = -\frac{1}{2}e^{u}+c$

$= -\frac{1}{2}e^{-2x}+c$

Ex 3

$\int \cos x \sin^2 x\,dx$

most complicated

$u = \sin x$

$du = \cos x\,dx$

$\Rightarrow \int u^2 du = \frac{u^3}{3}+c$

$= \frac{1}{3}\sin^3 x + c$

Ex4 $\int x\underline{\sqrt{x+1}}\,dx$

most complicated

$u = x+1 \Rightarrow du = dx$

so $\int x\sqrt{u}\,du$ — but we still have x!

Continue to use sub

$u = x+1 \Rightarrow x = u-1$ so

$\int (u-1)\sqrt{u}\,du = \int u^{3/2} - u^{1/2}\,du$

$= \dfrac{u^{5/2}}{5/2} - \dfrac{u^{3/2}}{3/2} + c$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

Ex 5 $\int x^5 (x^3+4)^3 dx$

let $u = x^3+4$ $du = 3x^2 dx$ $\frac{du}{3x^2} = dx$

$\int x^3 \cdot x^2 (x^3+4)^3 dx$

$\int x^3 (u)^3 \frac{du}{3\cancel{x^2}} \cancel{x^2} = \frac{1}{3}\int x^3 u^3 du$

cont to use sub

$\frac{1}{3}\int (u-4)u^3 du = \frac{1}{3}\int u^4 - 4u^3 du$

$= \frac{u^5}{15} - \frac{u^4}{3} + C$

$= \frac{1}{15}(x^3+4)^5 - \frac{1}{3}(x^3+4)^4 + C$

Definite integrals

Ex $\int_0^2 x\sqrt{4-x^2}\,dx$

let $u = 4-x^2$ so $du = -2x\,dx$

Now we switch the limits using the sub

$x=0 \quad u = 4-0 = 4$ $\qquad x=2 \quad u = 4-2^2 = 0$

$\Rightarrow \displaystyle\int_4^0 \sqrt{u}\,\frac{du}{-2} = +\frac{1}{2}\int_0^4 u^{1/2}\,du = \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2}\Big|_0^4$

$= \frac{1}{3}(4^{3/2} - 0) = 8/3$

Ex $\displaystyle\int_0^{\sqrt{2}} x\,e^{-x^2/2}\,dx$ $\qquad$ let $u = -x^2/2 \quad du = -x\,dx$

$x=0 \quad u=0 \quad x=\sqrt{2} \quad u=-1$

$\displaystyle\int_0^{-1} -e^u\,du = \int_{-1}^0 e^u\,du = e^u\Big|_{-1}^0 = 1-e^{-1}$

Ex $\displaystyle\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

$x=1 \quad u=3$

$x=3 \quad u=1$

$u = 3/x \qquad du = -\dfrac{3}{x^2}\,dx$

$$-\int_3^1 e^u \frac{du}{3} = \frac{1}{3}\int_1^3 e^u\,du = \frac{1}{3}\,e^u\Big|_1^3 = \frac{e^3 - e^1}{3}$$

Ex $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$x=0 \quad u = \cos 0 = 1$

$x = \dfrac{\pi}{4} \quad u = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$$\int_1^{\frac{1}{\sqrt{2}}} -\frac{du}{u^2} = \frac{1}{u}\Big|_1^{\frac{1}{\sqrt{2}}} = \sqrt{2} - 1$$