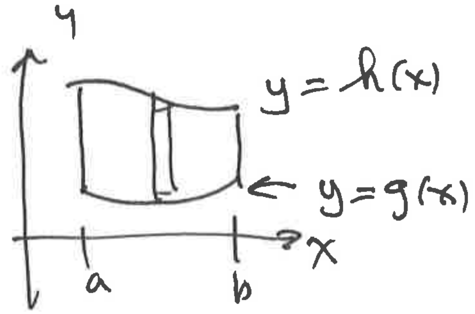
Math 2471 – Calc 3

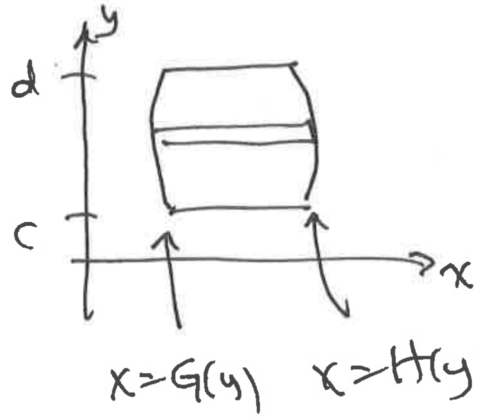
Double integrals

Type 1



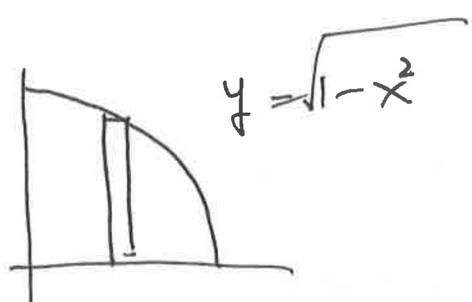
$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

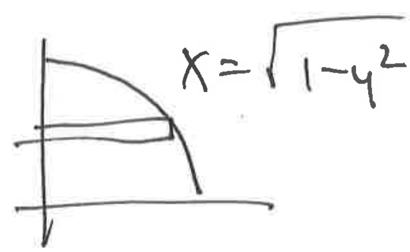
Type 2



$$\int_c^d \int_{G(y)}^{H(y)} f(x,y)\,dx\,dy$$

we left off with

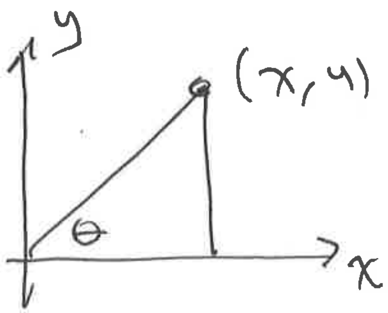




$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dy\,dx}{\sqrt{x^2+y^2}}, \qquad \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{dx\,dy}{\sqrt{x^2+y^2}}$$

The idea is switch to polar coord's

Recall



$x = r\cos\theta$ or $\sqrt{x^2+y^2} = r$

$y = r\sin\theta$ $\frac{y}{x} = \tan\theta$

Now circles become ~~etc~~ easy

$x^2+y^2 = 1 \Rightarrow r^2 = 1$ so $r = 1$ (we neglect −ve case)

So how does a double integral change?

$$\iint_R f(x,y)\,dA$$

there are 3 parts to this integral

(1) integrand

(2) dA

(3) R & the limits of integration.

(1) <u>integrand</u>

replace $x = r\cos\theta$, $y = r\sin\theta$

so $f(x,y) = f(r\cos\theta, r\sin\theta)$

In our example

$$\iint_R \frac{dA}{\sqrt{x^2+y^2}} \qquad f = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}}$$

$$= \frac{1}{r}$$

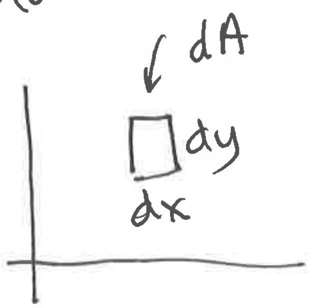
(much easier!)

(2) <u>dA</u>

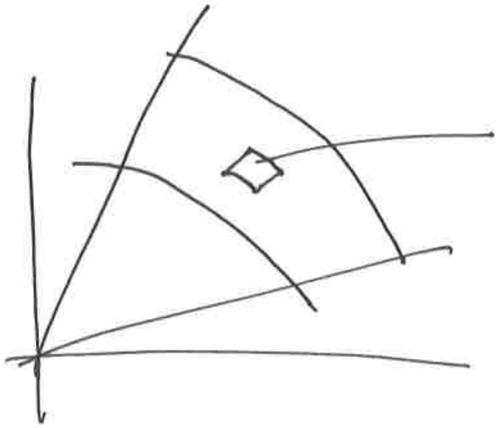
Now $dA = dx\,dy = dy\,dx$

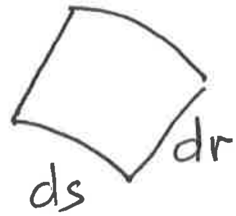
it's not just $dr\,d\theta$ or $d\theta\,dr$ (close though)

Consider



now consider a small area element in polar coord's





ds – small arc of circle

Now we have the following ratio

$$\frac{\text{arc circle}}{\text{perimeter of cir}} = \frac{\text{angle of arcle}}{\text{angle of circle}}$$

$$\frac{ds}{2\pi r} = \frac{d\theta}{2\pi} \Rightarrow ds = r\,d\theta$$

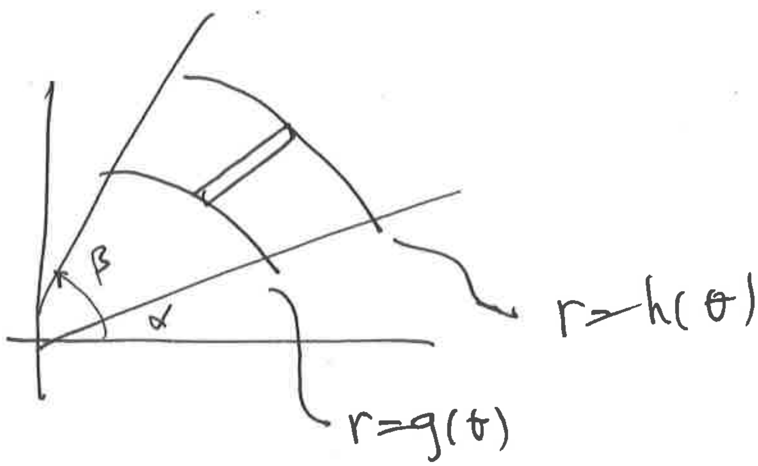
Now $dA = ds\,dr$

$= r\,dr\,d\theta$

So $\boxed{dA = r\,dr\,d\theta}$

Note: Extra r piece!

(3) R - Region & limits of integration



like double integrals

$$\int_{pt}^{pt} \int_{curve}^{curve} f \, dA$$

we still do curve to curve but inner curve to outer curve &

pt → pt is angle to angle

– very reminiscent of polar areas

so

$$\int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Now some examples

Ex 1

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dy\,dx}{\sqrt{x^2+y^2}}$$

(1) $\frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

(2) $dy\,dx = r\,dr\,d\theta$

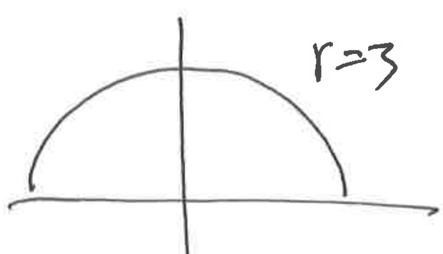
(3) $r = 0 \to 1$, $\theta = 0 \to \pi/2$

$$\int_0^{\pi/2} \int_0^1 \frac{\cancel{r}\,dr\,d\theta}{\cancel{r}} = \int_0^{\pi/2} r\Big|_0^1 d\theta = \int_0^{\pi/2} d\theta = \theta\Big|_0^{\pi/2}$$

$= \pi/2$ easy!

Briggs pg 992 #25

Ex 2 $\iint_R 2xy\,dA$ $\quad R: \{(x,y) \mid x^2+y^2 \le 9,\ y \ge 0\}$



$$\int_0^{\pi}\int_0^{3} 2\, r\cos\theta\, r\sin\theta\; r\,dr\,d\theta$$

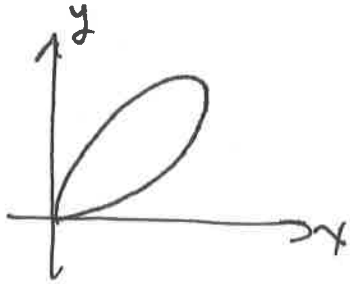
$$2\int_0^{\pi}\int_0^{3} r^3 \sin\theta\cos\theta\,dr\,d\theta$$

$$2\int_0^{\pi} \frac{r^4}{4}\Big|_0^3 \sin\theta\cos\theta\,d\theta = \frac{81}{2}\int_0^{\pi} \sin\theta\cos\theta\,d\theta$$

$$= \frac{81}{2}\,\frac{\sin^2\theta}{2}\Big|_0^{\pi} = 0$$

<u>Ex3</u> pg 992 # 34 $\quad \iint_R f(r,\theta)\,dA$

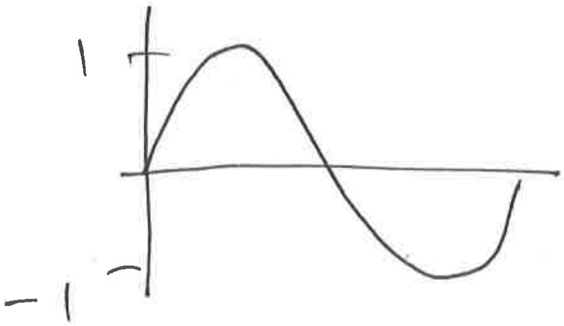
R - region inside leaf of the rose $r = 2\sin 2\theta$ in 1st quadrant (see next page for graph)



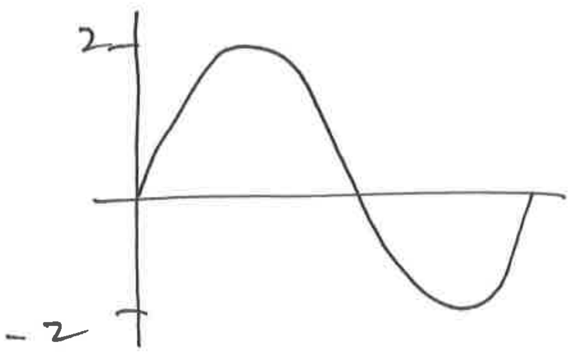
$$\int_0^{\pi/2}\int_{r=0}^{2\sin 2\theta} f(r,\theta)\, r\,dr\,d\theta$$

Recall drawing ~~the~~ polar graph's

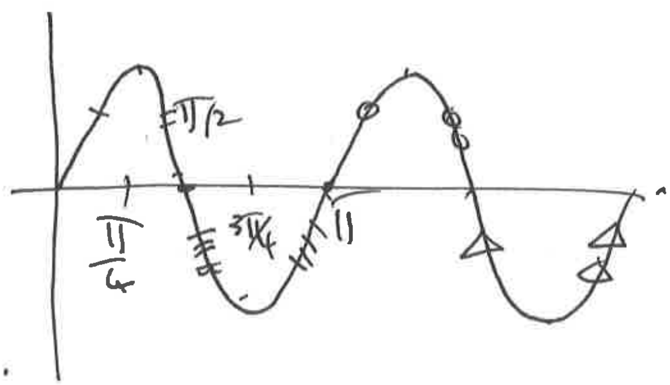
$r = \sin\theta$



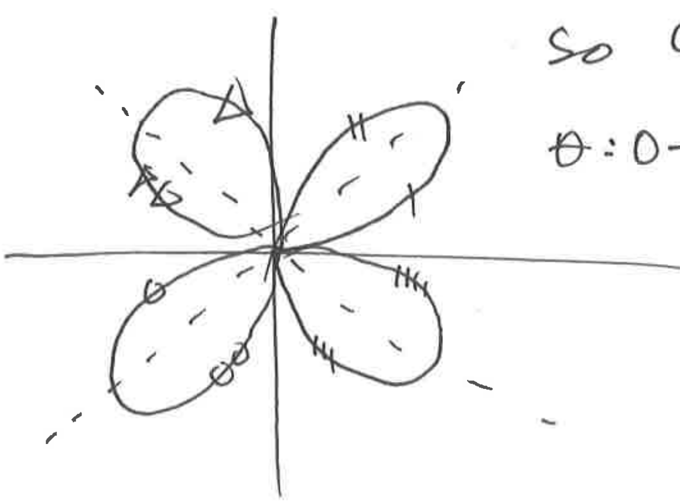
$r = 2\sin\theta$



$r = 2\sin 2\theta$



$\frac{2\pi}{8} = \frac{\pi}{4}$



So angle

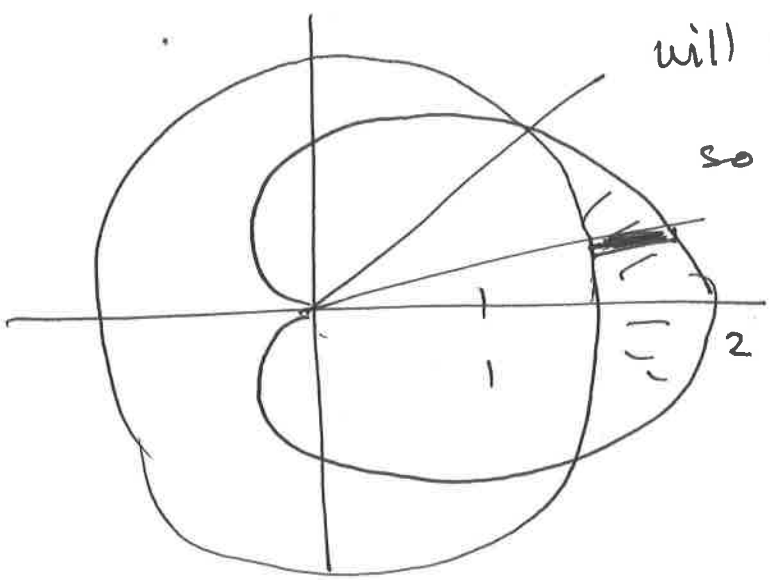
$\theta : 0 \to \pi/2$

Two Polar Curve's

Ex4 $\iint_R f(r,\theta)\,dA$

where R is the region outside $r = 3/2$ and inside

$r = 1 + \cos\theta$ (cardioid)



will need this angle

so $1 + \cos\theta = 3/2$

$\cos\theta = 1/2$

$\theta = \pi/3$

$$\int_{-\pi/3}^{\pi/3} \int_{r=3/2}^{1+\cos\theta} f(r,\theta)\, r\,dr\,d\theta$$