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***Research Article***

# On Hybrid Structures of Hypersoft Sets and Rough Sets

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## Abstract

Hypersoft sets, derived by transforming the approximate function in the structure of Molodtsov's soft set into a multi-attribute approximate function, and rough sets are effective mathematical tools for dealing with the uncertainties. This paper is devoted to making some contributions to the theory of rough hypersoft set and introducing the theory of hypersoft rough set, based on the hypersoft rough approximations with respect to the hypersoft approximation space. Furthermore, it is discussed the structural properties of hypersoft rough sets in detail.

**Keywords:** Rough set; Hypersoft set; Rough hypersoft set; Hypersoft rough set.

## Introduction

Rough set theory proposed by Pawlak in 1982 [1], as a useful tool for treating with imprecision information and uncertainty. Pawlak and Skowron [2] pointed out that these sets were successfully applied in the fields of medical diagnosis, pattern recognition, artificial intelligence, data mining and so on. In recent years, the rough set theory has aroused a great interest among more and more researchers. In 1999, Molodtsov [3] initiated the theory of soft sets, as a revolutionary mathematical method to deal with ambiguity and uncertainty.

The knowledge parameters in the structure of this set play a vital role in the inspection and interpretation of judged objects, and thus they are free from many difficulties. In the following years, the soft set theory was advanced in the theoretical aspects [4-8] and practical aspects such as decision making, game theory [9-14]. Moreover, many authors studied the extension models of soft sets such as intuitionistic fuzzy soft set, neutrosophic soft set, N-soft set [15-20]. Aktaş and Çağman [21] put forward that rough sets and soft sets are closely related notions. Feng et al. [22] proposed some hybrid models by combining the soft sets and rough sets. Ali [23] discussed the rough soft sets with different perspectives. Recently, the theories of rough soft set and soft rough set were improved in several directions [24-26].

In 2018, Smarandache [27] extended the concept of soft set to the hypersoft set by replacing the approximate function with the multi-argument approximate function, where its domain is described on the cartesian product of $n$ disjoint sets of parameters. Abbas et al. [28] and Saeed et al. [29] derived some operations on the hypersoft sets, and they put forward that these operations are more flexible than the soft set operations in several directions. The research works on the hypersoft sets have been progressing actively and rapidly in these years [30-34]. In 2021, Rahman et al. [35] proposed a combined model of the hypersoft set and the rough set, called rough hypersoft set.

This paper aims to contribute to the rough hypersoft set theory and to develop a new hybrid model of the hypersoft set and the rough set, called hypersoft rough set. Moreover, it is intended to address the axiomatic characterizations of hypersoft rough sets. This paper has layout as follows: Section 2 gives well-known definitions of the rough sets, the hypersoft sets and the rough hypersoft sets. Section 3 is dedicated to proposing some seminal notions and properties for the rough hypersoft sets. Section 4 introduces the concept of hypersoft rough set and related properties. Section 5 consists of the conclusions.

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## Preliminaries

In this section, some fundamental concepts related to rough sets, hypersoft sets and rough hypersoft sets are recalled.

From now on, $X$ is a non-empty set (universal set), and the power set of $X$ is denoted by $P(X)$.

***Rough Sets***

Suppose that $X$ and $Y$ are any two non-empty sets. The subset of cartesian product of $X$ and $Y$ is called a relation from $X$ to $Y$. Especially, the subset of cartesian product of $X \times Y$ is a relation on $X$. The relation $R$ on $X$ is said to be

1. reflexive when $(x_j, x_j) \in R$ for all $x_j \in X$.
2. symmetric when $(x_j, x_k) \in R \Rightarrow (x_k, x_j) \in R$ for all $x_j, x_k \in X$.
3. transitive when $(x_j, x_k) \in R$ and $(x_k, x_l) \in R \Rightarrow (x_j, x_l) \in R$ for all $x_j, x_k, x_l \in X$.

If $R$ is reflexive, symmetric and transitive then it is said to be an equivalence relation on $X$.

According to the equivalence relation $R$ on $X$, definition of an equivalence class of an element $x_j \in X$ is as follows:

$$[x_j]_R = \{x_k \in X |\ (x_j, x_k) \in R\}.$$

The theory of rough set initiated by Pawlak [1] provides a systematic method in handling vague notions induced by indiscernibility in situation with the incomplete information or a lack of knowledge.

**Definition 2.1.** ([1]) Let $X$ be any non-empty finite set and $R$ an equivalence relation on $R$. Then, $(X, R)$ is said to be (Pawlak) approximation space. If $V$ is a subset of $X$, then the sets

$$\underline{R}(V) = \{v | [v]_R \subseteq V\} \quad (1)$$

and

$$\overline{R}(V) = \{v | [v]_R \cap V \neq \emptyset\} \quad (2)$$

are called the lower and upper approximations of $V$ with respect to $(X, R)$, respectively. The positive region, negative region and boundary region of $V$ are defined as $pos_R(V) = \underline{R}(V)$, $neg_R(V) = X - \overline{R}(V) = (\overline{R}(V))^c$ and $bnd_R(V) = \overline{R}(V) - \underline{R}(V)$, respectively.

**Definition 2.2.** ([1,2]) Let $(X, R)$ be a Pawlak approximation space. A subset $V \subseteq X$ is said to be a definable if $\overline{R}(V) = \underline{R}(V)$, otherwise, i.e., if $bnd_R(V) \neq \emptyset$, it is called a rough (or inexact).

Sometimes a pair $R(V) = (\overline{R}(V), \underline{R}(V))$ is also termed to be a rough set. If the set $V \subseteq X$ is defined by a predicate $\mathcal{P}$ and $x \in X$, then there are the following.

1. $x \in \underline{R}(V)$ means that $x$ certainly has the property $\mathcal{P}$.
2. $x \in \overline{R}(V)$ means that $x$ possibly has the property $\mathcal{P}$.
3. $x \in neg_R(V)$ means that $x$ definitely does not have the property $\mathcal{P}$.

**Theorem 2.3.** ([1,2]) Let $(X, R)$ be a Pawlak approximation space and $V, V_1, V_2 \subseteq X$. Then, we have the axiomatic characterizations.

**1)** $\underline{R}(V) \subseteq V \subseteq \overline{R}(V)$.
**2)** $\underline{R}(\emptyset) = \emptyset = \overline{R}(\emptyset)$.
**3)** $\underline{R}(X) = X = \overline{R}(X)$.
**4)** $\underline{R}(\underline{R}(V)) = \underline{R}(V)$.
**5)** $\overline{R}(\overline{R}(V)) = \overline{R}(V)$.
**6)** $\overline{R}(\underline{R}(V)) = \underline{R}(V)$.
**7)** $\underline{R}(\overline{R}(V)) = \overline{R}(V)$.
**8)** $\underline{R}(V) = (\overline{R}(V^c))^c$.
**9)** $\overline{R}(V) = (\underline{R}(V^c))^c$.
**10)** $\underline{R}(V_1 \cap V_2) = \underline{R}(V_1) \cap \underline{R}(V_2)$.
**11)** $\overline{R}(V_1 \cap V_2) \subseteq \overline{R}(V_1) \cap \overline{R}(V_2)$.
**12)** $\underline{R}(V_1 \cup V_2) \supseteq \underline{R}(V_1) \cup \underline{R}(V_2)$.
**13)** $\overline{R}(V_1 \cup V_2) = \overline{R}(V_1) \cup \overline{R}(V_2)$.
**14)** $V_1 \subseteq V_2 \Rightarrow \underline{R}(V_1) \subseteq \underline{R}(V_2)$ and $\overline{R}(V_1) \subseteq \overline{R}(V_2)$.

***Hypersoft Sets***

Let $X$ be a universal set (or set of objects) and $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ be the pairwise disjoint sets of parameters with respect to $X$. Generally, the parameters are attributes, properties, characteristic of the objects in $X$. The concept of hypersoft set is described as follows.

**Definition 2.4.** ([27]) Let $A_i$ be the non-empty subset of $\mathcal{E}_i$ for each $i \in I = \{1,2, \ldots, n\}$. Then, the pair $(\Gamma, \prod_{i\in I} A_i)$ is called a hypersoft set over and $X$, where $\Gamma$ is mapping given by

$$\Gamma: \prod_{i\in I} A_i \longrightarrow P(X) \quad (3)$$

Also, $\varepsilon^i$ is an element of $A_i$ and $(\varepsilon^i)_{i\in I}$ is an element of $\prod_{i\in I} A_i = A_1 \times A_2 \times \ldots \times A_n$.

Notation 1. The set of all hypersoft sets over the universal set $X$ for $\prod_{i\in I} \mathcal{E}_i$ is denoted by $S_H\langle X, \prod_{i\in I} \mathcal{E}_i \rangle$.

**Example 2.5.** Let $X = \{x_1, x_2, \ldots, x_{12}\}$ be the set of objects having different sizes, colors and surfaces texture features. The pairwise disjoint sets of parameters are $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ such that $\mathcal{E}_1 = \{\varepsilon_1^1 = very\ small,\ \varepsilon_2^1 = small, \varepsilon_3^1 = average,\ \varepsilon_4^1 = large,\ \varepsilon_5^1 = very\ large\}$ represents the size ranges, $\mathcal{E}_2 = \{\varepsilon_1^2 = blackish,\ \varepsilon_2^2 = dark\ brown,\ \varepsilon_3^2 = yellowish,\ \varepsilon_4^2 = reddish\}$ represents the color space and $\mathcal{E}_3 = \{\varepsilon_1^3 = course,\ \varepsilon_2^3 = fine, \varepsilon_3^3 = extra\ fine\}$ represents the surface texture granularity. Assuming that $A_1 = \{\varepsilon_3^1, \varepsilon_4^1, \varepsilon_5^1\} \subseteq \mathcal{E}_1$, $A_2 = \{\varepsilon_1^2, \varepsilon_4^2\} \subseteq \mathcal{E}_2$ and $A_3 = \{\varepsilon_1^3, \varepsilon_3^3\} \subseteq \mathcal{E}_3$. Then, one creates the following hypersoft set.

$$(\Gamma, \textstyle\prod_{i=1}^3 A_i) = \left\{\begin{array}{l} ((\varepsilon_3^1, \varepsilon_1^2, \varepsilon_1^3), \{x_1, x_{10}\}), \\ ((\varepsilon_3^1, \varepsilon_1^2, \varepsilon_3^3), \emptyset), \\ ((\varepsilon_3^1, \varepsilon_4^2, \varepsilon_3^3), \{x_2\}), \\ ((\varepsilon_3^1, \varepsilon_4^2, \varepsilon_3^3), \{x_8\}), \\ ((\varepsilon_4^1, \varepsilon_1^2, \varepsilon_1^3), \{x_6, x_7\}), \\ ((\varepsilon_4^1, \varepsilon_1^2, \varepsilon_3^3), \emptyset), \\ ((\varepsilon_4^1, \varepsilon_4^2, \varepsilon_1^3), \{x_4\}), \\ ((\varepsilon_4^1, \varepsilon_4^2, \varepsilon_3^3), \{x_4, x_5, x_{12}\}), \\ ((\varepsilon_5^1, \varepsilon_1^2, \varepsilon_1^3), \emptyset), \\ ((\varepsilon_5^1, \varepsilon_1^2, \varepsilon_3^3), \{x_9\}), \\ ((\varepsilon_5^1, \varepsilon_4^2, \varepsilon_1^3), \emptyset), \\ ((\varepsilon_5^1, \varepsilon_4^2, \varepsilon_3^3), \{x_3\}) \end{array}\right\} \quad (4)$$

**Definition 2.6.** ([28]) Let $(\Gamma, \prod_{i\in I} A_i) \in S_H\langle X, \prod_{i\in I} \mathcal{E}_i\rangle$.

**a)** If $\Gamma((\varepsilon^i)_{i\in I}) = \emptyset$ for all $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$ then it is called a relative null hypersoft set, denoted by $\mathcal{N}_{\langle X, \prod_{i\in I} A_i\rangle}$. If $\prod_{i\in I} A_i = \prod_{i\in I} \mathcal{E}_i$ then it is said to be a null hypersoft set, simply denoted by $\mathcal{N}_{\langle X\rangle}$.

**b)** If $\Gamma((\varepsilon^i)_{i\in I}) = X$ for all $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$ then it is called a relative whole hypersoft set, denoted by $\mathcal{W}_{\langle X, \prod_{i\in I} A_i\rangle}$. If $\prod_{i\in I} A_i = \prod_{i\in I} \mathcal{E}_i$ then it is said to be an absolute hypersoft set, simply denoted by $\mathcal{W}_{\langle X\rangle}$.

**Definition 2.7.** ([28]) Let $(\Gamma^1, \prod_{i\in I} A_i)$, $(\Gamma^2, \prod_{i\in I} B_i) \in S_H\langle X, \prod_{i\in I} \mathcal{E}_i\rangle$.

**a)** $(\Gamma^1, \prod_{i\in I} A_i)$ is said to be a hypersoft subset of $(\Gamma^2, \prod_{i\in I} B_i)$, denoted by $(\Gamma^1, \prod_{i\in I} A_i) \,\tilde{\subseteq}\, (\Gamma^2, \prod_{i\in I} B_i)$, if $A_i \subseteq B_i$ for each $i \in I$ and $\Gamma^1((\varepsilon^i)_{i\in I}) \subseteq \Gamma^2((\varepsilon^i)_{i\in I})$ for all $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$.

**b)** Two hypersoft sets $(\Gamma^1, \prod_{i\in I} A_i)$ and $(\Gamma^2, \prod_{i\in I} B_i)$ are called equal, denoted by $(\Gamma^1, \prod_{i\in I} A_i) = (\Gamma^2, \prod_{i\in I} B_i)$, if $(\Gamma^1, \prod_{i\in I} A_i) \,\tilde{\subseteq}\, (\Gamma^2, \prod_{i\in I} B_i)$ and $(\Gamma^1, \prod_{i\in I} A_i) \,\tilde{\supseteq}\, (\Gamma^2, \prod_{i\in I} B_i)$.

**Definition 2.8.** ([28]) Let $(\Gamma, \prod_{i\in I} A_i) \in S_H\langle X, \prod_{i\in I} \mathcal{E}_i\rangle$. Then the complement of $(\Gamma, \prod_{i\in I} A_i)$, denoted by $(\Gamma, \prod_{i\in I} A_i)^{\tilde{c}}$, is defined as

$$(\Gamma, \textstyle\prod_{i\in I} A_i)^{\tilde{c}} = (\Gamma^{\tilde{c}}, \prod_{i\in I} A_i) \quad (5)$$

where $\Gamma^{\tilde{c}}((\varepsilon^i)_{i\in I})$ is the complement of $\Gamma((\varepsilon^i)_{i\in I})$ for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$.

**Definition 2.9.** ([28]) Let $(\Gamma^1, \prod_{i\in I} A_i)$, $(\Gamma^2, \prod_{i\in I} B_i) \in S_H\langle X, \prod_{i\in I} \mathcal{E}_i\rangle$.

**a)** The restricted intersection of $(\Gamma^1, \prod_{i\in I} A_i)$ and $(\Gamma^2, \prod_{i\in I} B_i)$ is denoted and defined by $(\Gamma^3, \prod_{i\in I} C_i) = (\Gamma^1, \prod_{i\in I} A_i) \Cap (\Gamma^2, \prod_{i\in I} B_i)$ where $C_i = A_i \cap B_i$ for each $i \in I$ and

$$\Gamma^3((\varepsilon^i)_{i\in I}) = \Gamma^1((\varepsilon^i)_{i\in I}) \cap \Gamma^2((\varepsilon^i)_{i\in I}) \quad (6)$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$. If $C_i$ is an empty set for some $i$, then $(\Gamma^1, \prod_{i\in I} A_i) \Cap (\Gamma^2, \prod_{i\in I} B_i)$ is a (relative) null hypersoft set.

**b)** The extended intersection of $(\Gamma^1, \prod_{i\in I} A_i)$ and $(\Gamma^2, \prod_{i\in I} B_i)$ is denoted and defined by $(\Gamma^3, \prod_{i\in I} C_i) = (\Gamma^1, \prod_{i\in I} A_i) \sqcap (\Gamma^2, \prod_{i\in I} B_i)$ where $C_i = A_i \cup B_i$ for each $i \in I$ and

$$\Gamma^3((\varepsilon^i)_{i\in I}) = \begin{cases} \Gamma^1((\varepsilon^i)_{i\in I}) & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \\ \Gamma^2((\varepsilon^i)_{i\in I}), & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} B_i \\ \Gamma^1((\varepsilon^i)_{i\in I}) \cap \Gamma^2((\varepsilon^i)_{i\in I}), & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \cap \prod_{i\in I} B_i \end{cases} \quad (7)$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$.

**Definition 2.10.** ([28]) Let $(\Gamma^1, \prod_{i\in I} A_i)$, $(\Gamma^2, \prod_{i\in I} B_i) \in S_H\langle X, \prod_{i\in I} \mathcal{E}_i\rangle$.

**a)** The restricted union of $(\Gamma^1, \prod_{i\in I} A_i)$ and $(\Gamma^2, \prod_{i\in I} B_i)$ is denoted and defined by $(\Gamma^3, \prod_{i\in I} C_i) = (\Gamma^1, \prod_{i\in I} A_i) \Cup (\Gamma^2, \prod_{i\in I} B_i)$ where $C_i = A_i \cap B_i$ for each $i \in I$ and

$$\Gamma^3((\varepsilon^i)_{i\in I}) = \Gamma^1((\varepsilon^i)_{i\in I}) \cup \Gamma^2((\varepsilon^i)_{i\in I}) \quad (8)$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$.

**b)** The extended union of $(\Gamma^1, \prod_{i\in I} A_i)$ and $(\Gamma^2, \prod_{i\in I} B_i)$ is denoted and defined by $(\Gamma^3, \prod_{i\in I} C_i) = (\Gamma^1, \prod_{i\in I} A_i) \sqcup (\Gamma^2, \prod_{i\in I} B_i)$ where $C_i = A_i \cup B_i$ for each $i \in I$ and

$$\Gamma^3((\varepsilon^i)_{i\in I}) = \begin{cases} \Gamma^1((\varepsilon^i)_{i\in I}) & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \\ \Gamma^2((\varepsilon^i)_{i\in I}), & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} B_i \\ \Gamma^1((\varepsilon^i)_{i\in I}) \cup \Gamma^2((\varepsilon^i)_{i\in I}), & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \cap \prod_{i\in I} B_i \end{cases} \quad (9)$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$.

**Definition 2.11.** Let $(\Gamma^1, \prod_{i\in I} A_i), (\Gamma^2, \prod_{i\in I} B_i) \in S_H\langle X, \prod_{i\in I} \mathcal{E}_i\rangle$.

**a)** The restricted difference of $(\Gamma^1, \prod_{i\in I} A_i)$ and $(\Gamma^2, \prod_{i\in I} B_i)$ is denoted and defined by $(\Gamma^3, \prod_{i\in I} C_i) = (\Gamma^1, \prod_{i\in I} A_i) \tilde{\setminus}_r (\Gamma^2, \prod_{i\in I} B_i)$ where $C_i = A_i \cap B_i$ for each $i \in I$ and

$$\Gamma^3((\varepsilon^i)_{i\in I}) = \Gamma^1((\varepsilon^i)_{i\in I}) \setminus \Gamma^2((\varepsilon^i)_{i\in I}) \quad (10)$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$. If $C_i$ is an empty set for some $i$, then $(\Gamma^1,\prod_{i\in I} A_i)\tilde{\setminus}_r(\Gamma^2,\prod_{i\in I} B_i)$ is described to be $(\Gamma^1,\prod_{i\in I} A_i)$. ([28])

**b)** The extended difference of $(\Gamma^1,\prod_{i\in I} A_i)$ and $(\Gamma^2,\prod_{i\in I} B_i)$ is denoted and defined by $(\Gamma^3,\prod_{i\in I} C_i) = (\Gamma^1,\prod_{i\in I} A_i)\tilde{\setminus}_\varepsilon(\Gamma^2,\prod_{i\in I} B_i)$ where $C_i = A_i \cup B_i$ for each $i \in I$ and

$$\Gamma^3((\varepsilon^i)_{i\in I}) = \begin{cases} \Gamma^1((\varepsilon^i)_{i\in I}) & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \\ \emptyset, & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} B_i \\ \Gamma^1((\varepsilon^i)_{i\in I})\setminus\Gamma^2((\varepsilon^i)_{i\in I}), & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \cap \prod_{i\in I} B_i \end{cases} \quad (11)$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$.

***Rough Hypersoft Sets***

In 2021, Rahman et al. ([35]) introduced the concepts of lower rough approximation and upper rough approximation of a hypersoft set with respect to the Pawlak approximation space as follows.

**Definition 2.12.** ([35]) Let $(X,R)$ be a Pawlak approximation space and $(\Gamma,\prod_{i\in I} A_i) \in S_H\langle X,\prod_{i\in I} \mathcal{E}_i\rangle$.

The lower and upper rough approximations of $(\Gamma,\prod_{i\in I} A_i)$ with respect to the Pawlak approximation space $(X,R)$ are respectively denoted by $\underline{R}((\Gamma,\prod_{i\in I} A_i)) = (\underline{\Gamma},\prod_{i\in I} A_i)$ and $\overline{R}((\Gamma,\prod_{i\in I} A_i)) = (\overline{\Gamma},\prod_{i\in I} A_i)$, which are hypersoft sets over $X$ with the set-valued mapping given by

$$\underline{\Gamma}((\varepsilon^i)_{i\in I}) = \underline{R}(\Gamma((\varepsilon^i)_{i\in I})) \quad (12)$$

and

$$\overline{\Gamma}((\varepsilon^i)_{i\in I}) = \overline{R}(\Gamma((\varepsilon^i)_{i\in I})) \quad (13)$$

where $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$.

**Definition 2.13.** ([35]) Let $(X,R)$ be a Pawlak approximation space and $(\Gamma,\prod_{i\in I} A_i) \in S_H\langle X,\prod_{i\in I} \mathcal{E}_i\rangle$. Also, $\underline{R}((\Gamma,\prod_{i\in I} A_i))$ and $\overline{R}((\Gamma,\prod_{i\in I} A_i))$ be lower and upper approximations of $(\Gamma,\prod_{i\in I} A_i)$. If $\underline{R}((\Gamma,\prod_{i\in I} A_i))= \overline{R}((\Gamma,\prod_{i\in I} A_i))$, then the hypersoft set $(\Gamma,\prod_{i\in I} A_i)$ is called a definable; otherwise it is called a rough hypersoft set.

**Theorem 2.14.** ([35]) Let $(X,R)$ be a Pawlak approximation space and $(\Gamma,\prod_{i\in I} A_i) \in S_H\langle X,\prod_{i\in I} \mathcal{E}_i\rangle$.Then, we have the following.

**1)** $\underline{R}((\Gamma,\prod_{i\in I} A_i)) \tilde{\subseteq} (\Gamma,\prod_{i\in I} A_i) \tilde{\subseteq} \overline{R}((\Gamma,\prod_{i\in I} A_i))$

**2)** $\underline{R}(\mathcal{N}_{\langle X,\prod_{i\in I} A_i\rangle}) = \mathcal{N}_{\langle X,\prod_{i\in I} A_i\rangle} = \overline{R}(\mathcal{N}_{\langle X,\prod_{i\in I} A_i\rangle})$

**3)** $\underline{R}(\mathcal{W}_{\langle X,\prod_{i\in I} A_i\rangle}) = \mathcal{W}_{\langle X,\prod_{i\in I} A_i\rangle} = \overline{R}(\mathcal{W}_{\langle X,\prod_{i\in I} A_i\rangle})$

**4)** $\underline{R}\left(\underline{R}((\Gamma,\prod_{i\in I} A_i))\right) = \underline{R}((\Gamma,\prod_{i\in I} A_i))$

**5)** $\overline{R}(\overline{R}((\Gamma,\prod_{i\in I} A_i))) \subseteq \overline{R}((\Gamma,\prod_{i\in I} A_i))$

**6)** $\underline{R}(\overline{R}((\Gamma,\prod_{i\in I} A_i))) \subseteq \overline{R}((\Gamma,\prod_{i\in I} A_i))$

**7)** $\overline{R}(\underline{R}((\Gamma,\prod_{i\in I} A_i))) \subseteq \underline{R}((\Gamma,\prod_{i\in I} A_i))$

## Some Aspects on Rough Hypersoft Sets

In this section, we present some novel concepts and theoretical results improving the rough hypersoft sets.

**Definition 3.1.** Let $\underline{R}((\Gamma,\prod_{i\in I} A_i))$ and $\overline{R}((\Gamma,\prod_{i\in I} A_i))$ be lower and upper approximations of $(\Gamma,\prod_{i\in I} A_i)$ with respect to the Pawlak approximation space $(X,R)$. Then, the positive region, negative region and boundary region of $(\Gamma,\prod_{i\in I} A_i)$ are described as

$pos_R((\Gamma,\prod_{i\in I} A_i)) = \underline{R}((\Gamma,\prod_{i\in I} A_i))$,

$neg_R((\Gamma,\prod_{i\in I} A_i)) = (\overline{R}((\Gamma,\prod_{i\in I} A_i)))^{\tilde{c}}$, and

$bnd_R((\Gamma,\prod_{i\in I} A_i)) = \overline{R}((\Gamma,\prod_{i\in I} A_i))\tilde{\setminus}\underline{R}((\Gamma,\prod_{i\in I} A_i))$

respectively.

**Example 3.2.** Consider the hypersoft set $(\Gamma,\prod_{i=1}^3 A_i)$ over $X$ given in Example 2.5. Also, let $R$ be an equivalence relation on $X$ where the equivalence clases are as follows:

$[x_1]_R = \{x_1,x_{10}\}$, $[x_2]_R = \{x_2\}$, $[x_3]_R = \{x_3\}$, $[x_4]_R = \{x_4,x_5\}$, $[x_6]_R = \{x_6,x_7\}$ and $[x_8]_R = \{x_8,x_9,x_{11},x_{12}\}$.

Then, the lower and upper rough approximations of $(\Gamma,\prod_{i=1}^3 A_i)$ with respect to the Pawlak approximation space $(X,R)$ are

$$\underline{R}\left((\Gamma,\prod_{i=1}^3 A_i)\right) = \left\{\begin{array}{l} ((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_1^3),\{x_1,x_{10}\}), \\ ((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset), \\ ((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),\{x_2\}), \\ ((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),\emptyset), \\ ((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_1^3),\{x_6,x_7\}), \\ ((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset), \\ ((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_1^3),\emptyset), \\ ((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_3^3),\{x_4,x_5\}), \\ ((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_1^3),\emptyset), \\ ((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset), \\ ((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_1^3),\emptyset), \\ ((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_3^3),\{x_3\}) \end{array}\right\} \quad (14)$$

and

$$\overline{R}\left((\Gamma,\prod_{i=1}^{3}A_i)\right)=\left\{\begin{array}{l}((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_1^3),\{x_1,x_{10}\}),\\((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset),\\((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),\{x_2\}),\\((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),\{x_8,x_9,x_{11},x_{12}\}),\\((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_1^3),\{x_6,x_7\}),\\((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset),\\((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_1^3),\{x_4,x_5\}),\\((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_3^3),\{x_4,x_5,x_8,x_9,x_{11},x_{12}\}),\\((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_1^3),\emptyset),\\((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_3^3),\{x_8,x_9,x_{11},x_{12}\}),\\((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_1^3),\emptyset),\\((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_3^3),\{x_3\})\end{array}\right\} \quad (15)$$

Since $\underline{R}\left((\Gamma,\prod_{i=1}^{3}A_i)\right)\neq\overline{R}\left((\Gamma,\prod_{i=1}^{3}A_i)\right)$, we can say that $(\Gamma,\prod_{i=1}^{3}A_i)$ is rough hypersoft set. Furthermore, the positive region of $(\Gamma,\prod_{i=1}^{3}A_i)$ is

$$pos_R\left((\Gamma,\prod_{i=1}^{3}A_i)\right)=\underline{R}\left((\Gamma,\prod_{i=1}^{3}A_i)\right) \quad (16)$$

the negative region of $(\Gamma,\prod_{i=1}^{3}A_i)$ is

$$neg_R\left((\Gamma,\prod_{i=1}^{3}A_i)\right)=\left\{\begin{array}{l}((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_1^3),X-\{x_1,x_{10}\}),\\((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_3^3),X),\\((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_2\}),\\((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_8,x_9,x_{11},x_{12}\}),\\((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_1^3),X-\{x_6,x_7\}),\\((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_3^3),X),\\((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_1^3),X-\{x_4,x_5\}),\\((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_4,x_5,x_8,x_9,x_{11},x_{12}\}),\\((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_1^3),X),\\((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_3^3),X-\{x_8,x_9,x_{11},x_{12}\}),\\((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_1^3),X),\\((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_3\})\end{array}\right\} \quad (17)$$

and the boundary region of $(\Gamma,\prod_{i=1}^{3}A_i)$ is

$$bnd_R\left((\Gamma,\prod_{i=1}^{3}A_i)\right)=\left\{\begin{array}{l}((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_1^3),\emptyset),\\((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset),\\((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),\emptyset),\\((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),\{x_8,x_9,x_{11},x_{12}\}),\\((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_1^3),\emptyset),\\((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset),\\((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_1^3),\{x_4,x_5\}),\\((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_3^3),\{x_8,x_9,x_{11},x_{12}\}),\\((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_1^3),\emptyset),\\((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_3^3),\{x_8,x_9,x_{11},x_{12}\}),\\((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_1^3),\emptyset),\\((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_3^3),\emptyset)\end{array}\right\} \quad (18)$$

Now, we present a few new properties for the lower and upper rough approximations of hypersoft set.

**Proposition 3.3.** Let $(X,R)$ be a Pawlak approximation space and $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$. Then, we have the following equalities.

**1)** $\underline{R}\left((\Gamma,\prod_{i\in I}A_i)\right)=\left(\overline{R}\left((\Gamma,\prod_{i\in I}A_i)^{\tilde{c}}\right)\right)^{\tilde{c}}$.

**2)** $\overline{R}\left((\Gamma,\prod_{i\in I}A_i)\right)=\left(\underline{R}\left((\Gamma,\prod_{i\in I}A_i)^{\tilde{c}}\right)\right)^{\tilde{c}}$.

**Proof.** Let $(X,R)$ be a Pawlak approximation space and $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$.

**1)** From the definitions of upper rough approximation and complement of hypersoft set,

$$\overline{\Gamma^c}\left((\varepsilon^i)_{i\in I}\right)=\{x\in X\,|\,[x]_R\cap\Gamma^c\left((\varepsilon^i)_{i\in I}\right)\neq\emptyset\} \quad (19)$$

and then

$$\begin{aligned}(\overline{\Gamma^c}((\varepsilon^i)_{i\in I}))^c&=X-\{x\in X\,|\,[x]_R\cap\Gamma^c\left((\varepsilon^i)_{i\in I}\right)\neq\emptyset\}\\&=X-\{x\in X\,|\,[x]_R\nsubseteq\Gamma\left((\varepsilon^i)_{i\in I}\right)\}\\&=\{x\in X\,|\,[x]_R\subseteq\Gamma\left((\varepsilon^i)_{i\in I}\right)\}\\&=\underline{\Gamma}\left((\varepsilon^i)_{i\in I}\right)\end{aligned} \quad (20)$$

for each $(\varepsilon^i)_{i\in I}\in\prod_{i\in I}A_i$. We deduce that

$$\underline{R}\left((\Gamma,\prod_{i\in I}A_i)\right)=\left(\overline{R}\left((\Gamma,\prod_{i\in I}A_i)^{\tilde{c}}\right)\right)^{\tilde{c}} \quad (21)$$

**2)** It can be proved similar to the proof of (1).

**Example 3.4.** Consider the hypersoft set $(\Gamma,\prod_{i=1}^{3}A_i)$ in Example 2.5 and $\underline{R}\left((\Gamma,\prod_{i=1}^{3}A_i)\right)$ and $\overline{R}\left((\Gamma,\prod_{i=1}^{3}A_i)\right)$ in Example 3.2. The complement of $(\Gamma,\prod_{i=1}^{3}A_i)$ is

 

$$(\Gamma,\prod_{i=1}^{3} A_i)^{\tilde{c}} = \left\{\begin{array}{l}((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_1^3),X-\{x_1,x_{10}\}),\\ ((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_3^3),X),\\ ((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_2\}),\\ ((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_8\}),\\ ((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_1^3),X-\{x_6,x_7\}),\\ ((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_3^3),X),\\ ((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_1^3),X-\{x_4\}),\\ ((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_4,x_5,x_{12}\}),\\ ((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_1^3),X),\\ ((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_3^3),X-\{x_9\}),\\ ((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_1^3),X),\\ ((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_3\})\end{array}\right\} \quad (22)$$

The upper rough approximation of $(\Gamma,\prod_{i=1}^{3} A_i)^{\tilde{c}}$ is found as equation (23).

Thus, we obtain that $\left(\overline{R}\left((\Gamma,\prod_{i=1}^{3} A_i)^{\tilde{c}}\right)\right)^{\tilde{c}} = \underline{R}\left((\Gamma,\prod_{i=1}^{3} A_i)\right)$ from Eqs. (14) and (23). Similarly, it can be shown that $\left(\underline{R}\left((\Gamma,\prod_{i=1}^{3} A_i)^{\tilde{c}}\right)\right)^{\tilde{c}} = \overline{R}\left((\Gamma,\prod_{i=1}^{3} A_i)\right)$.

In ([35]), the authors proposed several properties for the lower rough approximations and the upper rough approximations of the intersection and union of the hypersoft sets $(\Gamma^1,\prod_{i\in I} A_i)$ and $(\Gamma^2,\prod_{i\in I} A_i)$. By replicating these properties from different perspectives, we discuss the following properties for the lower rough approximations and the upper rough approximations of the restricted/extended intersection and restricted/extended union of the hypersoft sets $(\Gamma^1,\prod_{i\in I} A_i)$ and $(\Gamma^2,\prod_{i\in I} B_i)$.

$$\overline{R}((\Gamma,\prod_{i=1}^{3} A_i)^{\tilde{c}}) = \left\{\begin{array}{l}((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_1^3),X-\{x_1,x_{10}\}),\\ ((\varepsilon_3^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset),\\ ((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_2\}),\\ ((\varepsilon_3^1,\varepsilon_4^2,\varepsilon_3^3),\emptyset),\\ ((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_1^3),X-\{x_6,x_7\}),\\ ((\varepsilon_4^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset),\\ ((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_1^3),\emptyset),\\ ((\varepsilon_4^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_4,x_5\}),\\ ((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_1^3),\emptyset),\\ ((\varepsilon_5^1,\varepsilon_1^2,\varepsilon_3^3),\emptyset),\\ ((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_1^3),\emptyset),\\ ((\varepsilon_5^1,\varepsilon_4^2,\varepsilon_3^3),X-\{x_3\})\end{array}\right\} \quad (23)$$

**Theorem 3.5.** Let $(X,R)$ be a Pawlak approximation space and $(\Gamma^1,\prod_{i\in I} A_i)$, $(\Gamma^2,\prod_{i\in I} B_i) \in S_H\langle X,\prod_{i\in I} \mathcal{E}_i\rangle$. Then, we have the following operational properties.

**1)**
$$\underline{R}\left((\Gamma^1,\prod_{i\in I} A_i) \Cap (\Gamma^2,\prod_{i\in I} B_i)\right) = \underline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \Cap \underline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$$

**2)**
$$\underline{R}\left((\Gamma^1,\prod_{i\in I} A_i) \sqcap (\Gamma^2,\prod_{i\in I} B_i)\right) = \underline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \sqcap \underline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$$

**3)** $\overline{R}\left((\Gamma^1,\prod_{i\in I} A_i) \Cap (\Gamma^2,\prod_{i\in I} B_i)\right) \tilde{\subseteq} \overline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \Cap \overline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$.

**4)** $\overline{R}\left((\Gamma^1,\prod_{i\in I} A_i) \sqcap (\Gamma^2,\prod_{i\in I} B_i)\right) \tilde{\subseteq} \overline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \sqcap \overline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$.

**5)** $\underline{R}\left((\Gamma^1,\prod_{i\in I} A_i) \Cup (\Gamma^2,\prod_{i\in I} A_i)\right) \tilde{\supseteq} \underline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \Cup \underline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$.

**6)** $\underline{R}\left((\Gamma^1,\prod_{i\in I} A_i) \sqcup (\Gamma^2,\prod_{i\in I} B_i)\right) \tilde{\supseteq} \underline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \sqcup \underline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$.

**7)** $\overline{R}\left((\Gamma^1,\prod_{i\in I} A_i) \Cup (\Gamma^2,\prod_{i\in I} B_i)\right) = \overline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \Cup \overline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$.

**8)** $\overline{R}\left((\Gamma^1,\prod_{i\in I} A_i) \sqcup (\Gamma^2,\prod_{i\in I} B_i)\right) = \overline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \sqcup \overline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$.

**9)**
$$(\Gamma^1,\prod_{i\in I} A_i) \tilde{\subseteq} (\Gamma^2,\prod_{i\in I} B_i) \Rightarrow \underline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \tilde{\subseteq} \underline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$$

**10)**
$$(\Gamma^1,\prod_{i\in I} A_i) \tilde{\subseteq} (\Gamma^2,\prod_{i\in I} B_i) \Rightarrow \overline{R}\left((\Gamma^1,\prod_{i\in I} A_i)\right) \tilde{\subseteq} \overline{R}\left((\Gamma^2,\prod_{i\in I} B_i)\right)$$

**Proof.** Let $(X,R)$ be a Pawlak approximation space and $(\Gamma^1,\prod_{i\in I} A_i)$, $(\Gamma^2,\prod_{i\in I} B_i) \in S_H\langle X,\prod_{i\in I} \mathcal{E}_i\rangle$.

**1)**

Let $(\Gamma^1,\prod_{i\in I} A_i) \Cap (\Gamma^2,\prod_{i\in I} B_i) = (\Gamma^3,\prod_{i\in I} C_i)$. Then, $C_i = A_i \cap B_i$ for each $i \in I$ and $\Gamma^3((\varepsilon^i)_{i\in I}) = \Gamma^1((\varepsilon^i)_{i\in I}) \cap \Gamma^2((\varepsilon^i)_{i\in I})$ for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$. By Eq. (12), we can write

$$\underline{\Gamma^3}((\varepsilon^i)_{i\in I}) = \underline{R}(\Gamma^3((\varepsilon^i)_{i\in I}) = \underline{R}(\Gamma^1((\varepsilon^i)_{i\in I}) \cap \Gamma^2((\varepsilon^i)_{i\in I}))$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$. From Theorem 2.3, we have

$$\underline{R}\left(\Gamma^1((\varepsilon^i)_{i\in I}) \cap \Gamma^2((\varepsilon^i)_{i\in I})\right) = \underline{R}(\Gamma^1((\varepsilon^i)_{i\in I})) \cap \underline{R}(\Gamma^2((\varepsilon^i)_{i\in I}))$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$. So, we deduce that

$$\underline{\Gamma^3}((\varepsilon^i)_{i\in I}) = \underline{\Gamma^1}((\varepsilon^i)_{i\in I}) \cap \underline{\Gamma^2}((\varepsilon^i)_{i\in I})$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$. Consequently, we have

$\underline{R}\big((\Gamma^1,\prod_{i\in I} A_i) \Cap (\Gamma^2,\prod_{i\in I} B_i)\big)$

$= \underline{R}\big((\Gamma^1,\prod_{i\in I} A_i)\big) \Cap \underline{R}\big((\Gamma^2,\prod_{i\in I} B_i)\big)$.

**2)**
Let $(\Gamma^1,\prod_{i\in I} A_i) \sqcap (\Gamma^2,\prod_{i\in I} B_i) = (\Gamma^3,\prod_{i\in I} C_i)$. By using similar techniques, it is obtained that

$$\underline{\Gamma^3}\big((\varepsilon^i)_{i\in I}\big) = \begin{cases} \underline{\Gamma^1}\big((\varepsilon^i)_{i\in I}\big) & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \\ \underline{\Gamma^2}\big((\varepsilon^i)_{i\in I}\big), & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} B_i \\ \underline{\Gamma^1}\big((\varepsilon^i)_{i\in I}\big) \cap \underline{\Gamma^2}\big((\varepsilon^i)_{i\in I}\big), & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \cap \prod_{i\in I} B_i \end{cases}$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i = A_i \cup B_i$.

So, we have

$\underline{R}\big((\Gamma^1,\prod_{i\in I} A_i) \sqcap (\Gamma^2,\prod_{i\in I} B_i)\big)$
$= \underline{R}\big((\Gamma^1,\prod_{i\in I} A_i)\big) \sqcap \underline{R}\big((\Gamma^2,\prod_{i\in I} B_i)\big)$.

**3)** By considering Eq. (13), it can be demonstrated similar to the proof of (1).

**4)** By considering Eq. (13), it can be demonstrated similar to the proof of (2).

**5)**
Let $(\Gamma^1,\prod_{i\in I} A_i) \Cup (\Gamma^2,\prod_{i\in I} B_i) = (\Gamma^3,\prod_{i\in I} C_i)$. Then, $C_i = A_i \cap B_i$ for each $i \in I$ and $\Gamma^3((\varepsilon^i)_{i\in I}) = \Gamma^1((\varepsilon^i)_{i\in I}) \cup \Gamma^2((\varepsilon^i)_{i\in I})$ for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$. By Eq. (12), we have

$$\underline{\Gamma^3}\big((\varepsilon^i)_{i\in I}\big) = \underline{R}(\Gamma^3((\varepsilon^i)_{i\in I})) = \underline{R}(\Gamma^1((\varepsilon^i)_{i\in I}) \cup \Gamma^2((\varepsilon^i)_{i\in I}))$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$. By Theorem 2.3, we can write

$$\underline{R}\Big(\Gamma^1((\varepsilon^i)_{i\in I}) \cap \Gamma^2((\varepsilon^i)_{i\in I})\Big) \supseteq \underline{R}(\Gamma^1((\varepsilon^i)_{i\in I})) \cap \underline{R}(\Gamma^2((\varepsilon^i)_{i\in I}))$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$ and so

$$\underline{\Gamma^3}\big((\varepsilon^i)_{i\in I}\big) = \underline{\Gamma^1}\big((\varepsilon^i)_{i\in I}\big) \cup \underline{\Gamma^2}\big((\varepsilon^i)_{i\in I}\big)$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i$. Consequently, we obtain

$\underline{R}\big((\Gamma^1,\prod_{i\in I} A_i) \Cup (\Gamma^2,\prod_{i\in I} B_i)\big)$

$\widetilde{\supseteq}\, \underline{R}\big((\Gamma^1,\prod_{i\in I} A_i)\big) \Cup \underline{R}\big((\Gamma^2,\prod_{i\in I} B_i)\big)$.

**6)**
Let $(\Gamma^1,\prod_{i\in I} A_i) \sqcup (\Gamma^2,\prod_{i\in I} B_i) = (\Gamma^3,\prod_{i\in I} C_i)$. By using similar techniques, it is obtained that

$$\underline{\Gamma^3}\big((\varepsilon^i)_{i\in I}\big) = \begin{cases} \underline{\Gamma^1}\big((\varepsilon^i)_{i\in I}\big) & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \\ \underline{\Gamma^2}\big((\varepsilon^i)_{i\in I}\big), & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} B_i \\ \underline{\Gamma^1}\big((\varepsilon^i)_{i\in I}\big) \cup \underline{\Gamma^2}\big((\varepsilon^i)_{i\in I}\big), & \text{if } (\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i \cap \prod_{i\in I} B_i \end{cases}$$

for each $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} C_i = A_i \cup B_i$.By considering Theorem 2.3, it is deduced that

$\underline{R}\big((\Gamma^1,\prod_{i\in I} A_i) \sqcup (\Gamma^2,\prod_{i\in I} B_i)\big)$

$\widetilde{\supseteq}\, \underline{R}\big((\Gamma^1,\prod_{i\in I} A_i)\big) \sqcup \underline{R}\big((\Gamma^2,\prod_{i\in I} B_i)\big)$.

**7)** By considering Eq. (13), it can be demonstrated similar to the proof of (5).

**8)** By considering Eq. (13), it can be demonstrated similar to the proof of (6).

**9)** Assume that $(\Gamma^1,\prod_{i\in I} A_i) \,\widetilde{\subseteq}\, (\Gamma^2,\prod_{i\in I} B_i)$. Then, by Definition 2.7, $A_i \subseteq B_i$ for each $i \in I$ i.e., $\prod_{i\in I} A_i \subseteq \prod_{i\in I} B_i)$ and $\Gamma^1((\varepsilon^i)_{i\in I}) \subseteq \Gamma^2((\varepsilon^i)_{i\in I})$ for all $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$.

Hence, we obtain that

$$\begin{aligned}\underline{\Gamma^1}\big((\varepsilon^i)_{i\in I}\big) &= \{x \in X \mid [x]_R \subseteq \Gamma^1\big((\varepsilon^i)_{i\in I}\big)\} \\ &\subseteq \{x \in X \mid [x]_R \subseteq \Gamma^2\big((\varepsilon^i)_{i\in I}\big)\} \\ &= \underline{\Gamma^2}\big((\varepsilon^i)_{i\in I}\big)\end{aligned}$$

for all $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$. Thus, we conclude that $\underline{R}\big((\Gamma^1,\prod_{i\in I} A_i)\big) \,\widetilde{\subseteq}\, \underline{R}\big((\Gamma^2,\prod_{i\in I} B_i)\big)$.

**10)** This property is proved similar to the proof of (9).

**Example 3.6.** We consider the hypersoft set $(\Gamma,\prod_{i=1}^{3} A_i) = (\Gamma^1,\prod_{i=1}^{3} A_i)$ in Example 2.5, and the equivalence relation $R$ on $X$, $\underline{R}((\Gamma,\prod_{i=1}^{3} A_i)) = \underline{R}((\Gamma^1,\prod_{i=1}^{3} A_i))$ and $\overline{R}((\Gamma,\prod_{i=1}^{3} A_i)) = \overline{R}((\Gamma^1,\prod_{i=1}^{3} A_i))$ in Example 3.2. Assuming that another hypersoft set over $X$ for $B_1 = \{\varepsilon_2^1, \varepsilon_4^1\}$, $B_2 = \{\varepsilon_1^2, \varepsilon_4^2\}$ and $B_3 = \{\varepsilon_1^3\}$ is

$$(\Gamma^2,\prod_{i=1}^{3} B_i) = \left\{\begin{matrix} \big((\varepsilon_2^1, \varepsilon_1^2, \varepsilon_1^3), \{x_9, x_{10}\}\big), \\ \big((\varepsilon_2^1, \varepsilon_4^2, \varepsilon_1^3), \{x_4, x_8, x_{12}\}\big), \\ \big((\varepsilon_4^1, \varepsilon_1^2, \varepsilon_1^3), \{x_6, x_7, x_{11}\}\big), \\ \big((\varepsilon_4^1, \varepsilon_4^2, \varepsilon_1^3), \{x_2, x_3, x_5\}\big) \end{matrix}\right\} \quad (24)$$

Then, we obtain that

$\underline{R}\Big((\Gamma^1,\prod_{i=1}^{3} A_i) \sqcup (\Gamma^2,\prod_{i=1}^{3} B_i)\Big) \,\widetilde{\supseteq}\, \underline{R}\Big((\Gamma^1,\prod_{i=1}^{3} A_i)\Big) \sqcup \underline{R}\Big((\Gamma^2,\prod_{i=1}^{3} B_i)\Big)$ and $\overline{R}\Big((\Gamma^1,\prod_{i=1}^{3} A_i) \sqcup (\Gamma^2,\prod_{i=1}^{3} B_i)\Big) = \overline{R}\Big((\Gamma^1,\prod_{i=1}^{3} A_i)\Big) \sqcup \overline{R}\Big((\Gamma^2,\prod_{i=1}^{3} B_i)\Big)$.

### Hypersoft Rough Sets

In this section, we introduce the hypersoft lower approximation and hypersoft upper approximation of a subset of universal set with respect to the hypersoft approximation space. Also, we investigate their characteristic features.

**Definition 4.1.** Let $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$. Also, let $(X,R)$ be a Pawlak approximation space. If $R$ is taken as $(\Gamma,\prod_{i\in I}A_i)$ then Pawlak approximation space is said to be a hypersoft approximation space and denoted by $\mathfrak{P}=(X,(\Gamma,\prod_{i\in I}A_i))$.

**Definition 4.2.** Let $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$, $V\subseteq X$ and $\mathfrak{P}=(X,(\Gamma,\prod_{i\in I}A_i))$ be a hypersoft approximation space. Then,

$$\underline{appr}_{\mathfrak{P}}(V)=\{x\in X|\exists(\varepsilon^i)_{i\in I}\in\prod_{i\in I}A_i,\ [x\in\Gamma((\varepsilon^i)_{i\in I})\subseteq V]\} \quad (25)$$

and

$$\overline{appr}_{\mathfrak{P}}(V)=\{x\in X|\exists(\varepsilon^i)_{i\in I}\in\prod_{i\in I}A_i,\ [x\in\Gamma((\varepsilon^i)_{i\in I}),\Gamma((\varepsilon^i)_{i\in I})\cap V\neq\emptyset]\} \quad (26)$$

are called hypersoft $\mathfrak{P}$-lower approximation and hypersoft $\mathfrak{P}$-upper approximation of $V$ with respect to the hypersoft approximation space $\mathfrak{P}$, respectively.

**Definition 4.3.** Let $\underline{appr}_{\mathfrak{P}}(V)$ and $\overline{appr}_{\mathfrak{P}}(V)$ be hypersoft $\mathfrak{P}$-lower and $\mathfrak{P}$-upper approximations of $V\subseteq X$ with respect to the hypersoft approximation space $\mathfrak{P}$, respectively. Then,

$pos_{\mathfrak{P}}(V)=\underline{appr}_{\mathfrak{P}}(V)$,

$neg_{\mathfrak{P}}(V)=X-\overline{appr}_{\mathfrak{P}}(V)=(\overline{appr}_{\mathfrak{P}}(V))^c$,

$bnd_{\mathfrak{P}}(V)=\overline{appr}_{\mathfrak{P}}(V)-\underline{appr}_{\mathfrak{P}}(V)$

are called the hypersoft $\mathfrak{P}$-positive region, hypersoft $\mathfrak{P}$-negative region and hypersoft $\mathfrak{P}$-boundary region of $V$, respectively.

**Definition 4.4.** Let $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$ and $\mathfrak{P}=(X,(\Gamma,\prod_{i\in I}A_i))$ be a hypersoft approximation space. Also, Then, $\underline{appr}_{\mathfrak{P}}(V)$ and $\overline{appr}_{\mathfrak{P}}(V)$ be hypersoft $\mathfrak{P}$-lower and $\mathfrak{P}$-upper approximations of $V\subseteq X$ with respect to the hypersoft approximation space $\mathfrak{P}$, respectively. If $\underline{appr}_{\mathfrak{P}}(V)\neq\overline{appr}_{\mathfrak{P}}(V)$ then $V$ is said to be a hypersoft $\mathfrak{P}$-rough set, otherwise it is called a hypersoft $\mathfrak{P}$-definable.

From the above definitions, it can be said that $V\subseteq X$ is a hypersoft $\mathfrak{P}$-definable if $bnd_{\mathfrak{P}}(V)=\emptyset$. Furthermore, it is clear that $\underline{appr}_{\mathfrak{P}}(V)\subseteq V$ and $\underline{appr}_{\mathfrak{P}}(V)\subseteq\overline{appr}_{\mathfrak{P}}(V)$ for all $V\subseteq X$. However, $V\subseteq\overline{appr}_{\mathfrak{P}}(V)$ does not true in general, as illustrated by the following example.

**Example 4.5.** We consider the hypersoft set $(\Gamma,\prod_{i=1}^{3}A_i)$ over $X$ in Example 2.5. Also, let $\mathfrak{P}=(X,(\Gamma,\prod_{i=1}^{3}A_i))$ be a hypersoft approximation space. Assume that $V_1=\{x_1,x_3,x_5,x_6,x_7,x_{10},x_{11}\}\subseteq X$. Then, the hypersoft $\mathfrak{P}$-lower and $\mathfrak{P}$-upper approximations of $V_1$ with respect to the hypersoft approximation space $\mathfrak{P}$ are respectively

$$\underline{appr}_{\mathfrak{P}}(V_1)=\{x_1,x_3,x_6,x_7,x_{10}\} \quad (27)$$

and

$$\overline{appr}_{\mathfrak{P}}(V_1)=\{x_1,x_3,x_4,x_5,x_6,x_7,x_{10},x_{12}\} \quad (28)$$

Since $\underline{appr}_{\mathfrak{P}}(V_1)\neq\overline{appr}_{\mathfrak{P}}(V_1)$, $V_1$ is a hypersoft $\mathfrak{P}$-rough set. Moreover, it is easy to see that $\underline{appr}_{\mathfrak{P}}(V_1)\subseteq V_1$, $\underline{appr}_{\mathfrak{P}}(V_1)\subseteq\overline{appr}_{\mathfrak{P}}(V_1)$, $V_1\not\subseteq\overline{appr}_{\mathfrak{P}}(V_1)$ and $V_1\not\supseteq\overline{appr}_{\mathfrak{P}}(V_1)$.

Besides, hypersoft $\mathfrak{P}$-positive region, hypersoft $\mathfrak{P}$-negative region and hypersoft $\mathfrak{P}$-boundary region of $V_1$ are $pos_{\mathfrak{P}}(V_1)=\{x_1,x_3,x_6,x_7,x_{10}\}$, $neg_{\mathfrak{P}}(V_1)=\{x_2,x_8,x_9,x_{11}\}$ and $bnd_{\mathfrak{P}}(V_1)=\{x_4,x_5,x_{12}\}\neq\emptyset$.

**Proposition 4.6.** Let $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$ and $\mathfrak{P}=(X,(\Gamma,\prod_{i\in I}A_i))$ be a hypersoft approximation space.

**1)** For all $V\subseteq X$,

$$\underline{appr}_{\mathfrak{P}}(V)==\bigcup_{(\varepsilon^i)_{i\in I}\in\prod_{i\in I}A_i}\{\Gamma((\varepsilon^i)_{i\in I})|\ \Gamma((\varepsilon^i)_{i\in I})\subseteq V\}.$$

**2)** For all $V\subseteq X$,

$$\overline{appr}_{\mathfrak{P}}(V)==\bigcup_{(\varepsilon^i)_{i\in I}\in\prod_{i\in I}A_i}\{\Gamma((\varepsilon^i)_{i\in I})|\ \Gamma((\varepsilon^i)_{i\in I})\cap V\neq\emptyset\}.$$

**Proof.** They are obvious from Definition 4.2.

**Definition 4.7.** Let $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$, $V\subseteq X$ and $\mathfrak{P}=(X,(\Gamma,\prod_{i\in I}A_i))$ be a hypersoft approximation space. Then,

**a)** $V$ is called roughly hypersoft $\mathfrak{P}$-definable if $\underline{appr}_{\mathfrak{P}}(V)\neq\emptyset$ and $\overline{appr}_{\mathfrak{P}}(V)\neq X$.

 

**b)** $V$ is called internally hypersoft $\mathfrak{P}$-definable if $\underline{appr}_{\mathfrak{P}}(V) = \emptyset$ and $\overline{appr}_{\mathfrak{P}}(V) \neq X$.

**c)** $V$ is called externally hypersoft $\mathfrak{P}$-definable if $\underline{appr}_{\mathfrak{P}}(V) \neq \emptyset$ and $\overline{appr}_{\mathfrak{P}}(V) = X$.

**d)** $V$ is called totally hypersoft $\mathfrak{P}$-definable if $\underline{appr}_{\mathfrak{P}}(V) = \emptyset$ and $\overline{appr}_{\mathfrak{P}}(V) = X$.

**Theorem 4.8.** Let $(\Gamma, \prod_{i\in I} A_i) \in S_H\langle X, \prod_{i\in I} \mathcal{E}_i\rangle$ and $\mathfrak{P} = (X, (\Gamma, \prod_{i\in I} A_i))$ be a hypersoft approximation space and $V, V_1, V_2 \subseteq X$. Then, we have the following:

**1)** $\underline{appr}_{\mathfrak{P}}(\emptyset) = \overline{appr}_{\mathfrak{P}}(\emptyset) = \emptyset$.

**2)** $\underline{appr}_{\mathfrak{P}}(X) = \overline{appr}_{\mathfrak{P}}(X) = \bigcup_{(\varepsilon^i)_{i\in I}\in\prod_{i\in I} A_i} \Gamma((\varepsilon^i)_{i\in I})$.

**3)** $V_1 \subseteq V_2 \Rightarrow \underline{appr}_{\mathfrak{P}}(V_1) \subseteq \underline{appr}_{\mathfrak{P}}(V_2)$.

**4)** $V_1 \subseteq V_2 \Rightarrow \overline{appr}_{\mathfrak{P}}(V_1) \subseteq \overline{appr}_{\mathfrak{P}}(V_2)$.

**5)** $\underline{appr}_{\mathfrak{P}}(V_1 \cap V_2) \subseteq \underline{appr}_{\mathfrak{P}}(V_1) \cap \underline{appr}_{\mathfrak{P}}(V_2)$.

**6)** $\overline{appr}_{\mathfrak{P}}(V_1 \cap V_2) \subseteq \overline{appr}_{\mathfrak{P}}(V_1) \cap \overline{appr}_{\mathfrak{P}}(V_2)$.

**7)** $\underline{appr}_{\mathfrak{P}}(V_1 \cup V_2) \supseteq \underline{appr}_{\mathfrak{P}}(V_1) \cup \underline{appr}_{\mathfrak{P}}(V_2)$.

**8)** $\overline{appr}_{\mathfrak{P}}(V_1 \cup V_2) = \overline{appr}_{\mathfrak{P}}(V_1) \cup \overline{appr}_{\mathfrak{P}}(V_2)$.

**Proof.** Let $(\Gamma, \prod_{i\in I} A_i) \in S_H\langle X, \prod_{i\in I} \mathcal{E}_i\rangle$ and $\mathfrak{P} = (X, (\Gamma, \prod_{i\in I} A_i))$ be a hypersoft approximation space and $V, V_1, V_2 \subseteq X$.

**1)** This is straightforward, so omitted.

**2)** It is easily seen from Proposition 4.6 by replacing $V$ with $X$.

**3)** Assume that $V_1 \subseteq V_2$. If $x \in \overline{appr}_{\mathfrak{P}}(V_1)$ then there exists some $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$ such that $x \in \Gamma((\varepsilon^i)_{i\in I}) \subseteq V_1$. Since $V_1 \subseteq V_2$, we have $x \in \Gamma((\varepsilon^i)_{i\in I}) \subseteq V_2$, and so $x \in \overline{appr}_{\mathfrak{P}}(V_2)$. Therefore, we conclude that $\underline{appr}_{\mathfrak{P}}(V_1) \subseteq \underline{appr}_{\mathfrak{P}}(V_2)$ if $V_1 \subseteq V_2$.

**4)** This is similar to the proof of (3).

**5)** Since $V_1 \cap V_2 \subseteq V_1$ and $V_1 \cap V_2 \subseteq V_2$, we can say from the assertion (3) that $\underline{appr}_{\mathfrak{P}}(V_1 \cap V_2) \subseteq \underline{appr}_{\mathfrak{P}}(V_1)$ and $\underline{appr}_{\mathfrak{P}}(V_1 \cap V_2) \subseteq \underline{appr}_{\mathfrak{P}}(V_2)$. Hence, we have $\underline{appr}_{\mathfrak{P}}(V_1 \cap V_2) \subseteq \underline{appr}_{\mathfrak{P}}(V_1) \cap \underline{appr}_{\mathfrak{P}}(V_2)$.

**6)** By considering the assertion (4), it can be demonstrated similar to the proof of (5).

**7)** Since $V_1 \cup V_2 \supseteq V_1$ and $V_1 \cup V_2 \supseteq V_2$, we can write from the assertion (3) that $\underline{appr}_{\mathfrak{P}}(V_1 \cup V_2) \supseteq \underline{appr}_{\mathfrak{P}}(V_1)$ and $\underline{appr}_{\mathfrak{P}}(V_1 \cup V_2) \supseteq \underline{appr}_{\mathfrak{P}}(V_2)$. So, we obtain that $\underline{appr}_{\mathfrak{P}}(V_1 \cup V_2) \supseteq \underline{appr}_{\mathfrak{P}}(V_1) \cup \underline{appr}_{\mathfrak{P}}(V_2)$.

**8)** By proceeding with similar computations in the proof of (7), we can show that $\overline{appr}_{\mathfrak{P}}(V_1 \cup V_2) \supseteq \overline{appr}_{\mathfrak{P}}(V_1) \cup \overline{appr}_{\mathfrak{P}}(V_2)$.

Now we must prove that $\overline{appr}_{\mathfrak{P}}(V_1 \cup V_2) \subseteq \overline{appr}_{\mathfrak{P}}(V_1) \cup \overline{appr}_{\mathfrak{P}}(V_2)$. Let $x \in \overline{appr}_{\mathfrak{P}}(V_1 \cup V_2)$. By Definition 4.2, there exists some $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$ such that $x \in \Gamma((\varepsilon^i)_{i\in I})$ and $\Gamma((\varepsilon^i)_{i\in I}) \cap (V_1 \cup V_2) \neq \emptyset$. Therefore, we have that either $\Gamma((\varepsilon^i)_{i\in I}) \cap V_1 \neq \emptyset$ or $\Gamma((\varepsilon^i)_{i\in I}) \cap V_2 \neq \emptyset$. It follows that $x \in \overline{appr}_{\mathfrak{P}}(V_1)$ or $x \in \overline{appr}_{\mathfrak{P}}(V_2)$. This implies that $\overline{appr}_{\mathfrak{P}}(V_1 \cup V_2) \subseteq \overline{appr}_{\mathfrak{P}}(V_1) \cup \overline{appr}_{\mathfrak{P}}(V_2)$.

Consequently, we have $\overline{appr}_{\mathfrak{P}}(V_1 \cup V_2) = \overline{appr}_{\mathfrak{P}}(V_1) \cup \overline{appr}_{\mathfrak{P}}(V_2)$.

**Example 4.9.** We consider the hypersoft set $(\Gamma, \prod_{i=1}^{3} A_i)$ over $X$ in Example 2.5 and the hypersoft $\mathfrak{P}$-lower and $\mathfrak{P}$-upper approximations ($\underline{appr}_{\mathfrak{P}}(V_1)$ and $\overline{appr}_{\mathfrak{P}}(V_1)$) of $V_1$ with respect to $\mathfrak{P} = (X, (\Gamma, \prod_{i=1}^{3} A_i))$ in Example 4.5.

Assume that $V_2 = \{x_2, x_4, x_6, x_7, x_{12}\} \subseteq X$. Then, the hypersoft $\mathfrak{P}$-lower and $\mathfrak{P}$-upper approximations of $V_1$ with respect to the hypersoft approximation space $\mathfrak{P}$ are respectively

$$\underline{appr}_{\mathfrak{P}}(V_2) = \{x_2, x_4, x_6, x_7\} \quad (29)$$

and

$$\overline{appr}_{\mathfrak{P}}(V_2) = \{x_2, x_4, x_5, x_6, x_7, x_{12}\} \quad (30)$$

Hence, by considering Eqs. (27)-(30), we have

$\underline{appr}_{\mathfrak{P}}(V_1) \cup \underline{appr}_{\mathfrak{P}}(V_2) = \{x_1, x_2, x_3, x_4, x_6, x_7, x_{10}\}$

and

$\overline{appr}_{\mathfrak{P}}(V_1) \cup \overline{appr}_{\mathfrak{P}}(V_2) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_{10}, x_{12}\}$.

On the other hand, for $V_1 \cup V_2 = X - \{x_8, x_9, x_{11}\}$, the hypersoft $\mathfrak{P}$-lower and $\mathfrak{P}$-upper approximations of $V_1 \cup V_2$ with respect to the hypersoft approximation space $\mathfrak{P}$ are

$\underline{appr}_{\mathfrak{P}}(V_1 \cup V_2) = \overline{appr}_{\mathfrak{P}}(V_1 \cup V_2) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_{10}, x_{12}\}$.

Thus, we conclude that $\underline{appr}_{\mathfrak{P}}(V_1 \cup V_2) \supseteq \underline{appr}_{\mathfrak{P}}(V_1) \cup \underline{appr}_{\mathfrak{P}}(V_2)$ and $\overline{appr}_{\mathfrak{P}}(V_1 \cup V_2) = \overline{appr}_{\mathfrak{P}}(V_1) \cup \overline{appr}_{\mathfrak{P}}(V_2)$.

Remark 1. It can be usually said that $\underline{appr}_{\mathfrak{P}}(V) \neq (\overline{appr}_{\mathfrak{P}}(V^c))^c$ and $\overline{appr}_{\mathfrak{P}}(V) \neq (\underline{appr}_{\mathfrak{P}}(V^c))^c$. For instance, we consider Example 4.9. For $V_2^c = \{x_1, x_3, x_5, x_8, x_9, x_{10}, x_{11}\}$, the hypersoft $\mathfrak{P}$-lower and $\mathfrak{P}$-upper approximations of $V_2^c$ with respect to the hypersoft approximation space $\mathfrak{P}$ are respectively

$$\underline{appr}_{\mathfrak{P}}(V_2^c) = \{x_1, x_3, x_8, x_9, x_{10}\} \tag{31}$$

and

$$\overline{appr}_{\mathfrak{P}}(V_2^c) = \{x_1, x_3, x_4, x_5, x_8, x_9, x_{10}, x_{12}\} \tag{32}$$

By Eqs. (29)-(32), we obtain that $\underline{appr}_{\mathfrak{P}}(V_2) \neq (\overline{appr}_{\mathfrak{P}}(V_2^c))^c$ and $\overline{appr}_{\mathfrak{P}}(V_2) \neq (\underline{appr}_{\mathfrak{P}}(V_2^c))^c$.

**Proposition 4.10.** Let $(\Gamma^1, \prod_{i\in I} A_i)$, $(\Gamma^2, \prod_{i\in I} A_i) \in S_H\langle X, \prod_{i\in I} \mathcal{E}_i\rangle$ such that $(\Gamma^1, \prod_{i\in I} A_i) \widetilde{\subseteq} (\Gamma^2, \prod_{i\in I} A_i)$. Also, let $\mathfrak{P}_1 = (X, (\Gamma^1, \prod_{i\in I} A_i))$ and $\mathfrak{P}_2 = (X, (\Gamma^2, \prod_{i\in I} A_i))$ be two hypersoft approximation spaces and $V \subseteq X$. Then, $\overline{appr}_{\mathfrak{P}_1}(V) \subseteq \overline{appr}_{\mathfrak{P}_2}(V)$.

**Proof.** Suppose that $(\Gamma^1, \prod_{i\in I} A_i) \widetilde{\subseteq} (\Gamma^2, \prod_{i\in I} A_i)$. Let $x \in \overline{appr}_{\mathfrak{P}_1}(V)$. Then, by Definition 4.2, there exists some $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$ such that $x \in \Gamma^1((\varepsilon^i)_{i\in I})$ and $\Gamma^1((\varepsilon^i)_{i\in I}) \cap V \neq \emptyset$. Since $\Gamma^1((\varepsilon^i)_{i\in I}) \subseteq \Gamma^2((\varepsilon^i)_{i\in I})$ for all $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$, we obtain that $x \in \Gamma^2((\varepsilon^i)_{i\in I})$ and $\Gamma^2((\varepsilon^i)_{i\in I}) \cap V \neq \emptyset$. So, $x \in \overline{appr}_{\mathfrak{P}_2}(V)$. Thus, we deduce that $\overline{appr}_{\mathfrak{P}_1}(V) \subseteq \overline{appr}_{\mathfrak{P}_2}(V)$.

Note that generally, $\underline{appr}_{\mathfrak{P}_1}(V) \nsubseteq \underline{appr}_{\mathfrak{P}_2}(V)$ and $\underline{appr}_{\mathfrak{P}_1}(V) \nsupseteq \underline{appr}_{\mathfrak{P}_2}(V)$. Especially, if $\Gamma^2((\varepsilon^i)_{i\in I}) \subseteq V$ whenever $\Gamma^1((\varepsilon^i)_{i\in I}) \subseteq V$ for $(\varepsilon^i)_{i\in I} \in \prod_{i\in I} A_i$, then $\underline{appr}_{\mathfrak{P}_1}(V) \subseteq \underline{appr}_{\mathfrak{P}_2}(V)$.

This special condition can easily be demonstrated from the definitions. To illustrate them, the following example is given.

**Example 4.11.** We consider the hypersoft set $(\Gamma^1, \prod_{i=1}^3 A_i)$ over $X$ in Example 2.5 and the hypersoft $\mathfrak{P}_1$-lower and $\mathfrak{P}_1$-upper approximations $(\underline{appr}_{\mathfrak{P}_1}(V_1)$ and $\overline{appr}_{\mathfrak{P}_1}(V_1))$ of $V_1$ with respect to $\mathfrak{P}_1 = (X, (\Gamma^1, \prod_{i=1}^3 A_i))$ in Example 4.5. Assume that

$$(\Gamma^2, \prod_{i=1}^3 A_i) = \left\{\begin{array}{l} ((\varepsilon_3^1, \varepsilon_1^2, \varepsilon_1^3), \{x_1, x_9, x_{10}\}), \\ ((\varepsilon_3^1, \varepsilon_1^2, \varepsilon_3^3), \{x_3, x_9\}), \\ ((\varepsilon_3^1, \varepsilon_4^2, \varepsilon_3^3), \{x_2, x_4\}), \\ ((\varepsilon_3^1, \varepsilon_4^2, \varepsilon_3^3), \{x_8\}), \\ ((\varepsilon_4^1, \varepsilon_1^2, \varepsilon_1^3), \{x_6, x_7\}), \\ ((\varepsilon_4^1, \varepsilon_1^2, \varepsilon_3^3), \emptyset), \\ ((\varepsilon_4^1, \varepsilon_4^2, \varepsilon_1^3), \{x_4\}), \\ ((\varepsilon_4^1, \varepsilon_4^2, \varepsilon_3^3), \{x_3, x_4, x_5, x_{12}\}), \\ ((\varepsilon_5^1, \varepsilon_1^2, \varepsilon_1^3), \{x_{11}\}), \\ ((\varepsilon_5^1, \varepsilon_1^2, \varepsilon_3^3), \{x_9\}), \\ ((\varepsilon_5^1, \varepsilon_4^2, \varepsilon_1^3), \emptyset), \\ ((\varepsilon_5^1, \varepsilon_4^2, \varepsilon_3^3), \{x_3\}) \end{array}\right\} \tag{33}$$

By Definition 2.7, it is obvious that $(\Gamma^1, \prod_{i=1}^3 A_i) \widetilde{\subseteq} (\Gamma^2, \prod_{i=1}^3 A_i)$. Moreover, the hypersoft $\mathfrak{P}_2$-lower and $\mathfrak{P}_2$-upper approximations of $V_1$ with respect to $\mathfrak{P}_2 = (X, (\Gamma^2, \prod_{i=1}^3 A_i))$ are respectively

$$\underline{appr}_{\mathfrak{P}_2}(V_1) = \{x_3, x_6, x_7, x_{11}\} \tag{34}$$

and

$$\overline{appr}_{\mathfrak{P}_2}(V_1) = X - \{x_2, x_8\} \tag{35}$$

So, by Eqs. (27), (28), (34) and (35), we deduce that $\overline{appr}_{\mathfrak{P}_1}(V_1) \subseteq \overline{appr}_{\mathfrak{P}_2}(V_1)$, $\underline{appr}_{\mathfrak{P}_1}(V_1) \nsubseteq \underline{appr}_{\mathfrak{P}_2}(V_1)$ and $\underline{appr}_{\mathfrak{P}_1}(V_1) \nsupseteq \underline{appr}_{\mathfrak{P}_2}(V_1)$.

Especially, we take the hypersoft set

$$(\Gamma^3, \prod_{i=1}^3 A_i) = \left\{\begin{array}{l} ((\varepsilon_3^1, \varepsilon_1^2, \varepsilon_1^3), \{x_1, x_6, x_{10}\}), \\ ((\varepsilon_3^1, \varepsilon_1^2, \varepsilon_3^3), \{x_3\}), \\ ((\varepsilon_3^1, \varepsilon_4^2, \varepsilon_3^3), \{x_2\}), \\ ((\varepsilon_3^1, \varepsilon_4^2, \varepsilon_3^3), \{x_8, x_9\}), \\ ((\varepsilon_4^1, \varepsilon_1^2, \varepsilon_1^3), \{x_6, x_7\}), \\ ((\varepsilon_4^1, \varepsilon_1^2, \varepsilon_3^3), \emptyset), \\ ((\varepsilon_4^1, \varepsilon_4^2, \varepsilon_1^3), \{x_4\}), \\ ((\varepsilon_4^1, \varepsilon_4^2, \varepsilon_3^3), \{x_3, x_4, x_5, x_{12}\}), \\ ((\varepsilon_5^1, \varepsilon_1^2, \varepsilon_1^3), \{x_{11}\}), \\ ((\varepsilon_5^1, \varepsilon_1^2, \varepsilon_3^3), \{x_9\}), \\ ((\varepsilon_5^1, \varepsilon_4^2, \varepsilon_1^3), \emptyset), \\ ((\varepsilon_5^1, \varepsilon_4^2, \varepsilon_3^3), \{x_3\}) \end{array}\right\} \tag{36}$$

 

It obvious that $(\Gamma^1,\prod_{i=1}^{3}A_i)\,\widetilde{\subseteq}\,(\Gamma^3,\prod_{i=1}^{3}A_i)$ and $\Gamma^3((\varepsilon^i)_{i\in I})\subseteq V_1$ whenever $\Gamma^1((\varepsilon^i)_{i\in I})\subseteq V_1$ for $(\varepsilon^i)_{i\in I}\in\prod_{i\in I}A_i$. Then, we obtain $\underline{appr}_{\mathfrak{P}_3}(V_1)=\{x_1,x_3,x_6,x_7,x_{10},x_{11}\}$, and so $\underline{appr}_{\mathfrak{P}_1}(V_1)\subseteq\underline{appr}_{\mathfrak{P}_3}(V_1)$.

**Definition 4.12.** Let $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$ and $\mathfrak{P}=(X,(\Gamma,\prod_{i\in I}A_i))$ be a hypersoft approximation space and $V_1,V_2\subseteq X$.

**a)** The lower hypersoft rough equal relation is described as

$$V_1\simeq_{\mathfrak{P}}V_2\Leftrightarrow\underline{appr}_{\mathfrak{P}}(V_1)=\underline{appr}_{\mathfrak{P}}(V_2)\qquad(37)$$

**b)** The upper hypersoft rough equal relation is described as

$$V_1\overline{\simeq}_{\mathfrak{P}}V_2\Leftrightarrow\overline{appr}_{\mathfrak{P}}(V_1)=\overline{appr}_{\mathfrak{P}}(V_2)\qquad(38)$$

**c)** The hypersoft rough equal relation is described as

$$V_1\approx_{\mathfrak{P}}V_2\Leftrightarrow\underline{appr}_{\mathfrak{P}}(V_1)=\underline{appr}_{\mathfrak{P}}(V_2)\text{ and }\overline{appr}_{\mathfrak{P}}(V_1)=\overline{appr}_{\mathfrak{P}}(V_2)\qquad(39)$$

It is easy to see that these binary relations are all equivalence relations over $P(X)$.

**Theorem 4.13.** Let $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$ and $\mathfrak{P}=(X,(\Gamma,\prod_{i\in I}A_i))$ be a hypersoft approximation space and $V_1,V_2,V_3,V_4\subseteq X$. Then, we have the following:

**1)** $V_1\overline{\simeq}_{\mathfrak{P}}V_2\Leftrightarrow V_1\overline{\simeq}_{\mathfrak{P}}(V_1\cup V_2)\overline{\simeq}_{\mathfrak{P}}V_2$.

**2)** $V_1\overline{\simeq}_{\mathfrak{P}}V_2,V_3\overline{\simeq}_{\mathfrak{P}}V_4\Rightarrow(V_1\cup V_3)\overline{\simeq}_{\mathfrak{P}}(V_2\cup V_4)$.

**3)** $V_1\overline{\simeq}_{\mathfrak{P}}V_2\Rightarrow V_1\cup(X-V_2)\overline{\simeq}_{\mathfrak{P}}X$.

**4)** $V_1\subseteq V_2,V_2\overline{\simeq}_{\mathfrak{P}}\emptyset\Rightarrow V_1\overline{\simeq}_{\mathfrak{P}}\emptyset$.

**5)** $V_1\subseteq V_2,V_1\overline{\simeq}_{\mathfrak{P}}X\Rightarrow V_2\overline{\simeq}_{\mathfrak{P}}X$.

**Proof.** Let $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$ and $\mathfrak{P}=(X,(\Gamma,\prod_{i\in I}A_i))$ be a hypersoft approximation space and $V_1,V_2,V_3,V_4\subseteq X$.

**1)** Assume that $V_1\overline{\simeq}_{\mathfrak{P}}V_2$. By Definition 4.12 (b), we have $\overline{appr}_{\mathfrak{P}}(V_1)=\overline{appr}_{\mathfrak{P}}(V_2)$. From Theorem 4.8, it is known that $\overline{appr}_{\mathfrak{P}}(V_1\cup V_2)=\overline{appr}_{\mathfrak{P}}(V_1)\cup\overline{appr}_{\mathfrak{P}}(V_2)$. Thereby, we obtain $\overline{appr}_{\mathfrak{P}}(V_1\cup V_2)=\overline{appr}_{\mathfrak{P}}(V_1)=\overline{appr}_{\mathfrak{P}}(V_2)$. Consequently, $V_1\overline{\simeq}_{\mathfrak{P}}V_2\Rightarrow V_1\overline{\simeq}_{\mathfrak{P}}(V_1\cup V_2)\overline{\simeq}_{\mathfrak{P}}V_2$. Conversely, if $V_1\overline{\simeq}_{\mathfrak{P}}(V_1\cup V_2)\overline{\simeq}_{\mathfrak{P}}V_2$ then it is obvious that $V_1\overline{\simeq}_{\mathfrak{P}}V_2$ from the transitivity of $\overline{\simeq}_{\mathfrak{P}}$.

**2)** Suppose that $V_1\overline{\simeq}_{\mathfrak{P}}V_2$ and $V_3\overline{\simeq}_{\mathfrak{P}}V_4$. By Definition 4.12 (b), we can write $\overline{appr}_{\mathfrak{P}}(V_1)=\overline{appr}_{\mathfrak{P}}(V_2)$ and $\overline{appr}_{\mathfrak{P}}(V_3)=\overline{appr}_{\mathfrak{P}}(V_4)$. By considering Theorem 4.8, we have $\overline{appr}_{\mathfrak{P}}(V_1\cup V_3)=\overline{appr}_{\mathfrak{P}}(V_1)\cup\overline{appr}_{\mathfrak{P}}(V_3)$ and $\overline{appr}_{\mathfrak{P}}(V_2\cup V_4)=\overline{appr}_{\mathfrak{P}}(V_2)\cup\overline{appr}_{\mathfrak{P}}(V_4)$. Thereby, we conclude that $\overline{appr}_{\mathfrak{P}}(V_1\cup V_3)=\overline{appr}_{\mathfrak{P}}(V_2\cup V_4)$, and so $(V_1\cup V_3)\overline{\simeq}_{\mathfrak{P}}(V_2\cup V_4)$.

**3)** Let $V_1\overline{\simeq}_{\mathfrak{P}}V_2$. Since $\overline{appr}_{\mathfrak{P}}(V_1)=\overline{appr}_{\mathfrak{P}}(V_2)$, we can write $\overline{appr}_{\mathfrak{P}}(V_1\cup(X-V_2))=\overline{appr}_{\mathfrak{P}}(V_1)\cup\overline{appr}_{\mathfrak{P}}(X-V_2)$ and $\overline{appr}_{\mathfrak{P}}(X)=\overline{appr}_{\mathfrak{P}}(V_2)\cup\overline{appr}_{\mathfrak{P}}(X-V_2)$. Thus, we deduce that $\overline{appr}_{\mathfrak{P}}(V_1\cup(X-V_2))=\overline{appr}_{\mathfrak{P}}(V_2)\cup\overline{appr}_{\mathfrak{P}}(X-V_2)=\overline{appr}_{\mathfrak{P}}(X)$ and so $V_1\cup(X-V_2)\overline{\simeq}_{\mathfrak{P}}X$.

**4)** Let us assume that $V_1\subseteq V_2$ and $V_2\overline{\simeq}_{\mathfrak{P}}\emptyset$. From Definition 4.12 (b) and Theorem 4.8, we have $\overline{appr}_{\mathfrak{P}}(V_1)\subseteq\overline{appr}_{\mathfrak{P}}(V_2)=\overline{appr}_{\mathfrak{P}}(\emptyset)=\emptyset$. It is obvious that $\overline{appr}_{\mathfrak{P}}(\emptyset)\subseteq\overline{appr}_{\mathfrak{P}}(V_1)$. Hence, we obtain $V_1\overline{\simeq}_{\mathfrak{P}}\emptyset$.

**5)** Let us assume that $V_1\subseteq V_2$ and $V_1\overline{\simeq}_{\mathfrak{P}}X$. From Definition 4.12 (b) and Theorem 4.8, we have $\overline{appr}_{\mathfrak{P}}(V_2)\supseteq\overline{appr}_{\mathfrak{P}}(V_1)=\overline{appr}_{\mathfrak{P}}(X)$. Since $V_2\subseteq X$, it is obvious that $\overline{appr}_{\mathfrak{P}}(V_2)\subseteq\overline{appr}_{\mathfrak{P}}(X)$. Hence, we obtain $V_2\overline{\simeq}_{\mathfrak{P}}X$.

**Definition 4.14.** Let $(\Gamma,\prod_{i\in I}A_i)\in S_H\langle X,\prod_{i\in I}\mathcal{E}_i\rangle$. If $(\varepsilon_\alpha^i)_{i\in I},(\varepsilon_\beta^i)_{i\in I}\in\prod_{i\in I}A_i$, there exists $(\varepsilon_\gamma^i)_{i\in I}\in\prod_{i\in I}A_i$ such that $\Gamma((\varepsilon_\gamma^i)_{i\in I})=\Gamma((\varepsilon_\alpha^i)_{i\in I})\cap\Gamma((\varepsilon_\beta^i)_{i\in I})$ whenever $\Gamma((\varepsilon_\alpha^i)_{i\in I})\cap\Gamma((\varepsilon_\beta^i)_{i\in I})\neq\emptyset$ then $(\Gamma,\prod_{i\in I}A_i)$ is termed to be an intersection complete hypersoft set.

**Proposition 4.15.** Let $(\Gamma,\prod_{i\in I}A_i)$ be an intersection complete hypersoft set over $X$ and $\mathfrak{P}=(X,(\Gamma,\prod_{i\in I}A_i))$ be a hypersoft approximation space and $V_1, V_2 \subseteq X$. Then we have, for all $V_1, V_2 \subseteq X$

$$\underline{appr}_{\mathfrak{P}}(V_1 \cap V_2) = \underline{appr}_{\mathfrak{P}}(V_1) \cap \underline{appr}_{\mathfrak{P}}(V_2) \quad (40)$$

**Proof.** From Theorem 4.8,

$$\underline{appr}_{\mathfrak{P}}(V_1 \cap V_2) \subseteq \underline{appr}_{\mathfrak{P}}(V_1) \cap \underline{appr}_{\mathfrak{P}}(V_2)$$

holds for every $(\Gamma, \prod_{i \in I} A_i)$ (does not need to be intersection complete). To complete the proof, it is sufficient to demonstrate reverse inclusion. Let $x \in \underline{appr}_{\mathfrak{P}}(V_1) \cap \underline{appr}_{\mathfrak{P}}(V_2)$. Then, there exists $(\varepsilon_\alpha^i)_{i \in I}, (\varepsilon_\beta^i)_{i \in I} \in \prod_{i \in I} A_i$ such that $x \in \Gamma((\varepsilon_\alpha^i)_{i \in I})$ $\subseteq V_1$ and $x \in \Gamma((\varepsilon_\beta^i)_{i \in I}) \subseteq V_2$. Since $(\Gamma, \prod_{i \in I} A_i)$ is an intersection complete hypersoft set over $X$, we conclude that there exists $(\varepsilon_\gamma^i)_{i \in I} \in \prod_{i \in I} A_i$ such that

$$x \in \Gamma((\varepsilon_\gamma^i)_{i \in I}) = \Gamma((\varepsilon_\alpha^i)_{i \in I}) \cap \Gamma((\varepsilon_\beta^i)_{i \in I}) = V_1 \cap V_2$$

Hence, we obtain that $x \in \underline{appr}_{\mathfrak{P}}(V_1 \cap V_2)$. Consequently, we have

$$\underline{appr}_{\mathfrak{P}}(V_1 \cap V_2) \supseteq \underline{appr}_{\mathfrak{P}}(V_1) \cap \underline{appr}_{\mathfrak{P}}(V_2)$$

Thus, the proof is completed.

Now, the following results on lower hypersoft equal relations can be verified.

**Theorem 4.16.** Let $(\Gamma, \prod_{i \in I} A_i) \in S_H\langle X, \prod_{i \in I} \mathcal{E}_i \rangle$ and $\mathfrak{P} = (X, (\Gamma, \prod_{i \in I} A_i))$ be a hypersoft approximation space and $V_1, V_2, V_3, V_4 \subseteq X$. Then, we have the following:

**1)** $V_1 \simeq_{\mathfrak{P}} V_2 \Leftrightarrow V_1 \simeq_{\mathfrak{P}} (V_1 \cap V_2) \simeq_{\mathfrak{P}} V_2$.

**2)** $V_1 \simeq_{\mathfrak{P}} V_2, V_3 \simeq_{\mathfrak{P}} V_4 \Rightarrow (V_1 \cap V_3) \simeq_{\mathfrak{P}} (V_2 \cap V_4)$.

**3)** $V_1 \simeq_{\mathfrak{P}} V_2 \Rightarrow V_1 \cap (X - V_2) \simeq_{\mathfrak{P}} \emptyset$.

**4)** $V_1 \subseteq V_2, V_2 \simeq_{\mathfrak{P}} \emptyset \Rightarrow V_1 \simeq_{\mathfrak{P}} \emptyset$.

**5)** $V_1 \subseteq V_2, V_1 \simeq_{\mathfrak{P}} X \Rightarrow V_2 \simeq_{\mathfrak{P}} X$.

**Proof.** By considering Definitions 4.12 (a), 4.14 and Theorem 4.8, it can be proved similar to the proof of Theorem 4.13.

## Conclusions

The rough set theory emerges as a strong theory and has various applications in many fields. On the other hand, the hypersoft sets are a powerful mathematical tool for modelling various types of uncertainty. In this paper, several new ideas for the rough hypersoft sets were proposed. Moreover, a new combined model of hypersoft sets and rough sets was investigated and accordingly the hypersoft rough sets were introduced. Some characteristic properties of hypersoft rough sets were discussed. Developing the rough hypersoft sets and hypersoft rough sets in theoretical aspects such as deriving their structural properties in more general frameworks, as well as researching their practical applications may be future research topics.

## Conflict of interest

Author declared no conflict of interests.

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