Ann. Univ. Ferrara - Sez. VII - Sc. Mat.
Vol. XL, 71-95 (1994)

On the Falling of Objects in Non-Newtonian Fluids.

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1. – Introduction.

The study of objects falling under the influence of gravity in a fluid occupies a central place in fluid dynamics. The seminal work of Stokes [1] on the falling of a sphere in a Navier-Stokes fluid has been followed by a plethora of extensions that include objects of various shapes as well as assemblages of bodies (cf. Happel & Brenner [2]). While the problem for a body of general shape is quite daunting, it becomes tractable if it is assumed to be falling slowly, i.e., the flow being non-inertial, and furthermore if the body is a body of revolution or «slender body» (cf. Basset [3]). Also of considerable interest is the translation of a liquid sphere inside a liquid (cf. Hadamard [4]).

While the Navier-Stokes model explains the behavior of water and other such fluids exceptionally well, at least in the laminar regime, it fails to capture even the essential features of many polymeric and biological fluids. There are many fluids that exhibit normal stress differences in simple shear flow (cf. Truesdell and Noll [5]) and the Navier-Stokes model cannot describe this phenomenon. Amongst the numerous models that have been proposed to describe the non-Newtonian behaviour of these fluids, a class that has attracted significant attention is that of the fluids of the differential type. In these fluids, the stress is determined by knowledge of very little of the deformation history of the fluid; the current value of the Rivlin-Ericksen tensors A_n being sufficient. Fluid models of the differential type have been found to be useful in describing the behavior of dilute polymeric liquids and biological fluids. Of course, the classical Navier- Stokes model is also a fluid of the differential type as are the popular power-law models or the generalized New-

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tonian model. A simple fluid of the differential type that is capable of exhibiting both the normal stress differences is the fluid of second grade, and thus this model has been studied in great detail during the past two decades.

The Cauchy stress T in an incompressible fluid of second grade is related to the fluid motion in the following manner:

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 , \tag{1.1}$$

where

$$A_1 = (\operatorname{grad} v) + (\operatorname{grad} v)^T , \tag{1.2}$$

and

$$A_2 = \frac{d}{dt} A_1 + A_1 (\operatorname{grad} v) + (\operatorname{grad} v)^T A_1 . \tag{1.3}$$

In the above equations, v denotes the velocity field, d/dt the usual material time derivative, $-pI$ is the spherical part of the stress due to the constraint of incompressibility, μ is the viscosity and α_1 and α_2 the normal stress moduli.

Dunn and Fosdick [6] studied in detail the dynamic and thermodynamic response of fluids modeled by (1.1)-(1.3) and found that if the fluids are to be compatible with thermodynamics in the sense that all motions meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy is a minimum in equilibrium, then

$$\mu \geqslant 0 , \quad \alpha_1 \geqslant 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0 . \tag{1.4}$$

They showed that if $\mu > 0$, $\alpha_1 < 0$, and $\alpha_1 + \alpha_2 = 0$, then the rest state of the fluid is unstable. Later, Fosdick and Rajagopal [7] proved that if $\mu > 0$, and $\alpha_1 < 0$, then the fluid exhibits anomalous stability behaviour that is unacceptable in any reasonable fluid, without the restriction that $\alpha_1 + \alpha_2 = 0$. More recently, Galdi, Padula, and Rajagopal [8] showed that if $\mu > 0$ and $\alpha_1 > 0$, the rest state of a second grade fluid in an unbounded domain is conditionally stable, while it is unstable if $\mu > 0$, and $\alpha_1 < 0$, irrespective of the sign of α_2.

The result that $\alpha_1 \geqslant 0$ was in keeping with the indication of earlier work by Ting [9] on the existence of solutions for the flows of such fluids. The validity of the restrictions (1.4) have unfortunately been the object of much debate and we shall not get into a discussion of the same here. We refer the reader to an exhaustive article by Dunn and Rajagopal [10] where all the relevant issues are discussed at length. Recently, there has been a great deal of interest in studying issues related to existence and uniqueness to the equations of motion for a fluid of second grade. Galdi and Sequiera [11] and Galdi *et al.* [12] have proved existence and uniqueness for small data for the homogeneous boundary condition $v = 0$.

In this paper, we shall consider the falling of objects in an incompressible fluid of second grade. It might be appropriate to mention that it has been found that the addition of parts per million of a polymer to water causes a significant reduction in the drag of objects moving in the fluid. However, as these results pertain to turbulent flow, the study of an object moving slowly in a fluid of second grade cannot shed much light on the problem. On the other hand, such a study might be the first logical step in efforts to fully understanding the problem of drag reduction due to the addition of polymers to water.

The composition of the paper is as follows. First, we study the problem of a thin, but heavy, lamina falling under the influence of gravity between two vertical plates, the space being occupied by a fluid of second grade. This is followed by the study of a heavy cylinder falling under the influence of gravity inside a vertical tube filled with a fluid of second grade.

2. – Governing equations.

If we substitute the stress T into the balance of linear momentum

$$\operatorname{div} T + \varrho b = \varrho \frac{dv}{dt}\,, \tag{2.1}$$

we obtain, in the case of a conservative body force field $b = -\operatorname{grad}\phi$, (cf. Rajagopal [13])

$$\begin{aligned}\mu \Delta v + \alpha_1 \Delta v_t + \alpha_1(\Delta w \times v) + {}&\\ +(\alpha_1 + \alpha_2)\{A_1 \Delta v + +2\operatorname{div}[(\operatorname{grad} v)(\operatorname{grad} v)^T]\} - \varrho(w \times v) - \varrho v_t = \operatorname{grad} P\,,&\end{aligned} \tag{2.2}$$

where

$$P = p - \alpha_1(V \cdot \Delta v) - \frac{(2\alpha_1 + \alpha_2)}{4}\,|A_1|^2 + \frac{1}{2}\varrho|v|^2 + \varrho\phi\,, \tag{2.3}$$

and Δ is the Laplacian, the subscript t indicates partial differentiation with respect to time, $|A_1|$ the trace norm of A_1 and

$$w = \operatorname{curl} v\,. \tag{2.4}$$

As the fluid is incompressible, it can undergo only isochoric motions, therefore

$$\operatorname{div} v = 0\,. \tag{2.5}$$

3. – Heavy thin lamina falling under gravity between parallel vertical plates.

Consider a heavy thin lamina falling from rest under the action of gravity at time $t = 0$ between two parallel vertical walls filled with a fluid of second grade, in such a manner that the plane of the lamina is parallel to the walls.

Let σ be the mass per unit area of the plate, $2d$ its thickness, ϱ the density of the liquid, ν the kinematic viscosity, v and V respectively the velocity of the liquid and the plate; let the plate be positioned halfway between the walls which are at a distance $2s$ apart. We assume that the velocity field for the liquid is of the kind

$$\boldsymbol{v} = (0,\, 0,\, v(x,\, t))\,,$$

and that the pressure gradient is the same as in the static case. It is in fact customary to make the more stringent assumption that body forces can be ignored as far as the fluid is concerned and that the pressure gradient is zero (cf. Carslaw and Jaeger [17]). Our assumption includes the usual hypothesis as a special case, for if body forces are ignored, the pressure field would be a constant in the static problem. The governing momentum equation becomes

$$\varrho\,\frac{\partial v}{\partial t} = \mu\,\frac{\partial^2 v}{\partial x^2} + \alpha_1\,\frac{\partial^3 v}{\partial t\,\partial x^2}\,, \qquad -s < x < -d\,,\ \ d < x < s\,,\ \ t > 0\,, \tag{3.1}$$

with boundary conditions

$$v = 0\,, \qquad x = \pm s\,, \qquad t > 0\,, \tag{3.2}$$

$$v = V\,, \qquad x = \pm d\,, \qquad t > 0\,, \tag{3.3}$$

and initial condition

$$v = 0\,, \qquad -s \leqslant x \leqslant -d\,, \qquad d \leqslant x \leqslant s\,, \qquad t = 0\,, \tag{3.4}$$

where

$$\boldsymbol{V} = (0,\, 0,\, V(t))\,,$$

is the velocity of the lamina. From (3.1) we obtain, after applying the Laplace transform

$$L\{f(\xi, \tau)\} := \int_0^\infty e^{-p\tau} f(\xi, \tau)\, d\tau,$$

the subsidiary equation:

$$\varrho p \bar{v} = (\mu + \alpha_1 p) \frac{d^2 \bar{v}}{dx^2}, \qquad -s < x < \pm d, \quad d < x < s, \tag{3.5}$$

under the boundary conditions

$$\bar{v} = 0, \qquad x = \pm s, \qquad t > 0, \tag{3.6}$$

$$\bar{v} = \bar{V}, \qquad x = \pm d, \qquad t > 0. \tag{3.7}$$

The solution to the two-point boundary value problem (3.5)-(3.7) is:

$$\bar{v} = \bar{V} \frac{\sinh (s - x) \sqrt{\dfrac{\varrho p}{\mu + \alpha_1 p}}}{\sinh (s - d) \sqrt{\dfrac{\varrho p}{\mu + \alpha_1 p}}}, \qquad d < x < s, \qquad -s < x < d. \tag{3.8}$$

Now let us set $s - d = h$.

Motion of the lamina.

The equation of motion for the lamina is

$$\sigma \frac{dV}{dt} = \sigma g + 2 \left[\mu \frac{\partial v}{\partial x} + \alpha_1 \frac{\partial^2 v}{\partial t\, \partial x} \right]_{x = d}, \tag{3.9}$$

with initial condition

$$V = 0, \qquad \text{when} \qquad t = 0,$$

leading to the subsidiary equation:

$$\sigma p \bar{V} = \frac{\sigma g}{p} + 2(\mu + \alpha_1 p) \frac{d\bar{v}}{dx} \bigg|_{x = d}. \tag{3.10}$$

Using (3.8) into (3.10) and solving for $\bar{V}$, we obtain

$$\bar{V} = \frac{\sigma g}{p} \left[\sigma p + 2 \sqrt{(\mu + \alpha_1 p) \varrho p} \coth h \sqrt{\frac{\varrho p}{\mu + \alpha_1 p}} \right]^{-1}. \tag{3.11}$$

The bar indicates the Laplace transform of V. The inverse transform of (3.11) will be computed with the complex inversion formula

$$V(t) = \frac{1}{2\pi i} \int_L \bar{V} e^{pt}\, dp\,, \tag{3.12}$$

where L is a path defined by $\operatorname{Re} p = \text{const.} > 0$ leaving all the singularities of $\bar{V}$ to the left (cf. Churcill [14]).

At this juncture, let us summarize some remarks made by Ting [11] which are relevant to the problem on hand and those that follow. The complex inversion integral (3.12) can be evaluated, in general, by applying the Cauchy residue theorem. However, an extension of this theorem is necessary because the integrand in (3.12) possesses infinitely many singularities concentrated in the interval $[-\mu/\alpha_1, 0]$ and $p = -\mu/\alpha_1$ is an essential singularity. Towards this purpose, we consider the integral

$$V^* = \frac{1}{2\pi i} \int_{L+C+C_N} \bar{V} e^{pt}\, dp\,. \tag{3.13}$$

The integration contour is reported in fig. 1. C is a half circle of radius R to the left of the path L and C_N a curve enclosing all the singularities p_n for $n > N$; C and C_N are connected by two portions of straight lines parallel to the real axis where $\operatorname{Re} p < -\mu/\alpha_1$. The curves C_N are constructed such that they do not cut any singularity.

By the residue theorem, for all finite N

$$\frac{1}{2\pi i} \int_{L+C+C_N} \bar{V} e^{pt}\, dp = \sum_{n=0}^{N} R_n\,, \tag{3.14}$$

where R_n is the residue of $\bar{V}(p, t)$ at the pole p_n. We will show that the line integrals over Γ and C_N vanish respectively as R and N tend to infinity, whereas those on the horizontal straight lines cancel each other, for the integrand is continuous there.

By taking the limit of (3.14) as N and R go to infinity, we will formally have

$$V(t) = \frac{1}{2\pi i} \int_L \bar{V} e^{pt}\, dp = \sum_{n=0}^{\infty} R_n\,. \tag{3.15}$$

The formal solution obtained in (3.15) is to be checked a posteriori that it actually satisfies the given initial-boundary value problem. By the Inversion

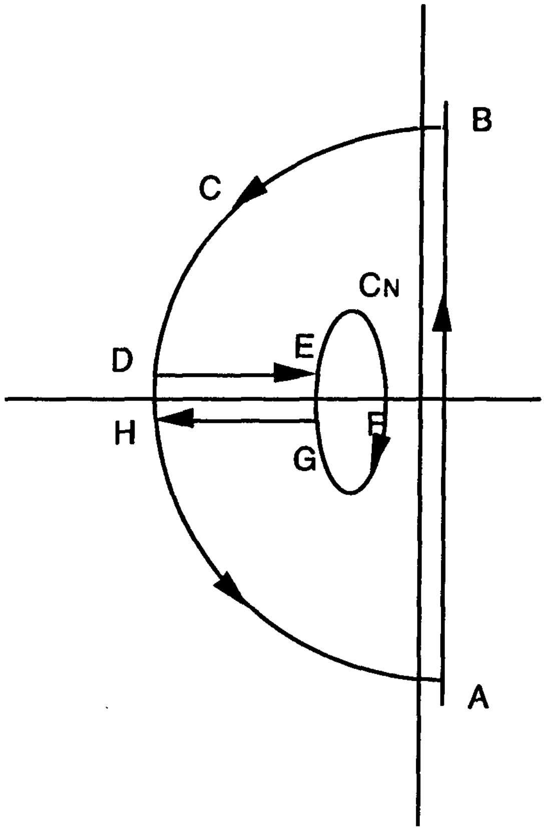

Figure 1.

Theorem, we find

$$(3.16)\qquad V(t) = \frac{\sigma g}{2\pi i}\int_L \frac{1}{p}\,\frac{e^{pt}\,dp}{\sigma p + 2\sqrt{(\mu + \alpha_1 p)\varrho p}\coth h\sqrt{\dfrac{\varrho p}{\mu + \alpha_1 p}}} =$$

$$= \frac{\sigma g}{2\pi i}\int_L \frac{1}{\sigma p^2}\,\frac{e^{pt}\,dp}{1 + \dfrac{2\varrho h}{\sigma}\,\dfrac{\coth\zeta}{\zeta}}$$

where

$$\zeta = h\sqrt{\frac{\varrho p}{\mu + \alpha_1 p}}\,.$$

We shall always choose the branch of ζ such that

$$\zeta = \varphi + i\psi \qquad -\frac{\pi}{2} < \arg\zeta \leqslant \frac{\pi}{2}\,. \tag{3.17}$$

The singularities of the integrand (3.16) are given by the zeros of the denominator, namely

$$p = 0\,, \qquad 1 + \frac{2\varrho h}{\sigma}\,\frac{\coth\zeta}{\zeta} = 0\,. \tag{3.18}$$

It is easy to show that $p = 0$ is a simple pole; to find the other poles, we set $\zeta = i\lambda$ in (3.18) and find

$$1 - \frac{2\varrho h}{\sigma}\,\frac{\cot\lambda}{\lambda} = 0\,, \tag{3.19}$$

or $\lambda\tan\lambda = k$, where $k = 2\varrho h/\sigma$. The zeros of (3.19) are all simple, real, and infinitely many(1) and can be ordered in an increasing sequence $\{\lambda_n\}$; also

$$\lim_{n\to\infty}\lambda_n = \infty\,.$$

Thus the poles of the integrand in (3.16) (other than $p = 0$) are given by

$$p_n = -\frac{\mu\lambda_n^2}{\varrho h^2 + \alpha_1\lambda_n^2}\,, \qquad n = 1, 2, \ldots, \tag{3.19}$$

where λ_n are the real positive roots of (3.19).

We will show that the integral (3.16) converges: to this end, we need an estimate for $|\overline{V}(p)|$. All the singularities are concentrated in the finite interval $-\mu/\alpha_1 \leqslant p \leqslant 0$ with $p = -\mu/\alpha_1$ as the point of accumulation. By our choice, $\varphi \geqslant 0$ and $\varphi = 0$ if and only if

$$p = -\frac{\mu\psi^2}{\varrho h^2 + \alpha_1\psi^2}\,. \tag{3.20}$$

Consequently, if $\operatorname{Re} p < 0$, then

$$\varphi > 0\,, \qquad \mu + \alpha_1 p \neq 0\,, \qquad 1 + k\,\frac{\coth\zeta}{\zeta} \neq 0\,.$$

(1) *Op. cit.*, Carslaw and Jaeger, p. 171.

so that the function $\bar{V}(p)$ is guaranteed to be analytic in the right-hand side of the complex plane. Hence for $\operatorname{Re} p > 0$, after setting

$$f(p) = \frac{1}{p^2\left(1 + k\,\dfrac{\coth\zeta}{\zeta}\right)},$$

we have

$$|f(p)| = \frac{1}{\left|p^2\left(1 + k\,\dfrac{\coth\zeta}{\zeta}\right)\right|} = \frac{1}{|p|^2}\,\frac{1}{\left|1 + k\,\dfrac{\coth\zeta}{\zeta}\right|}. \tag{3.21}$$

Now,

$$\left|1 + k\,\frac{\coth\zeta}{\zeta}\right| \geqslant \left|1 - \left|k\,\frac{\coth\zeta}{\zeta}\right|\right|, \tag{3.22}$$

and

$$\left|\frac{\cosh\zeta}{\sinh\zeta}\right| = \frac{|\exp[(\varphi + i\psi)] + \exp[-(\varphi + i\psi)]|}{|\exp[(\varphi + i\psi)] - \exp[-(\varphi + i\psi)]|} = \tag{3.23}$$

$$= \frac{|1 + \exp[-2(\varphi + i\psi)]|}{|1 - \exp[-2(\varphi + i\psi)]|} \leqslant \frac{1 + \exp(-2\varphi)}{1 - \exp(-2\varphi)} = \text{const.}\,.$$

Notice that

$$\left|k\,\frac{\coth\zeta}{\zeta}\right| = k\,|\varrho p|\,\left|\frac{\coth\zeta}{[\varrho p(\mu + \alpha_1 p)]^{1/2}}\right|, \tag{3.24}$$

which, in combination with (3.23), implies that

$$\left|k\,\frac{\coth\zeta}{\zeta}\right| \leqslant \text{const.}\,, \tag{3.25}$$

and, by virtue of (3.22), that

$$\left|1 - \left|k\,\frac{\coth\zeta}{\zeta}\right|\right| \geqslant \text{const}\,. \tag{3.26}$$

We conclude that

$$|\bar{V}| \leqslant \frac{\text{const.}}{|p|^2}\,. \tag{3.27}$$

We now need to prove that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_c \overline{V} e^{pt}\, dp = 0\,. \tag{3.28}$$

To do so, we show that in the left-hand plane $\overline{V}(p)$ tends to zero uniformly. Now

$$\left| k\,\frac{\coth\zeta}{\zeta} \right| = k\,|\varrho p|\;|p|\;\left| \frac{\coth\zeta}{p\,[\varrho p(\mu+\alpha_1 p)]^{1/2}} \right|. \tag{3.29}$$

Ting [9] has shown that

$$\left| \frac{\coth\zeta}{p\,[\varrho p(\mu+\alpha_1 p)]^{1/2}} \right| \leqslant \frac{\text{const.}}{p\,[\varrho p(\mu+\alpha_1 p)]^{1/2}}\;\frac{\coth\left(\dfrac{\varrho}{\alpha_1}\right)^{1/2} h}{\sinh\left(\dfrac{\varrho}{\alpha_1}\right)^{1/2} h} = O\!\left(\frac{1}{|p|^2}\right). \tag{3.30}$$

Hence for a large enough radius, in virtue of (3.29) and (3.30), it follows that

$$\left| k\,\frac{\coth\zeta}{\zeta} \right| \leqslant \text{const.}\,, \tag{3.31}$$

and, as a consequence of (3.22),

$$\left| 1 + k\,\frac{\coth\zeta}{\zeta} \right| \geqslant \text{const.}. \tag{3.32}$$

The last inequality and (3.31) yield

$$|\overline{V}| \leqslant \frac{\text{const.}}{|p|^2}\,.$$

As this restriction is valid throughout the whole complex plane, it also guarantees that the initial condition is satisfied. Hence, for large R

$$V(t) = \frac{1}{2\pi i} \int_L \overline{V}(p)\, e^{pt}\, dp = -\frac{1}{2\pi i} \int_{C_N} \overline{V}(p)\, e^{pt}\, dp + \sum_{n=0}^{N} R_n\,. \tag{3.33}$$

Let us compute the residues R_n: for any finite n, p_n is a simple pole, therefore, at $p_0 = 0$

$$R_0 = \frac{\sigma g h}{2\varrho\nu}\,,$$

and

$$R_n = -\frac{4g\varrho h^3}{\nu\sigma}\frac{\exp(p_n t)}{\lambda_n^2(\lambda_n^2 + k^2 + k)}, \qquad (n = 1, 2, \ldots).$$

Thus

$$V = \frac{1}{2\pi i}\int_L \overline{V}(p)\, e^{pt}\, dp = -\frac{1}{2\pi i}\int_{C_N} \overline{V} e^{pt}\, dp + \tag{3.34}$$

$$+\frac{\sigma g h}{2\varrho\nu} - \frac{4g\varrho h^3}{\nu\sigma}\sum_{n=1}^{N}\frac{\exp(p_n t)}{\lambda_n^2(\lambda_{n2} + k^2 + k)}.$$

As $N \to \infty$, equation (3.34) formally yields

$$V = -\frac{1}{2\pi i}\lim_{N\to\infty}\int_{C_N}\overline{V}(p)\, e^{pt}\, dp + \sum_{n=0}^{\infty} R_n. \tag{3.35}$$

The existence of $V(t)$ in combination with the absolute and uniform convergence of the infinite series ensures the existence of the limit on the right-hand side of (3.35). We will show that

$$\lim_{N\to\infty}\int_{C_N}\overline{V}(p)\, e^{pt}\, dp \equiv 0. \tag{3.36}$$

Following Ting [11] we set

$$\left(\frac{\varrho p}{(\mu + \alpha_1 p)}\right)^{1/2} h = \tau\exp(i\eta), \qquad p + \frac{\mu}{\alpha_1} = \varepsilon\exp(i\theta),$$

$$\tau\sin\eta = \lambda_N\pi, \qquad (N = 1, 2, 3, \ldots).$$

Squaring both sides of the first equation, equating the corresponding real and imaginary parts, and using the last relationship, we have

$$\frac{h^2\varrho}{\alpha_1}\left(1 - \frac{1}{\varepsilon}\frac{\mu}{\alpha_1}\cos\theta\right) = \lambda_N^2\pi^2(\cot^2\eta - 1), \tag{3.37}$$

$$\frac{2}{\varepsilon}\frac{\mu h^2\varrho}{(\alpha_1)^2}\sin\theta = 2\lambda_N^2\pi^2\cot\eta. \tag{3.38}$$

If we eliminate $\cot\eta$ from the above equations, we obtain

$$\left(\frac{2h^2\varrho}{\alpha_1} + \lambda_N^2\pi^2\right)\varepsilon^2 - \left(\frac{2\mu h^2\varrho}{\alpha_1^2}\cos\theta\right)\varepsilon - \left(\frac{2\mu h\varrho}{\alpha_1^2}\right)^2\sin^2\theta = 0. \tag{3.39}$$

Solving for ε, we find that

$$(3.40)\qquad \varepsilon_N = \frac{\mu h^2 \varrho \cos\theta}{2\alpha_1(\alpha_1\lambda_N^2\pi^2/2 + h^2\varrho)} + \left\{\left[\frac{\mu h^2 \varrho \cos\theta}{2\alpha_1(\alpha_1\lambda_N^2\pi^2/2 + h^2\varrho)}\right]^2 + \frac{(\mu h\varrho\sin\theta)^2}{2\alpha_1^3(\alpha_1\lambda_N^2\pi^2/2 + h^2\varrho)}\right\}^{1/2} = O\left(\frac{1}{\lambda_N}\right),$$

which, for any given N, is the representation in polar coordinates of a closed curve around the accumulation point. Thus $\varepsilon_N \to 0$ as $N \to \infty$; if $\theta = 0$,

$$(3.41)\qquad \varepsilon_N = \frac{\mu h^2\varrho}{\alpha_1(\alpha_1\lambda_N^2\pi^2/2 + h^2\varrho)}, \qquad \varepsilon_N - \frac{\mu}{\alpha_1} = -\frac{\mu\lambda_N^2\pi^2}{2(\alpha_1\lambda_N^2\pi^2/2 + h^2\varrho)},$$

which implies that, for all integers N, the curves C_N do not cut any pole.

Let us evaluate the following quantities over the curves C_N

$$(3.42)\qquad \left|k\,\frac{\coth\zeta}{\zeta}\right| = k\varrho\,|p|^2\left|\frac{\coth\zeta}{p\,[\varrho p(\mu+\alpha_1 p)]^{1/2}}\right| \leqslant \text{const.}\ \frac{|p|^2}{|p|^{3/2}\varepsilon_N^{1/2}}\cdot$$

$$\cdot\left|\frac{1+\sinh^2(\lambda_N\pi\cot\eta)}{\sinh^2(\lambda_N\pi\cot\eta)+\sin^2(\lambda_N\pi)}\right|^{1/2} = \text{const.}\left(\frac{|p|}{\varepsilon_N}\right)^{1/2} = O\left(\frac{1}{\lambda_N}\right)^{-1/2}.$$

Hence,

$$(3.43)\qquad \left|1 + k\,\frac{\coth\zeta}{\zeta}\right| \geqslant O\left(\frac{1}{\lambda_N}\right)^{-1/2}.$$

Now

$$(3.44)\qquad \left|\int_{C_N} f(p)e^{pt}dp\right| \leqslant \int_{C_N} |f(p)|\,e^{|p|t}dp \leqslant \int_{-\pi}^{\pi}\frac{e^{|p|t}}{|p|^{3/2}}\,O\left(\frac{1}{\lambda_N}\right)^{1/2}\varepsilon_N\,d\theta =$$

$$= \int_{-\pi}^{\pi}\frac{e^{|p|t}}{|p|^{3/2}}\,O\left(\frac{1}{\lambda_N}\right)^{2/3}d\theta = O\left(\frac{1}{\lambda_N}\right)^{3/2}.$$

Hence, relationship (3.36) is proved. The solution to the mixed initial-boundary value problem (3.1)-(3.4) is

$$(3.45)\qquad V(t) = \frac{\sigma g h}{2\varrho\nu} - \frac{4g\varrho h^3}{\nu\sigma}\sum_{n=1}^{\infty}\frac{\exp(p_n t)}{\lambda_n^2(\lambda_n^2 + k^2 + k)}.$$

If we set $\alpha_1 = 0$ in the above expression, we recover the classical result for the Naiver-Stokes fluid (cf. Carslaw and Jaeger [17]). It is worthwhile to observe that when $\alpha_1 = 0$, the accumulation point for the singularities is at infinity. Thus, we would have to use a different contour from that employed here. However, it is interesting that we can recover the result form (3.45) by setting $\alpha_1 = 0$.

Notice that the solution for large times, i.e. $t \to \infty$ is

$$V(\infty) = \frac{\sigma g h}{2\mu}$$

which is exactly the same as that for a lamina falling in a Navier-Stokes fluid. However, the pressure field in the fluid and the shear stress on the lamina would be different. In all the problems considered in this paper we find that the large time behavior for the velocity is identical for both the fluid of second grade and the Navier-Stokes fluid.

Motion of the liquid.

On substituting (3.11) into (3.8), the Laplace transform of the velocity of the liquid is

$$\bar{v}(p, \zeta) = g \frac{\sinh (s - x)\, \zeta/h}{p^2 \left(1 + k \dfrac{\coth \zeta}{\zeta}\right) \sinh \zeta} . \tag{3.46}$$

Again, by the Inversion Theorem, following the line of reasoning in the previous section, we find that

$$v(x, t) = \frac{1}{2\pi i} \int_L \bar{v} e^{pt}\, dp = - \frac{1}{2\pi i} \lim_{N \to \infty} \int_{C_N} \bar{v}(p)\, e^{pt}\, dp + \tag{3.47}$$

$$- \frac{1}{2\pi i} \lim \int_C \bar{v}(p)\, e^{pt}\, dp + \sum_{n=0}^{\infty} R_n .$$

To verify the convergence of the integral (3.47) in x and t for $\operatorname{Re} p > 0$, we need an estimate for $|\bar{v}|$ given by

$$|\bar{v}| = g \frac{\left| \dfrac{\sinh (s - x)\, \zeta/h}{\sinh \zeta} \right|}{\left| p^2 \left(1 + k \dfrac{\coth \zeta}{\zeta}\right) \right|} . \tag{3.48}$$

We have already established in (3.25)-(3.29) that

$$\frac{1}{\left|p^2\left(1 + k\,\frac{\coth\zeta}{\zeta}\right)\right|} \leqslant \frac{\text{const.}}{|p^2|}\,.$$

Now observe that

$$(3.49)\qquad \left|\frac{\sinh (s-x)\,\zeta/h}{\sinh\zeta}\right| = \frac{\left|\exp\left(-\dfrac{x\zeta}{h}\right)(1 \pm \exp[-2\zeta(s-x)/h])\right|}{|1 \pm \exp(-2\zeta)|} \leqslant$$

$$\leqslant \frac{\exp\left(-\dfrac{\varphi x}{h}\right)(1 + \exp(-2\varphi(s-x)/h))}{1 \pm \exp(-2\varphi)} = \text{const.}\,.$$

Hence

$$(3.50)\qquad |\bar{v}| \leqslant \frac{\text{const.}}{|p|^2}\,,$$

which is sufficient to guarantee the convergence of the integral (3.47). Since

$$\lim_{|p|\to\infty} \zeta = \left(\frac{\varrho}{\alpha_1}\right)^{1/2} h\,,$$

for sufficiently large $|p|$ it follows that

$$(3.51)\qquad \lim_{|p|\to\infty} \frac{\sinh (s-x)\,\zeta/h}{\sinh\zeta} = \frac{\sinh\left[(s-x)\left(\dfrac{\varrho}{\alpha_1}\right)^{1/2}\right]}{\sinh\left[\left(\dfrac{\varrho}{\alpha_1}\right)^{1/2} h\right]}\,,$$

and, therefore, by virtue also of (3.32),

$$|\bar{v}| \leqslant O\left(\frac{1}{|p|^2}\right).$$

This means that the integral (3.47) vanishes on the semi-circle in the left hand plane and that the initial and boundary conditions are met.

Let us estimate the value of the integral along the curves C_N where (2)

$$\left| \frac{\sinh(s-x)\zeta/h}{\sinh\zeta} \right| \leqslant 1, \tag{3.52}$$

Inequalities (3.44) and (3.52) lead to

$$\left| \frac{1}{2\pi i} \int_{C_N} \bar{v} e^{pt}\, dp \right| \leqslant O\left(\frac{1}{\lambda_N}\right). \tag{3.53}$$

which means that, as N goes to infinity, the integral (3.53) vanishes. The poles of the integrand in (3.47) are the same as those of (3.12), therefore, after straightforward calculations, the velocity is computed as

$$v(x,t) = \frac{\sigma g}{2\varrho\nu}(s-x) - \frac{4\varrho g h^3}{\nu\sigma} \sum_{n=1}^{\infty} \frac{\exp(p_n t)}{\lambda_n^2(\lambda_n^2+k^2+k)} \frac{\sin\lambda_n(s-x)/h}{\sin\lambda_n}. \tag{3.54}$$

The shear stress is given by

$$\tau_{xz} = -\frac{\sigma g}{2} + \frac{4\varrho g h^2}{\nu\sigma} \sum_{n=1}^{\infty} \frac{(\mu+\alpha_1 p_n)}{\lambda_n^2(\lambda_n^2+k^2+k)} \frac{\cos\lambda_n(s-x)/h}{\sin\lambda_n} \exp(p_n t). \tag{3.55}$$

4. – Heavy cylinder falling under gravity inside a tube.

At time $t=0$ an infinitely long circular cylinder of radius R_0 is released from rest and falls vertically under gravity along its axis in a coaxial tube of radius R_1 filled with a second grade fluid.

Let σ be the weight per unit length of the inner cylinder, ϱ the density of the liquid, ν the kinematic viscosity, v and w respectively the velocity of the inner cylinder and the liquid. We shall assume the velocity field

$$\boldsymbol{w} = (0, 0, w(r,t)),$$

(2) *Op. cit.*, Ting, p. 18.

For the motion of the liquid the governing equations is (cf. [15]):

$$\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)-\varrho\frac{\partial w}{\partial t}=0, \tag{4.1}$$

$$R_0<r<R_1,\quad t>0,$$

with

$$w=0,\qquad r=R_1,\qquad t>0, \tag{4.2}$$

$$w=v,\qquad r=R_0,\qquad t>0, \tag{4.3}$$

$$w=0,\qquad R_0\leqslant r\leqslant R_1,\qquad t=0, \tag{4.4}$$

while for the motion of the cylinder,

$$\sigma\frac{dv}{dt}=\sigma g+2\pi R_0\left[\mu\frac{\partial w}{\partial r}+\alpha_1\frac{\partial^2 w}{\partial t\,\partial r}\right]_{r=R_0}. \tag{4.5}$$

Once again we have assumed that the pressure gradient is approximated by the static solution.

The subsidiary equation derived from (4.1) after applying the Laplace transform is

$$\frac{d^2\overline{w}}{dr^2}+\frac{1}{r}\frac{d\overline{w}}{dr}-\frac{\varrho p}{\mu+\alpha_1 p}\overline{w}=0,\qquad R_0<r<R_1, \tag{4.6}$$

with

$$\overline{w}=0,\qquad r=R_1,\qquad t>0, \tag{4.6a}$$

$$\overline{w}=\overline{v},\qquad r=R_0.\qquad t>0, \tag{4.6b}$$

The bar indicates the Laplace transform of w. The solution of the two-point boundary value problem (4.6)-(4.6b) is

$$\overline{w}=\overline{v}\,\frac{I_0(zr)K_0(zR_1)-I_0(zR_1)K_0(zr)}{I_0(zR_0)K_0(zR_1)-I_0(zR_1)K_0(zR_0)}, \tag{4.7}$$

where $z=\left(\dfrac{\varrho p}{\mu+\alpha_1 p}\right)^{1/2}$.

Motion of the cylinder.

The subsidiary equation corresponding to (4.5) is

$$\sigma p\overline{v}=\frac{\sigma g}{p}+2\pi R_0(\mu+\alpha_1 p)\left.\frac{d\overline{w}}{dr}\right|_{r=R_0}, \tag{4.8}$$

and, after introducing the value of $\overline{w}$ into (4.8) from (4.7) and solving for

$$\bar{v} = \frac{g}{p^2}\left[1 - \frac{2\pi\varrho}{\sigma}\,\frac{1}{z}\,\frac{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)}{I_0(zR_0)K_0(zR_1) - I_0(zR_1)K_0(zR_0)}\right]^{-1}. \tag{4.9}$$

Thus, by the Inversion Theorem,

$$v(t) = \frac{g}{2\pi i}\int_L \frac{e^{pt}}{p^2}\left[1 - \frac{2\pi\varrho}{\sigma}\,\frac{1}{z}\,\frac{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)}{I_0(zR_0)K_0(zR_1) - I_0(zR_1)K_0(zR_0)}\right]^{-1} dp, \tag{4.10}$$

and, in compact form,

$$v(t) = \frac{g}{2\pi i}\int_L \frac{e^{pt}}{p^2}\,\frac{N(z)}{D(z)}\,dp. \tag{4.11}$$

where

$$D(z) = z[I_0(zR_0)K_0(zR_1) - I_0(zR_1)K_0(zR_0)] + \\ -k[I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)], \tag{4.12}$$

and

$$N(z) = z[I_0(zR_0)K_0(zR_1) - I_0(zR_1)K_0(zR_0)], \tag{4.13}$$

Here $k = 2\pi\varrho/\sigma$. Let us estimate $|\bar{v}|$:

$$\left|\frac{g}{p^2}\left[1 - \frac{2\pi\varrho}{\sigma}\,\frac{1}{z}\,\frac{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)}{I_0(zR_0)K_0(zR_1) - I_0(zR_1)K_0(zR_0)}\right]^{-1}\right| \leqslant \\ \leqslant \frac{g}{|p|^2}\left|1 - \frac{2\pi\varrho}{\sigma}\,\frac{1}{|z|}\,\left|\frac{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)}{I_0(zR_0)K_0(zR_1) - I_0(zR_1)K_0(zR_0)}\right|\right|^{-1}. \tag{4.14}$$

The inequality $|a-b| \geqslant ||a| - |b||$ has been employed in deriving (4.14). It can be shown (cf. Bandelli and Rajagopal [16]) that

$$z\,\frac{I_0(zR_0)K_0(zR_1) - I_0(zR_1)K_0(zR_0)}{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)},$$

is bounded and, therefore,

$$|\bar{v}| \leqslant \frac{\text{const.}}{|p|^2},$$

which also ensures that the initial condition is met. Thus, the uniform convergence of the complex integral is assured. As in the previous section we will evaluate the integral

$$v^* = \frac{1}{2\pi i} \int_{L+C+C_N} v(p,t)\, e^{pt}\, dp = \sum_{n=0}^{N} R_n \, .$$

The singularities of the integrand in (4.11) are given by $p = 0$ (single pole) and by the zeros of the transcendental equation

$$D(z) = 0 \, . \tag{4.15}$$

In (4.15) set $z = i\lambda$ and use the identities

$$I_0(ix) = J_0(x) \, , \tag{4.15a}$$

$$I_1(ix) = iJ_1(x) \, , \tag{4.15b}$$

$$K_0(ix) = -\frac{1}{2}\pi i [J_0(x) - i\,Y_0(x)] \, , \tag{4.15c}$$

$$K_1(ix) = -\frac{1}{2}\pi [J_1(x) - i\,Y_1(x)] \, . \tag{4.15d}$$

to find

$$D(i\lambda R_0) = -\frac{1}{2}\pi i \{\lambda [J_0(\lambda R_0)\, Y_0(\lambda R_1) - J_0(\lambda R_1)\, Y_0(\lambda R_0)] -$$

$$-k [J_1(\lambda R_0)\, Y_0(\lambda R_1) - J_0(\lambda R_1)\, Y_1(\lambda R_0)]\} \, .$$

The transcendental equation leading to the roots of (4.15) is:

$$\lambda [J_0(\lambda R_0)\, Y_0(\lambda R_1) - J_0(\lambda R_1)\, Y_0(\lambda R_0)] - \tag{4.16}$$

$$-k [J_1(\lambda R_0)\, Y_0(\lambda R_1) - J_0(\lambda R_1)\, Y_1(\lambda R_0)] = 0 \, ,$$

or

$$\frac{Y_0(\lambda R_1)}{J_0(\lambda R_1)} = \frac{\lambda Y_0(\lambda R_0) - k Y_1(\lambda R_0)}{\lambda J_0(\lambda R_0) - k J_1(\lambda R_0)} \, . \tag{4.17}$$

It can be shown that the roots of this equation are all single, real, and infinitely many (3) (cf. [17]) and can be ordered in an increasing sequence $\{\lambda_n\}$.

(3) *Op. cit.*, Carslaw and Jaeger, p. 217.

Hence the poles of the integrand of (4.10) or (4.11) are

$$p_0 = 0\,, \qquad p_n = -\frac{\mu\lambda_n^2}{\varrho + \alpha_1\lambda_n^2}\,, \qquad (n = 1, 2, \ldots)$$

where λ_n is the n-th positive root of (4.16). Clearly, they are concentrated in the interval $-\mu/\alpha_1 \leqslant p \leqslant 0$ with $p = -\mu/\alpha_1$ as the accumulation point.

Straightforward computation leads to the residues R_n at p_n as follows

$$(4.18)\quad \begin{cases} R_0 = \dfrac{\sigma g}{2\pi\mu}\log\dfrac{R_1}{r}\,, \\[2ex] R_n = \dfrac{2g}{\nu}\dfrac{1}{\lambda_n^2}\dfrac{\exp(p_n t)}{-2 + kR_0 + \dfrac{\lambda_n^2 R_0}{k} - \dfrac{2}{\pi}\dfrac{J_0(\lambda_n R_0) - \dfrac{k}{\lambda_n}J_1(\lambda_n R_0)}{J_0(\lambda_n R_1)V_0(\lambda_n R_0)}} = \\[6ex] = \dfrac{2g}{\nu}\dfrac{1}{\lambda_n^2}\dfrac{\exp(p_n t)V_0(\lambda_n R_0)}{V_0(\lambda_n R_0)\left(kR_0 + \dfrac{\lambda_n^2 R_0}{k} - 2\right) - \dfrac{2}{\pi}\dfrac{J_0(\lambda_n R_0) - \dfrac{k}{\lambda_n}J_1(\lambda_n R_0)}{J_0(\lambda_n R_1)}}\,, \end{cases}$$

where

$$V_0(\lambda_n R_0) = J_0(\lambda_n R_0)Y_0(\lambda_n R_1) - J_0(\lambda_n R_1)Y_0(\lambda_n R_0) \qquad (n = 1, 2, \ldots)\,.$$

It easy to show that

$$(4.19)\quad V_0(\lambda_n R_0)\left(kR_0 + \frac{\lambda_n^2 R_0}{k} - 2\right) - \frac{2}{\pi}\frac{J_0(\lambda_n R_0) - (k/\lambda_n)J_1(\lambda_n R_0)}{J_0(\lambda_n R_1)} =$$

$$= \frac{-2\{(k^2 + \lambda_n^2 - 2(k/R_0))J_0^2(\lambda_n R_1) - [kJ_1(\lambda_n R_0) - \lambda_n J_0(\lambda_n R_0)]^2\}}{\pi\lambda_n J_0(\lambda_n R_1)[kJ_1(\lambda_n R_0) - \lambda_n J_0(\lambda_n R_0)]}\,.$$

By virtue of the last equality, R_n can also be expressed as

$$(4.20)\quad R_n =$$

$$= -\frac{\pi g}{\nu\lambda_n}\frac{V_0(\lambda_n R_0)J_0(\lambda_n R_1)[kJ_1(\lambda_n R_0) - \lambda_n J_0(\lambda_n R_0)]}{\left(k^2 + \lambda_n^2 - \dfrac{2k}{R_0}\right)J_0^2(\lambda_n R_1) - [kJ_1(\lambda_n R_0) - \lambda_n J_0(\lambda_n R_0)]^2}\exp(p_n t)\,.$$

Finally,

$$(4.21)\qquad v(t) = \frac{\sigma g}{2\pi\mu}\log\frac{R_1}{R_0} + \sum_{n=1}^{N}\frac{2\varrho g}{\mu\lambda_n^2}\cdot$$

$$\cdot\frac{\exp(p_n t)}{\left(kR_0 + \dfrac{\lambda_n^2 R_0}{k} - 2\right) - \dfrac{2}{\pi}\dfrac{J_0(\lambda_n R_0) - (k/\lambda_n)J_1(\lambda_n R_0)}{J_0(\lambda_n R_1)V_0(\lambda_n R_0)}} - \int_{C_N}\bar{v}e^{pt}\,dp\,.$$

By letting $N \to \infty$, we have formally

$$(4.22)\qquad v(t) = \frac{\sigma g}{2\pi\mu}\log\frac{R_1}{R_0} + \sum_{n=1}^{\infty}\frac{2g}{\nu}\frac{1}{\lambda_n^2}\cdot$$

$$\cdot\frac{\exp(p_n t)}{\left(kR_0 + \dfrac{\lambda_n^2 R_0}{k} - 2\right) - \dfrac{2}{\pi}\dfrac{J_0(\lambda_n R_0) - (k/\lambda_n)J_1(\lambda_n R_0)}{J_0(\lambda_n R_1)V_0(\lambda_n R_0)}} - \lim_{N\to\infty}\int_{C_N}\bar{v}e^{pt}\,dp\,.$$

Arguments similar to those used in the previous section lead to

$$\lim_{N\to\infty}\int_{C_N}\bar{v}e^{pt}\,dp = 0\,.$$

We now prove that the series in (4.22) converges absolutely and uniformly in r and t and is a valid expression for $v(t)$.

Since $J_1(\lambda R_0)Y_0(\lambda R_1) - J_0(\lambda R_1)Y_1(\lambda R_0)$ is bounded, as $\lambda \to \infty$ the transcendental equation (4.16) reduces to

$$(4.23)\qquad J_0(\lambda R_0)Y_0(\lambda R_1) - J_0(\lambda R_1)Y_0(\lambda R_0) \cong 0\,.$$

Therefore, if $\{\alpha_n\}$ $(n = 1, 2, \dots)$ denotes the increasing sequence of roots of the trascendental equation (4.23), it is

$$\lim_{n\to\infty}|\lambda_n - \alpha_n| = 0 \;\to\; O(\lambda_n) = O(\alpha_n)\,.$$

i.e., the roots of the transcendental equation (4.16) are the same order as those of the equation

$$(4.24)\qquad J_0(\lambda R_0)Y_0(\lambda R_1) - J_0(\lambda R_1)Y_0(\lambda R_0) = 0\,.$$

As $n \to \infty$ (4) (cf. [18]), $\alpha_n/n = O(1)$ and $\lambda_n/n = O(1)$. Now, with the aid of

(4) *Op. cit.*, Gray and Mathews, p. 261.

the asymptotic expansions (cf. Hildebrand [19])

$$J_p(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{p\pi}{2}\right), \tag{4.24a}$$

$$Y_p(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{p\pi}{2}\right), \tag{4.24b}$$

we find that

$$V_0(\lambda_n R_0) \cong \frac{2}{\pi\lambda_n \sqrt{R_0 R_1}} \sin[\lambda_n(R_0 - R_1)] = O\left(\frac{1}{\lambda_n}\right), \tag{4.25}$$

and

$$\frac{J_0(\lambda_n R_0)}{J_0(l_n R_1)} \cong \sqrt{\frac{R_1}{R_0}} \frac{\cos(\lambda_n R_0 - (\pi/4))}{\cos(\lambda_n R_1 - (\pi/4))} = O(1). \tag{4.26}$$

Thus, as $n \to \infty$

$$\frac{J_0(\lambda_n R_0) - (k/\lambda_n) J_1(\lambda_n R_0)}{J_0(\lambda_n R_1) V_0(\lambda_n R_0)} = \tag{4.27}$$

$$= \frac{\lambda_n J_0(\lambda_n R_0) - kJ_1(\lambda_n R_0)}{\lambda_n J_0(\lambda_n R_1) V_0(\lambda_n R_0)} \cong \frac{\text{const.}\, \lambda_n \cos(\lambda_n - \pi/4)}{\cos(\lambda_n R_1 - \pi/4) \sin[\lambda_n(R_0 - R_1)]} \cong O(\lambda_n),$$

because

$$\frac{kJ_1(\lambda_n R_0)}{\lambda_n J_0(\lambda_n R_1) V_0(\lambda_n R_0)} = O(1).$$

Therefore

$$\left(kR_0 + \frac{\lambda_n^2 R_0}{k} - 2\right) - \frac{2}{\pi} \frac{J_0(\lambda_n R_0) - (k/\lambda_n) J_1(\lambda_n R_0)}{J_0(\lambda_n R_1) V_0(\lambda_n R_0)} = O(\lambda_n^2), \tag{4.28}$$

and

$$\frac{2g}{\nu} \frac{1}{\lambda_n^2} \frac{\exp(p_n t) V_0(\lambda_n R_0)}{V_0(\lambda_n R_0)\left(kR_0 + \frac{\lambda_n^2 R_0}{k} - 2\right) - \frac{2}{\pi} \frac{J_0(\lambda R_0) - (k/\lambda_n) J_1(\lambda_n R_0)}{J_0(\lambda_n R_1)}} =$$

$$= O\left(\frac{1}{\lambda_n^4}\right).$$

Hence, the series (4.22) converges in t. Observe that $\cos(\lambda_n R_1 - \pi/4)$ and $\sin[\lambda_n(R_0 - R_1)]$ are $O(1)$ and do not vanish because they respectively represent the asymptotic expansions of $J_0(\lambda_n R_1)$ and $V_0(\lambda_n R_0)$ which themselves do not vanish. In fact, suppose that $J_0(\lambda R_1)$ would vanish for some λ_n, then the transcendental equation (4.17) would imply that

$$[\lambda_n J_0(\lambda_n R_0) - kJ_1(\lambda_n R_0)]\, Y_0(\lambda_n R_1) \equiv 0 \,. \tag{4.29}$$

Now, $Y_0(\lambda_n R_1)$ cannot vanish because Y_0 and J_0 do not have common zeros, hence

$$\lambda_n J_0(\lambda_n R_0) - kJ_1(\lambda_n R_0) = 0 \,, \tag{4.30}$$

which is not possible, for all the zeros of (4.30) are, in general, different from those of $J_0(\lambda_n R_1)$.

Now suppose that $V_0(\lambda R_0)$ vanishes for some λ_n, which, therefore, satisfies equation (4.24): equation (4.16) would imply that

$$J_1(\lambda_n R_0)\, Y_0(\lambda_n R_1) - J_0(\lambda_n R_1)\, Y_1(\lambda_n R_0) \equiv 0 \,. \tag{4.31}$$

Yet, equations (4.24) and (4.31) do not have any common roots; in fact, if they did, then by (4.24)

$$\frac{J_0(\lambda R_0)}{Y_0(\lambda R_0)} = \frac{J_0(\lambda R_1)}{Y_0(\lambda R_1)} \,, \tag{4.32}$$

and by (4.31)

$$\frac{J_1(\lambda R_0)}{Y_1(\lambda R_0)} = \frac{J_0(\lambda R_1)}{Y_0(\lambda R_1)} \,, \tag{4.33}$$

which combined would yield

$$\frac{J_0(\lambda R_0)}{Y_0(\lambda R_0)} = \frac{J_1(\lambda R_0)}{Y_1(\lambda R_0)} : \; J_0(\lambda R_0)\, Y_1(\lambda R_0) - J_1(\lambda R_0)\, Y_0(\lambda R_0) = 0 \,. \tag{4.34}$$

However,

$$J_n(\lambda R_0)\, Y_{n+1}(\lambda R_0) - J_{n+1}(\lambda R_0)\, Y_n(\lambda R_0) = -\frac{2}{\pi \lambda R_0} \,, \tag{4.35}$$

and thus $V_0(\lambda R_0)$ does not vanish for any λ_n.

The final expression of the velocity of the cylinder is

$$(4.36) \qquad v(t) = \frac{\sigma g}{2\pi\mu} \log \frac{R_1}{R_0} +$$

$$+ \sum_{n=1}^{\infty} \frac{2\varrho g}{\mu\lambda_n^2} \frac{\exp(p_n t)}{\left(kR_0 + \dfrac{\lambda_n^2 R_0}{k} - 2\right) - \dfrac{2}{\pi} \dfrac{J_0(\lambda_n R_0) - (k/\lambda_n) J_1(\lambda_n R_0)}{J_0(\lambda_n R_1) V_0(\lambda_n R_0)}} .$$

Notice that the terminal velocity of the cylinder is

$$v(t) = \frac{\sigma g}{2\pi\mu} \log \frac{R_1}{R_0} .$$

Motion of the liquid.

The velocity of the liquid is given by the Inversion Theorem as

$$(4.37) \qquad w(r, t) = \frac{g}{2\pi i} \int_L \frac{1}{p^2} \cdot$$

$$\cdot \frac{I_0(zr) K_0(zR_1) - I_0(zR_1) K_0(zr)}{z(I_0(zR_0) K_0(zR_1) - I_0(zR_1) K_0(zR_0)) - k(I_1(zR_0) K_0(zR_1) + I_0(zR_1) K_0(zR_0))} e^{pt} dp .$$

The integrand in (4.37) is formally identical to that of (4.20), with the difference that r replaces R_0 in the numerator. Therefore, following the same line of reasoning that led to (4.36), the integral (4.37) can be computed as

$$w(r, t) = \frac{\sigma g}{2\pi\mu} \log \frac{R_1}{r} - \sum_{n=1}^{\infty} \frac{2g}{\nu\lambda_n^2} \cdot$$

$$\cdot \frac{\exp(p_n t)}{2 - kR_0 - \dfrac{\lambda_n^2 R_0}{k} + \dfrac{2}{\pi} \dfrac{J_0(\lambda_n R_0) - (k/\lambda_n) J_1(\lambda_n R_0)}{J_0(\lambda_n R_1) V_0(\lambda_n r)}} =$$

$$= \frac{\sigma g}{2\pi\mu} \log \frac{R_1}{r} - \sum_{n=1}^{\infty} \frac{\pi g}{\nu\lambda_n} \cdot$$

$$\cdot \frac{V_0(\lambda_n r) J_0(\lambda_n R_1)[kJ_1(\lambda_n R_0) - \lambda_n J_0(\lambda_n R_0)]}{(k^2 + \lambda_n^2 - 2(k/R_0)) J_0^2(\lambda_n R_1) - [kJ_1(\lambda_n R_0) - \lambda_n J_0(\lambda_n R_0)]^2} \exp(p_n t),$$

where

$$V_0(\lambda_n r) = J_0(\lambda_n r) Y_0(\lambda_n R_1) - J_0(\lambda_n R_1) Y_0(\lambda_n R_0). \tag{4.38}$$

SUNTO

Si studia la caduta di una lamina fra due piastre parallele contenenti un fluido di secondo grado. La velocità della lamina e del fluido sono determinate risolvendo un problema misto al contorno — a valori iniziali per mezzo della trasformata di Laplace —. Si studia poi la caduta di un cilindro in un tubo contenente un fluido di secondo grado utilizzando ancora la trasformata di Laplace e anche in questo caso si determina la soluzione esatta.

ABSTRACT

The falling of a lamina in between two parallel plates containing a fluid of second grade is studied. The velocity of the lamina and the fluid are determined by solving the mixed initial—boundary value problem using Laplace transform. Explicit exact solutions are obtained for the velocity of the lamina and the fluid. Next, the falling of a cylinder in a tube containing a fluid of second grade is analyzed using Laplace transform, and once again exact solutions are found.

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Pervenuto in Redazione il 15 giugno 1995.