Math 6345 - Advanced ODEs

Elementary ODE Review

1 Linear Systems

1.1 Homogeneous equations

A linear system of equations

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy, \tag{1}$$

can be can be written as a matrix ODE

$$\frac{d\bar{x}}{dt} = A\bar{x} \tag{2}$$

where $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\bar{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we consider solutions of the form

$$\bar{x} = \bar{c}e^{\lambda t},$$

then after substitution into (2) we obtain

$$\lambda \bar{c}\, e^{\lambda t} = A\, \bar{c}\, e^{\lambda t}$$

from which we deduce

$$(\lambda I - A)\, \bar{c} = 0. \tag{3}$$

In order to have nontrivial solutions $\bar{c}$, we require that

$$|\lambda I - A| = 0. \tag{4}$$

This is the eigenvalue-eigenvector problem. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then (4) becomes

$$\lambda^2 - (a+d)\lambda + a\,d - b\,c = 0,$$

from which we have three cases:

(i) two distinct eigenvalues
(ii) two repeated eigenvalues,
(iii) two complex eigenvalues

We consider examples of each.

Example 1 − two distinct eigenvalues.
If
$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \bar{x} \tag{5}$$
then the characteristic equation is
$$\begin{vmatrix} \lambda - 1 & -1 \\ -2 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0,$$
from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 2$.

Case 1: $\lambda = -1$
From (3) we have
$$\begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
from which we obtain after expanding $2c_1 + c_2 = 0$ and we deduce the eigenvector
$$\bar{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Case 2: $\lambda = 2$
From (3) we have
$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
from which we obtain after expanding $c_1 - c_2 = 0$ and we deduce the eigenvector
$$\bar{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution to (5) is then given by
$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-\mathrm{t}} + \mathrm{c}_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{2\mathrm{t}}. \tag{6}$$

Example 2 − Two repeated eigenvalues.

If

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \bar{x} \tag{7}$$

then the characteristic equation is

$$\begin{vmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 3 \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0,$$

from which we obtain the eigenvalues $\lambda = 2$ and $\lambda = 2$.

Case 1: $\lambda = 2$

From (3) we have

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain after expanding $c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

so, we have one solution

$$\bar{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

For the second solution we might try

$$\bar{x}_2(t) = \bar{u}\, t\, e^{2t},$$

but substitution into (7) shows that $\bar{u}$ is identically zero! For the second solution, we look for solutions of the form

$$\bar{x}_2(t) = \bar{u}\, t\, e^{2t} + \bar{v}\, e^{2t}.$$

Substitution into (7) gives

$$\begin{aligned} (2I - A)\,\bar{u} &= 0, \\ (2I - A)\,\bar{v} &= -\bar{v}. \end{aligned}$$

The first of (8) is just the eigenvalue-vector problem we already solved, so

$$\bar{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

From the second of (8) we have

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

or

$$v_1 + v_2 = -1,$$

and so any solution of this will work. Here, we choose

$$\bar{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and so the second independent solution is

$$\bar{x}_2(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t\, e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$$

and the general solution is

$$\bar{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} t\, e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t\, e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right] \tag{8}$$

Example 3 — Two complex eigenvalues.
If

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \bar{x} \tag{9}$$

then the characteristic equation is

$$\begin{vmatrix} \lambda - 6 & 1 \\ -5 & \lambda - 4 \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0,$$

from which we obtain the eigenvalues $\lambda = 5 \pm 2i$.

Case 1: $\lambda = 5 + 2i$

From (3) we have

$$\begin{pmatrix} -1 + 2i & 1 \\ -5 & 1 + 2i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain after expanding $-(1 - 2i)c_1 + c_2 = 0$, the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix}.$$

Case 2: $\lambda = 5 - 2i$

From (3) we have

$$\begin{pmatrix} -1 - 2i & 1 \\ -5 & 1 - 2i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain after expanding $-(1 + 2i)c_1 + c_2 = 0$, the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix}.$$

Thus, the two solutions are

$$\bar{x}_1(t) = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t}, \quad \bar{x}_2(t) = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t}.$$

The general solution is therefore

$$\bar{x}(t) = k_1 \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t} + k_2 \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(5-2i)t}.$$

As this is still complex, it is necessary to do some more work. We re-write this as

$$\begin{aligned} \bar{x}(t) &= k_1 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} i \right] e^{5t} \left(\cos 2t + i \sin 2t\right) \\ &+ k_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} i \right] e^{5t} \left(\cos 2t - i \sin 2t\right). \end{aligned}$$

Expanding and letting

$$k_1 + k_2 + c_1, \quad (k_1 - k_2)\, i = c_2,$$

gives

$$\begin{aligned} \bar{x}(t) &= c_1 e^{5t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin 2t \right] \\ &+ c_2 e^{5t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos 2t. \right] \end{aligned} \tag{10}$$

1.2 The Fundamental Matrix

The solutions given in (6), (8) and (10) can all be written as

$$\bar{x} = \Phi \bar{c}. \tag{11}$$

In the first solution (6)

$$\Phi = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & e^{2t} \end{pmatrix}, \tag{12}$$

in the second solution (8)

$$\Phi = \begin{pmatrix} t\, e^{2t} & t\, e^{2t} \\ -t\, e^{2t} & -(t+1)\, e^{2t} \end{pmatrix}, \tag{13}$$

and, in the third solution (10)

$$\Phi = \begin{pmatrix} e^{5t} \cos 2t & e^{5t} \sin 2t \\ e^{5t} \left(\cos 2t + \sin 2t\right) & e^{5t} \left(\cos 2t - \sin 2t\right) \end{pmatrix}, \tag{14}$$

noting that the fundamental matrix satisfies the matrix ODE

$$\frac{d\Phi}{dt} = A\Phi \tag{15}$$

1.3 Variation of parameters for systems

We now wish to solve the nonhomogeneous system,

$$\frac{d\bar{x}}{dt} = A\bar{x} + \bar{f}(t). \tag{16}$$

The solution comprises of two parts: the complementary solution and the particular solution. The complementary solution is found by solving

$$\frac{d\bar{x}}{dt} = A\bar{x},$$

and the particular, by any method. For single equations, we introduced the variation of parameters. In this technique, we replaced the constants c_1 and c_2 in the complementary solutions with functions u and v and then create two equations for these unknowns. For systems, we do the same. The complementary solution is

$$\bar{x} = \Phi\bar{c}.$$

and for the particular solution, we seek a solution of the form

$$\bar{x} = \Phi\bar{u}. \tag{17}$$

where $\bar{u}$ is a vector function to be determined. Substitution of (17) into (16) and solving for $\dot{\bar{u}}$ gives

$$\bar{u} = \int \Phi^{-1}\bar{f}\,dt,$$

thus giving the particular solution as

$$\bar{x}_p(t) = \Phi \int \Phi^{-1}\bar{f}\,dt,$$

and the general solution as

$$\bar{x}(t) = \Phi\bar{c} + \Phi \int \Phi^{-1}\bar{f}\,dt.$$

Example 4

Consider

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \bar{x} + \begin{pmatrix} 6e^{2t} \\ 0 \end{pmatrix}. \tag{18}$$

The characteristic equation is

$$\begin{vmatrix} \lambda - 3 & -2 \\ -2 & \lambda + 2 \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0,$$

from which we obtain the eigenvalues $\lambda = 2$ and $\lambda = -1$.

Case 1: $\lambda = 2$

From (3) we have

$$\begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain after expanding $c_1 - 2c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Case 2: $\lambda = -1$

From (3) we have

$$\begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain after expanding $2c_1 - c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The complementary solution to (18) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \mathrm{e}^{2\mathrm{t}} + \mathrm{c}_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathrm{e}^{-\mathrm{t}}.$$

The associated fundamental matrix is then

$$\Phi = \begin{pmatrix} 2\,e^{2t} & e^{-t} \\ e^{2t} & 2\,e^{-t} \end{pmatrix}.$$

The determinant of Φ is $\det\Phi = 3e^{t}$, and the inverse is given by

$$\Phi^{-1} = \frac{1}{3e^{t}} \begin{pmatrix} 2\,e^{-t} & -e^{-t} \\ -e^{2t} & 2\,e^{2t} \end{pmatrix}.$$

This then gives

$$\begin{aligned} \Phi^{-1}\bar{f} &= \frac{1}{3e^{t}} \begin{pmatrix} 2\,e^{-t} & -e^{-t} \\ -e^{2t} & 2\,e^{2t} \end{pmatrix} \begin{pmatrix} 6e^{t} \\ 0 \end{pmatrix}, \\ &= \begin{pmatrix} 4\,e^{-t} \\ -2\,e^{2t} \end{pmatrix}. \end{aligned}$$

Integrating gives

$$\int \Phi^{-1}\bar{f}\,dt = \begin{pmatrix} -4\,e^{-t} \\ -e^{2t} \end{pmatrix},$$

and multiplying by Φ gives the particular solution

$$\begin{aligned}
\bar{x}_p(t) &= \Phi \int \Phi^{-1} \bar{f}\, dt \\
&= \begin{pmatrix} 2\,e^{2t} & e^{-t} \\ e^{2t} & 2\,e^{-t} \end{pmatrix} \begin{pmatrix} -4\,e^{-t} \\ -e^{2t} \end{pmatrix} \\
&= -\begin{pmatrix} 9\,e^{t} \\ 6e^{t} \end{pmatrix},
\end{aligned}$$

Therefore, the general solution is

$$\bar{x} = \begin{pmatrix} 2\,e^{2t} & e^{-t} \\ e^{2t} & 2\,e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \begin{pmatrix} 9 \\ 6 \end{pmatrix} e^{t},$$