BOUNDS FOR SYSTEMS OF LINES

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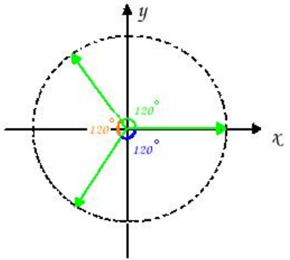
Outline

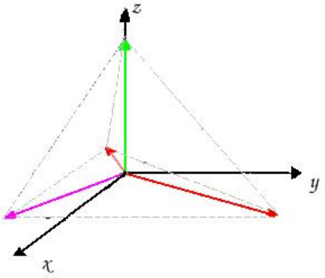
Question

How many equiangular unit vectors are there in $\mathbb{R}^2$ *and* $\mathbb{R}^3$*?*

$\mathcal{R}^2$



$\mathcal{R}^3$



Definition

Let V be the space of all polynomials and let $f, g \in V$. We define the inner product between f, g as

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x) dx,$$

where $w(x)$ is a weight function.

Jacobi polynomials

- In $\mathbb{R}^n$, there are some special type of polynomials called Jacobi polynomial which are given by

$$P_0(x) = 1,$$
$$P_1(x) = \frac{(n+2)(nx-1)}{2},$$
$$P_2(x) = \frac{n(n+6)\left((n+2)(n+4)x^2 - 6(n+2)x + 3\right)}{24},$$
$$\vdots$$
$$P_k(x) = \cdots$$
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- These polynomials form a basis for the linear space of all polynomials of degree $\leqslant k$.

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P_k(x) &= \cdots\\
&\vdots
\end{aligned}
$$

- These polynomials form a basis for the linear space of all polynomials of degree $\leqslant k$.
- This basis is orthogonal on $[0,1]$ under a suitable weight function.

Jacobi polynomials

example

In $\mathbb{R}^2$, *we have* $P_0(x) = 1, P_1(x) = 2(2x-1)$. *So*

$$\begin{aligned}
\langle P_0(x),\ P_1(x)\rangle &= \langle 1,\ 2(2x-1)\rangle \\
&= \int_0^1 2(2x-1)(1-x)^{-\frac{1}{2}}x^{-\frac{1}{2}}\,dx \\
&= -4\sqrt{x-x^2}\ ]_0^1 \\
&= 0
\end{aligned}$$

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- *In this example the suitable weight is* $(1-x)^{-\frac{1}{2}}x^{-\frac{1}{2}}$, *and* $P_0(x), P_1(x)$ *make a basis for the space of polynomials of degree* $\leqslant 1$.

Outline

1. Jacobi polynomials
2. Special bounds for A-set (Delsarte, Goethals and Seidel)
3. Absolute bounds for A-set (Delsarte, Goethals and Seidel)
4. Recent results

Definition

Let X be a set of unit vectors in $\mathbb{R}^n$ and

$$A = \{\alpha_1, \ldots, \alpha_s\}$$

a set of all non-negative numbers all less than 1.

Special bounds for A-set (Delsarte, Goethals and Seidel)

Definition

Let X be a set of unit vectors in $\mathbb{R}^n$ and

$$A = \{\alpha_1, \ldots, \alpha_s\}$$

a set of all non-negative numbers all less than 1.
X is called an A-set if for any $u, v \in X$,

$$|\langle u, v\rangle|^2 \in A.$$

Special bounds for A-set (Delsarte, Goethals and Seidel)

Theorem

Let $F(x)$ be a polynomial of degree k such that

$$\forall\alpha \in A \quad ; \quad F(\alpha) \leqslant 0.$$

Write $F(x) = a_0P_0(x) + a_1P_1(x) + \cdots + a_kP_k(x)$, and assume that $a_0 > 0$ and $a_i \geqslant 0$ for $1 \leqslant i \leqslant k$. Then

$$|X| \leqslant \frac{F(1)}{a_0}.$$

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$$|X| \leqslant \frac{F(1)}{a_0}.$$

- One of the best polynomials that satisfies all conditions in the previous theorem is the annihilator of A which is defined by

$$\left(\frac{x-\alpha_1}{1-\alpha_1}\right) \cdot \left(\frac{x-\alpha_2}{1-\alpha_2}\right) \cdot \cdots \cdot \left(\frac{x-\alpha_s}{1-\alpha_s}\right).$$

Special bounds for A-set (Delsarte, Goethals and Seidel)

example

In $\mathbb{R}^n$, *if* $A = \{\alpha\}, 0 \leqslant \alpha < \frac{1}{n}$, *we take*

$$\begin{aligned} F(x) = \frac{x-\alpha}{1-\alpha} &= a_0 P_0(x) + a_1 P_1(x) \\ &= a_0 + a_1 \frac{(n+2)(nx-1)}{2}, \end{aligned}$$

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then

$$a_1 = \frac{2}{(1-\alpha)n(n+2)}, \quad a_0 = \frac{1-n\alpha}{n-n\alpha}.$$

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Thus

$$|X| \leqslant \frac{n(1-\alpha)}{1-n\alpha}.$$

Special bounds for A-set (Delsarte, Goethals and Seidel)

example

In $\mathbb{R}^2$, *let* $A = \left\{\frac{1}{4}\right\}$, *so* $|X| \leqslant 3$ *and this bound is achieved!*

$$X=\left\{(1,0),\left(\frac{-1}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{-1}{2},\frac{-\sqrt{3}}{2}\right)\right\}$$

Special bounds for A-set (Delsarte, Goethals and Seidel)

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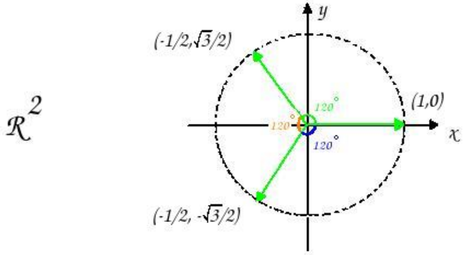
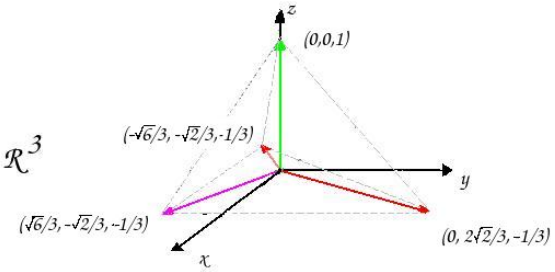
example

In $\mathbb{R}^3$*, let* $A = \left\{\frac{1}{9}\right\}$, *so* $|X| \leqslant 4$ *and this bound is achieved!*

$$X=\left\{(0,0,1),\left(\frac{-\sqrt{6}}{3},\frac{-\sqrt{2}}{3},\frac{-1}{3}\right),\left(\frac{\sqrt{6}}{3},\frac{-\sqrt{2}}{3},\frac{-1}{3}\right),\left(0,\frac{2\sqrt{2}}{3},\frac{-1}{3}\right)\right\}$$

Special bounds for *A*-set (Delsarte, Goethals and Seidel)





Special bounds for A-set (Delsarte, Goethals and Seidel)

example

In $\mathbb{R}^n$, $A = \{\alpha\}$, the following bounds are achieved!

n	2	3	4	5	6	7	15	19	20	21	22
α	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$
$\lvert X\rvert$	3	4	6	10	16	28	36	76	96	126	176

example

In $\mathbb{R}^n$, *let* $A = \{\alpha_1, \alpha_2\}, 0 \leqslant \alpha_1 \neq \alpha_2 < 1.$

example

In $\mathbb{R}^n$, *let* $A = \{\alpha_1, \alpha_2\}, 0 \leqslant \alpha_1 \neq \alpha_2 < 1$. *Thus we take*

$$\begin{aligned} F(x) &= \left(\frac{x-\alpha_1}{1-\alpha_1}\right)\cdot\left(\frac{x-\alpha_2}{1-\alpha_2}\right) \\ &= a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) \\ &= a_0 + a_1 \frac{(n+2)(nx-1)}{2} \\ &+ a_2 \frac{n(n+6)\left((n+2)(n+4)x^2 - 6(n+2)x + 3\right)}{24}. \end{aligned}$$

Special bounds for A-set (Delsarte, Goethals and Seidel)

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By equating the coefficients on both sides we get

$$a_2 = \frac{24}{(1-\alpha_1)(1-\alpha_2)n(n+2)(n+4)(n+6)},$$

Special bounds for A-set (Delsarte, Goethals and Seidel)

example

$$a_1 = \frac{12 - 2(\alpha_1 + \alpha_2)(n+4)}{n(n+2)(n+4)(1-\alpha_1)(1-\alpha_2)},$$

Special bounds for A-set (Delsarte, Goethals and Seidel)

example

$$a_1 = \frac{12 - 2(\alpha_1 + \alpha_2)(n+4)}{n(n+2)(n+4)(1-\alpha_1)(1-\alpha_2)},$$

$$a_0 = \frac{3 - (n+2)(\alpha_1 + \alpha_2) + n(n+2)\alpha_1\alpha_2}{n(n+2)(1-\alpha_1)(1-\alpha_2)}.$$

Special bounds for A-set (Delsarte, Goethals and Seidel)

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Thus

$$|X| \leqslant \frac{F(1)}{a_0} = \frac{n(n+2)(1-\alpha_1)(1-\alpha_2)}{3 - (n+2)(\alpha_1 + \alpha_2) + n(n+2)\alpha_1\alpha_2};$$

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Moreover, we must have $a_i \geqslant 0$ *for* $i = 1, 2$. *Thus*

$$\alpha_1 + \alpha_2 \leqslant \frac{6}{n+4} = K.$$

Outline

Absolute bounds for A-set (Delsarte, Goethals and Seidel)

- In the following theorem, a different bound is obtained for an A-set which depends only on the dimension of vector space and the number of elements in the set A.

Absolute bounds for A-set (Delsarte, Goethals and Seidel)

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- Note that this bound is independent of the nature of elements in the set A.

Theorem

In $\mathbb{R}^n$, for any A-set with $|A| = s$, we have

$$|X| \leqslant M_s,$$

where

$$M_s := P_0(1) + P_1(1) + \cdots + P_s(1) = \binom{n+2s-1}{n-1}.$$

Outline

1. Jacobi polynomials
2. Special bounds for A-set (Delsarte, Goethals and Seidel)
3. Absolute bounds for A-set (Delsarte, Goethals and Seidel)
4. Recent results

- In the case $A = \{0, \alpha\}$, where $0 < \alpha < 1$, by completely different methods, Calderbank, Cameron, Kantor and Seidel in 1996 proved that $|X| \leqslant \binom{n+2}{3}$, and

$$|X| \leqslant \frac{n(n+2)(1-\alpha)}{3-(n+2)\alpha}$$

which are the absolute bound and the special bound, respectively, obtained by Delsarte, Goethals and Seidel in 1974.

- In the case $A = \{0, \alpha\}$, Best, Kharaghani and Ramp revisited the inequality

$$|X| \leqslant \frac{n(n+2)(1-\alpha)}{3-(n+2)\alpha}$$

and they introduced a class of weighing matrices that they named mutually unbiased weighing matrices.

Unbiased weighing matrices (Best, Kharaghani, Ramp)

Definition

A matrix $W = [w_{ij}]_{n\times n}$ such that $w_{ij} \in \{0, -1, 1\}$ and $WW^t = pI_{n\times n}$ is called a weighing matrix of order n and weight p. It is denoted by $W(n, p)$.

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Definition

Let p be a perfect square. $W_1(n, p), W_2(n, p)$ are called unbiased if $W_1 W_2^t = \sqrt{p}W$, where W is a $W(n, p)$.

Unbiased weighing matrices (Best, Kharaghani, Ramp)

example

$$W_1(4,4)=\begin{pmatrix} 1 & 1 & 1 & - \\ 1 & 1 & - & 1 \\ - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \end{pmatrix},\quad W_2(4,4)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{pmatrix};$$

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Thus

$$W_1(4,4)W_2^t(4,4)=2\begin{pmatrix} 1 & 1 & 1 & - \\ 1 & 1 & - & 1 \\ 1 & - & - & - \\ 1 & - & 1 & 1 \end{pmatrix}.$$

Unbiased weighing matrices (Best, Kharaghani, Ramp)

- Think of each row of W as an n-dimensional vector.

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In $\mathbb{R}^4$, with $A = \left\{0, \frac{1}{4}\right\}$, we know that $|X| \leqslant 12$.
By normalizing each row of the unbiased weighing matrices in the previous example and adding the standard basis in $\mathbb{R}^4$, we find 12 vectors such that the inner product of each pair is either $\pm\frac{1}{2}$ or 0.

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By normalizing each row of the unbiased weighing matrices in the previous example and adding the standard basis in $\mathbb{R}^4$, we find 12 vectors such that the inner product of each pair is either $\pm\frac{1}{2}$ or 0.
Thus the square of these inner products belong to the set A.
Therefore this upper bound is achieved!

example

In $\mathbb{R}^7$, *with* $A = \left\{0, \frac{1}{4}\right\}$, *we know that* $|X| \leqslant 63$.

Unbiased weighing matrices (Best, Kharaghani, Ramp)

example

In $\mathbb{R}^7$, *with* $A = \left\{0, \frac{1}{4}\right\}$, *we know that* $|X| \leqslant 63$.
Best, Kharaghani and Ramp could find 8 unbiased weighing matrices $W(7,4)$ *that by adding standard basis in* $\mathbb{R}^7$ *give us 63 vectors in such a way that the square of inner product of each pair belongs to the set* A.

example

In $\mathbb{R}^8$, *with* $A = \left\{0, \frac{1}{4}\right\}$, *we know that* $|X| \leqslant 120$.

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example

In $\mathbb{R}^8$, *with* $A = \left\{0, \frac{1}{4}\right\}$, *we know that* $|X| \leqslant 120$.
They could also find 14 unbiased weighing matrices $W(8,4)$ *that by adding standard basis in* $\mathbb{R}^8$ *give us 120 vectors in such a way that the square of inner product of each pair belongs to the set* A.


Thank
you!