Wavelet-based 3D inversion for frequency-domain AEM data

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SUMMARY

In this paper, we propose a new wavelet-based 3D inversion method for frequency-domain AEM data. Instead of inverting the model in the space domain using a smoothing constraint, this new method recovers the model in the wavelet domain based on a sparsity constraint. In the wavelet domain, the model is represented by two types of coefficients, which contain both global smoothing and local fine-scale information of the model, meaning the wavelet domain inversion has inherent multiresolution. In order to accomplish the sparsity constraint on the generally sparse wavelet coefficients, we minimize an L_1-norm measure in the wavelet domain. The final inversion system is solved by an iteratively reweighted least squares method. We investigate different orders of Daubechies wavelets to accomplish our inversion algorithm, and test them on a synthetic frequency-domain AEM dataset. The results show that higher order wavelets have larger vanishing moments and regularity, which can deliver a more stable inversion process and give better local resolution, while the lower order wavelets are simpler and less smooth, and thus capable of recovering sharp discontinuities if the model is simple.

Keywords: inversion, wavelet domain, frequency-domain AEM, sparsity constraint

INTRODUCTION

For three-dimensional (3D) electromagnetic (EM) inversion, the geo-electrical model is generally parametrized into regular or unstructured cells. The dimensions of the model are usually not small, so the number of unknowns is much larger than the number of observed data, which always results in an ill-conditioned problem to be solved. In order to make the 3D inversion stable, some kind of regularization such as Tikhonov regularization needs to be applied (Aster et al., 2013). Such regularization methods usually use a smoothing operator to build a relationship between different cells. This technique indeed makes the inversion more stable, but if the regularization is strong, it often smears out small-scale features of the model even where there is sufficient data density to provide more detailed information. If we use a weak regularization, the resolution may be improved, but the inversion may not be stable.

For seismics, to get different multiscale features of the model according to the data's distribution, several adaptive seismic tomography methods have been proposed in which the cells are irregularly distributed to match with the uneven ray distribution (Abers and Roecker, 1991; Spakman and Bijwaard, 2001; Zhang and Thurber, 2005). These methods work well for recovering multiscale features of the model, but they are based purely on data and do not consider whether the model is appropriately parametrized. The wavelet representation of a signal has an inherent multiscale nature (Daubechies, 1992), and wavelet-based multiscale tomography methods have been proposed to recover the multiscale features of the model parametrized by regular cells (Chiao and Kuo, 2001; Chiao and Liang, 2003; Loris et al., 2007; Delost et al., 2008; Hung et al., 2011; Simons et al., 2011; Fang and Zhang, 2014).

In this research, we propose a new wavelet-based inversion method suitable for EM data. We test different kinds of basis functions and compare with other traditional methods on a synthetic frequency-domain AEM dataset, analysing the impact of key characters of the wavelet basis functions on the inversion result.

METHODOLOGY

General measure inversion

Generally, the objective function for a three-dimensional inversion problem can be expressed as (Farquharson, 2008):

$$\mathbf{\Phi} = \phi_d + \lambda \phi_m. \tag{1}$$

Here, ϕ_d is the measure of data misfit, which can be expressed in a general form

$$\phi_d = \phi_d(\mathbf{u}), \tag{2}$$

$$\mathbf{u} = \mathbf{W}_d(\mathbf{d}^{obs} - \mathbf{d}^{prd}), \tag{3}$$

where $\mathbf{d}^{obs}$ is the vector of observed data, $\mathbf{d}^{prd}$ is the predicted data computed using the model $\mathbf{m}$, and $\mathbf{W}_d$ is a diagonal matrix whose elements are the reciprocals of the estimates of the standard deviations of the noise in the observations. Also, ϕ_m is a measure of the complexity of the recovered model, having the form

$$\phi_m = \sum_k \alpha_k \phi_k(\mathbf{v}_k), \tag{4}$$

$$\mathbf{v}_k = \mathbf{W}_k(\mathbf{m} - \mathbf{m}_k^{ref}). \tag{5}$$

Here $\mathbf{m}$ is the vector of the recovered model parameters, $\mathbf{m}_k^{\mathrm{ref}}$ is the reference model, $\mathbf{W}_k$ is the smoothing operator matrix which is used to constrain the complexity of the recovered model, and α_k are the coefficients used to balance the contributions of different kinds of measure of the recovered model. The factor λ is the trade-off parameter which balances the contributions of the data misfit and the model complexity term.

A general form of ϕ_d and ϕ_k is

$$\phi(\mathbf{x}) = \sum_j \rho(x_j), \tag{6}$$

where x_j are the elements of the vector $\mathbf{x}$, which can be $\mathbf{u}$ or $\mathbf{v}_k$, and the summation is over all elements in each vector. There are numerous choices for the specific form of $\rho(x_j)$, such as the L_2-norm, L_1-norm and so on.

In the traditional inversion approach, differentiating both sides of equation (1) with respect to the space-domain model parameters and adopting the Taylor expansion gives the linear system of equations to be solved at each iteration, which is expressed as

$$\begin{aligned}&\left[\mathbf{J}^T\mathbf{W}_d^T\mathbf{R}_d\mathbf{W}_d\mathbf{J}^T + \lambda^n\sum_k \alpha_k\mathbf{W}_k^T\mathbf{R}_k\mathbf{W}_k\right]\delta\mathbf{m}\\ &= \mathbf{J}^T\mathbf{W}_d^T\mathbf{R}_d\mathbf{W}_d(\mathbf{d}^{obs} - \mathbf{d}^{n-1})\\ &\quad + \lambda^n\sum_k \alpha_k\mathbf{W}_k^T\mathbf{R}_k\mathbf{W}_k(\mathbf{m}_k^{ref} - \mathbf{m}^{n-1}),\end{aligned} \tag{7}$$

where the matrix $\mathbf{R}$ is a diagonal matrix, which has the form as

$$\mathbf{R} = \mathrm{diag}\{\rho'(x_j)/x_j,\quad \cdots\cdots,\quad \rho'(x_N)/x_N,\} \tag{8}$$

$\mathbf{J}$ is the sensitivity matrix in terms of the space-domain model parameters, which can be calculated by an adjoint modelling technique (Liu and Yin, 2016), $\mathbf{d}^{n-1}$ is the vector of data for the model $\mathbf{m}^{n-1}$ obtained from the previous iteration, and $\delta\mathbf{m}=\mathbf{m}^n-\mathbf{m}^{n-1}$.

Wavelet-domain inversion with sparsity constraint

In the traditional style of inversion, we usually use an L_2-norm measure for the model's complexity to get a smoothed result, or an L_1-norm or some other focusing measure to get a somewhat "focused" model with greater boundary resolution. In numerical tests, an L_2-norm inversion is typically more stable than an L_1-norm inversion, particularly if data density is low. It is hard to combine the advantages of L_2- and L_1-norms in one inversion.

The wavelet transform can divide one signal into two groups of coefficients: one is the approximation coefficients and the other is the detail coefficients. This means that this transform has the potential to resolve the model at different levels at the same time according to the data-set's distribution. This character of wavelet transform motivates us to do the inversion in the wavelet domain.

For a space-domain model, we use the wavelet transform to transform it into the wavelet domain. This can be expressed as

$$\tilde{\mathbf{m}} = \mathbf{W}_w\mathbf{m}, \tag{9}$$

where $\tilde{\mathbf{m}}$ is the vector of wavelet-domain coefficients, and $\mathbf{W}_w$ is the wavelet transform matrix. Since the wavelet transform usually demands that the dimension of the vector should be a power of 2, we first extend the three dimensions of $\mathbf{m}$ to be power of 2, then we apply the 1D pwt (partial wavelet transform) scheme of William et al. (1996) to each dimension of the 3D model sequentially. According to the chain rule, the sensitivity matrix in the wavelet domain is computed as

$$\tilde{\mathbf{J}} = \mathbf{J}\mathbf{W}_w^{-1}, \tag{10}$$

where $\mathbf{W}_w^{-1}$ is the inverse wavelet transform operator.

Since the wavelet coefficients are usually sparse, we use a sparsity constrain on the wavelet coefficients instead of the traditional constrain on the space-domain model (Fang and Zhang, 2014). According to Aster et al.(2013), the L_1-norm measure can lead to a sparse solution, so we simply use the L_1-norm measure on the wavelet coefficients and set k = 1 and α_1 = 1 in equation (1). For the data misfit term, we still use the L_2-norm. Then, the linear system to be solved at each iteration of the inversion becomes

$$\begin{aligned}&\left[\tilde{\mathbf{J}}^T\mathbf{W}_d^T\mathbf{R}_d\mathbf{W}_d\tilde{\mathbf{J}}^T + \lambda^n\mathbf{R}\right]\delta\tilde{\mathbf{m}}\\ &= \tilde{\mathbf{J}}^T\mathbf{W}_d^T\mathbf{R}_d\mathbf{W}_d(\mathbf{d}^{obs} - \mathbf{d}^{n-1})\\ &\quad + \lambda^n\mathbf{R}(\tilde{\mathbf{m}}^{ref} - \tilde{\mathbf{m}}^{n-1}),\end{aligned} \tag{11}$$

where the smoothing operator $\mathbf{W}_k$ is no longer needed because the wavelet-based representation of the model can give the smoothing constrain inherently. Since we use an L_2-norm measure for the data misfit term and an L_1-norm measure for the wavelet coefficient, $\mathbf{R}_d$ equals 2 times the unit matrix, while for $\mathbf{R}$, we use the specific form proposed by Ekblom (1987),

$$R_{jj} = (x_j^2 + \varepsilon^2)^{-\frac{1}{2}}, \tag{12}$$

$$x_j = \tilde{m}_j^{ref} - \tilde{m}_j^{n-1}, \tag{13}$$

where j is the index of the element in the vector $\mathbf{x}$, and ε is a very small real number which is used to guarantee the derivative exists at x = 0 is exists.

The matrices $\mathbf{R}$ and the sensitivity matrix $\mathbf{J}$ all depend on the model. They are updated at each iteration, meaning it is an IRLS (iteratively reweighted least squares) technique that is being used to solve the inversion problem. At each iteration, we use the conjugate gradient (CG) method to solve equation (11). After we obtain the update in the wavelet domain, the model in the space domain is updated as

$$\mathbf{m}^n = \mathbf{m}^{n-1} + \mathbf{W}_w^{-1}\delta\tilde{\mathbf{m}}. \tag{14}$$

In practice, in each iteration of the inversion, we set a threshold to zero out small or nearly zero wavelet coefficients. In the beginning of the inversion, we set a relatively large threshold value to recover the main features of the model, then we gradually reduce the threshold value to obtain more details of the model. This process can make the inversion more stable.

NUMERICAL EXAMPLES

The synthetic model is divided into 48 × 48 × 21 cells. The cell size in the *x* and *y* directions is 20 m, while in the *z* direction the thickness of the top layer is 5 m and the thicknesses of the underlying layers increases with a ratio of 1.2. We consider the frequency-domain AEM system of the Norway Geology Survey as an example. This system has three frequencies for its HCP coil array, namely 880 Hz, 6606 Hz and 34133 Hz with corresponding coil separations of 6 m, 6.3 m and 4.9 m, and two frequencies for its VCA coil array, namely 980 Hz and 7001 Hz for coil separations of 6 m and 6.3 m. The flight height is 30 m. The background half-space resistivity is 100 Ωm. A flat conductive plate of dimensions 360 m × 360 m × 6 m and a resistivity of 20 Ωm is located at the centre of the model, with a depth to its top of 5 m. At the depth of 18 m, there is a 60 m thick, 360 m long and 80 m high inclining plate with a resistivity of 10 Ωm (see Figure 1a). The survey stations are evenly distributed in the central area of the model with a spacing of 40 m by 100 m along 8 flight lines with 19 sites on each line, for a total of 152 survey locations. We apply the "moving footprint" method of Cox et al. (2010) to accelerate the forward modelling. For the example here we add 1% Gaussian noise to the synthetic data.

Here we investigate three different wavelet basis functions: the db2, db6 and db10 Daubechies wavelets. We compare the inversion results based on the wavelet-domain method with results based on traditional space-domain methods. For the space-domain inversion, we set the initial value of the trade-off parameter λ to be 10000, while for the wavelet domain inversion we set the initial value of λ to be 1000. If the data misfit reduces very slowly, we set λ to be ten percent of its previous value. After it reduces to 0.1, it will be kept constant in the subsequent iterations. For the L_1-norm type inversion, we set the ε in equation (12) to be $1.0{\times}10^{-4}$ and keep it constant in the inversion.

From Figure 1 we can see that the db10 result has nearly the same resolution as the L_1-norm inversion. This is because the db10 basis function has a larger vanishing moment than db2 and db6, so it has a more focused energy distribution. Also, the regularity of db10 is greater than db2 and db6, which defines the smoothness of the basis function, so the inversion based on db10 is more stable and suitable for complex models. Theoretically, if the model is simple and has quite clear discontinuities, the db2 wavelet can give sharper boundaries because its basis function has quite small regularity. As can be seen in Figure 1, the conductive plate recovered by db2 has better resolution than the other methods.

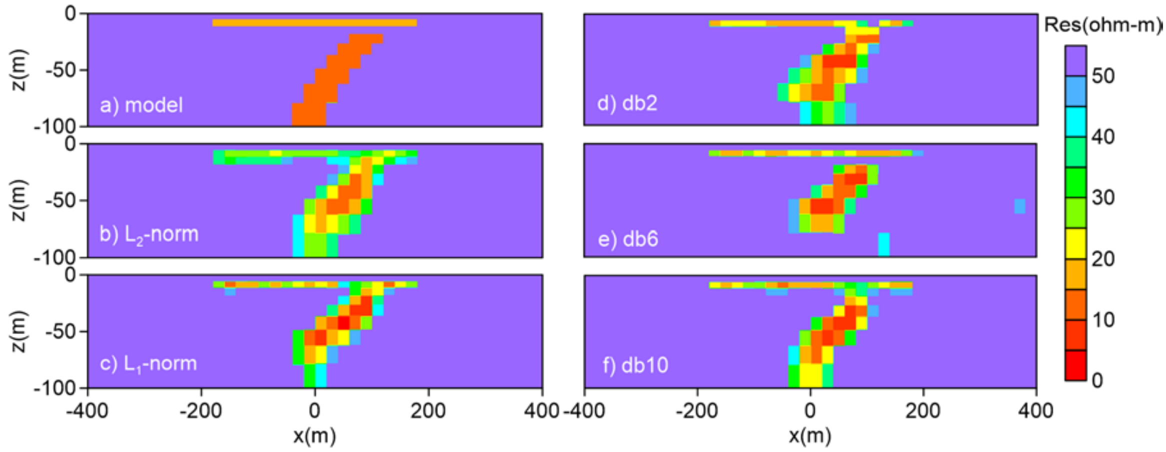

Figure 1. True model (a) and models constructed by (b) L_2-norm inversion in the space domain, (c) L_1-norm inversion in the space domain, (d) wavelet inversion using the db2 basis function, (e) wavelet inversion using db6, (f) wavelet inversion using db10.

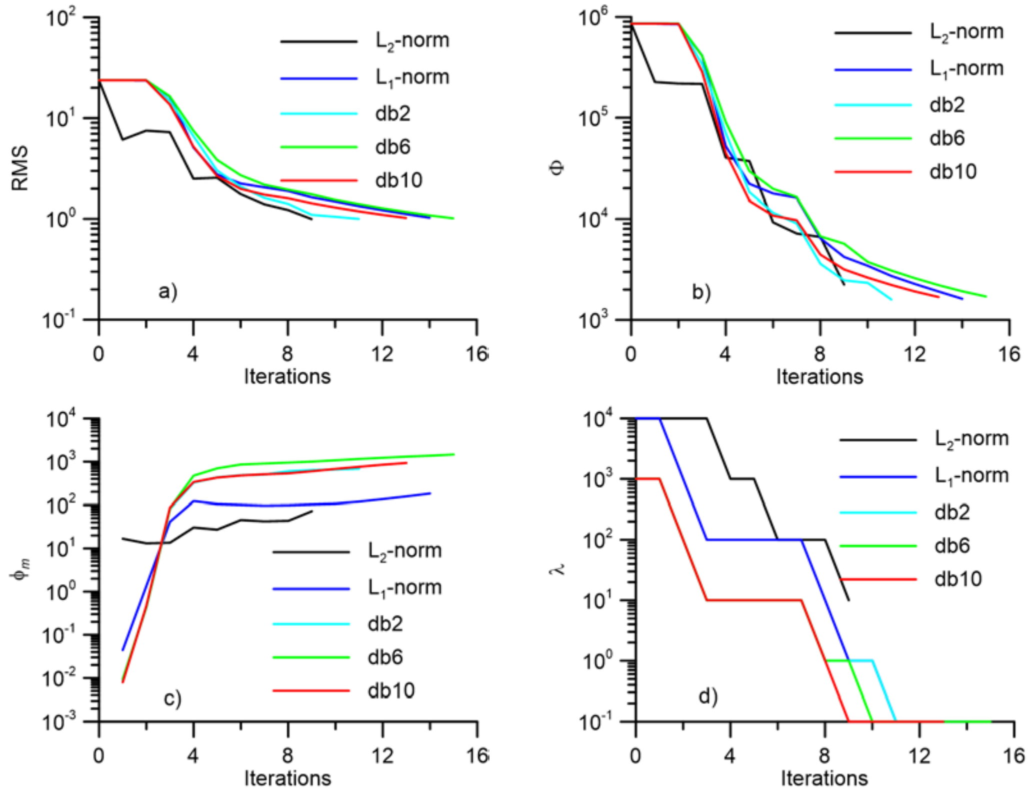

Figure 2. Inversion parameters vs. iteration for the inversions whose results are shown in Figure 1: (a) data misfit (RMS); (b) value of objective function, **Φ**; (c) ϕ_m; (d) λ.

From Figure 2 we can see that all the inversion methods can reduce the data misfit to the target level (an RMS of 1), which means the inversions have behaved correctly. However, for the L_1-norm based inversions, at the first 2-3 iterations the misfit reduces very slowly. This is because in the first several iterations the **R** calculated by formula (12) is quite large, so we can't obtain a large update for the model by solving equation (11). One possibility would be to increase ε to a large number in the first several iterations to get a large model update, then change ε to a smaller value in subsequent iterations. Through numerical testing, we found that we can't significantly accelerate the convergence of the inversion by using this technique, so we just keep ε small throughout the inversion for simplicity.

CONCLUSIONS

To make the 3D inversion of frequency-domain AEM data both stable and of good resolution where data coverage is good, we propose a new inversion method based on the wavelet transform and sparsity constraint. This new method takes advantages of the multiscale property of the wavelet representation. Synthetic tests show that converge is stable and that the approach can give as good resolution as L_1-norm approaches in the space domain. The method has the potential to improve inversion resolution for many other electromagnetic methods while not compromising the stability of the inversion.

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