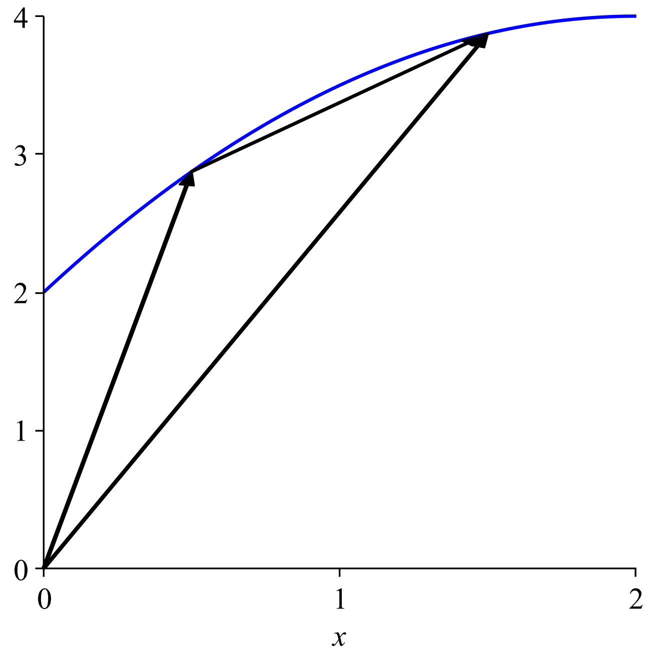
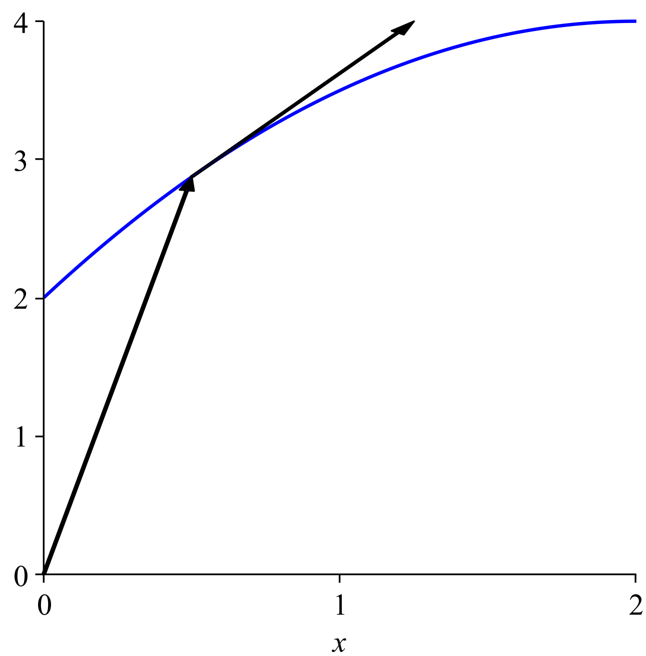
Calculus 3 - Vector Functions

Last class we considered $\vec{r}(t + \Delta t) - \vec{r}(t)$



then computed $\lim_{\Delta t \to 0} \dfrac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$



It is interesting to note the the derivative of the vector functions gives rise to a vector that is tangent to the curve. This we will define later. Now we consider the derivative analytically for the vector function

$$\vec{r}(t) = < f(t), g(t) >$$

$$\begin{aligned}
\vec{r}'(t) &= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{< f(t+\Delta t), g(t+\Delta t) > - < f(t), g(t) >}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \right\rangle \\
&= < f'(t), g'(t) >
\end{aligned} \tag{1}$$

so we have

$$\vec{r}'(t) = < f'(t), g'(t) > \tag{2}$$

and all the derivative rules from Calc 1 apply.

Example 1

If $\vec{r}(t) = < t, t^2 >$ find $\vec{r}'(t)$.

A simple calculation gives $\vec{r}'(t) = < 1, 2t >$

Example 2

If $\vec{r}(t) = < \cos t, \sin t >$ find $\vec{r}'(t)$.

A simple calculation gives $\vec{r}'(t) = < -\sin t, \cos t >$

Example 3

If $\vec{r}(t) = < t^2 - 1, 2t, t^2 + 1 >$ find $\vec{r}'(t)$ and $\vec{r}''(t)$.

A simple calculation gives $\vec{r}'(t) = < 2t, 2, 2t >$ and $\vec{r}''(t) = < 2, 0, 2 >$. A physical interpretation of $\vec{r}''(t)$ will be given later.

Integrals

As much as we can take derivatives of vector functions, we can integrate vector functions. So if $\vec{r}(t) =< f(t), g(t) >$ then for indefinite integrals

$$\int \vec{r}(t)dt =< \int f(t)dt + c_1, \int g(t)dt + c_2 > \tag{3}$$

where c_1 and c_2 are arbitrary functions and for definite integrals

$$\int_a^b \vec{r}(t)dt =< \int_a^b f(t)dt, \int_a^b g(t)dt > . \tag{4}$$

Example 4

If $\vec{r}(t) =< \cos t, \sin t >$ find $\int \vec{r}(t)dt$.

A simple calculation gives

$\int \vec{r}(t)dt =< \int \cos t dt + c_1, \int \sin t dt + c_2 >=< \sin t + c_1, -\cos t + c_2 >$

Example 5

If $\vec{r}(t) =< 3t^2, 2t, \frac{1}{t^2} >$ find $\int_1^2 \vec{r}(t)dt$.

A simple calculation gives

$$\begin{aligned} \int_1^2 \vec{r}(t)dt &= \left\langle \int_1^2 3t^2 dt, \int_1^2 2t dt, \int_1^2 \frac{1}{t^2} dt \right\rangle \\ &= \left\langle t^3 \Big|_1^2, \ t^2 \Big|_1^2, \ -\frac{1}{t} \Big|_1^2 \right\rangle \\ &= \left\langle 7, 3, \frac{1}{2} \right\rangle \end{aligned} \tag{5}$$

In calculus 2 we introduced two ways of multiplying vectors: the dot product and cross product.

<u>*Dot Product*</u> The dot product of two vectors $\vec{u} =< u_1, u_2 >$ and $\vec{v} =< v_1, v_2 >$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

or in 3D where $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

The alternate definition is

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

where $\|\vec{u}\|$ and $\|\vec{v}\|$ is the magnitude of the two vectors and θ is the angle between the vectors.

<u>*Cross Product*</u>

Given vectors $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$ we define the cross product between two vectors as

$$\vec{u} \times \vec{v} =< u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 >$$

Now we define the cross product

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors and c a scalar (a number)

$$\begin{aligned}
(i) &\quad \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u} \\
(ii) &\quad \vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} \\
(iii) &\quad c(\vec{u}\cdot\vec{v}) = (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v}) \\
(iv) &\quad \vec{0}\cdot\vec{v} = 0 \\
(v) &\quad \vec{u}\cdot\vec{u} = \|\vec{u}\|^2
\end{aligned}$$

Properties of the Cross Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors and c a scalar (a number)

$$\begin{aligned}
(i) &\quad \vec{u}\times\vec{v} = -\vec{v}\times\vec{u} \\
(ii) &\quad \vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w} \\
(iii) &\quad c(\vec{u}\times\vec{v}) = (c\vec{u})\times\vec{v} = \vec{u}\times(c\vec{v}) \\
(iv) &\quad \vec{0}\times\vec{v} = \vec{0} \\
(v) &\quad \vec{u}\times\vec{u} = \vec{0} \\
(vi) &\quad \vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\times\vec{w}
\end{aligned}$$

With the introduction of derivative of vector functions

$$\vec{r}(t) =< f(t), g(t) > \quad \text{then} \quad \vec{r}\,'(t) =< f'(t), g'(t) > \tag{6}$$

We have the following derivative rules. Let $\vec{u}$ and $\vec{v}$ be vector functions and $f(t)$ a differentiable scalar function, and $\vec{c}$ a constant vector, then

$$
\begin{aligned}
(i) &\quad \frac{d}{dt}\vec{c} = \vec{0} \\
(ii) &\quad \frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \vec{u}'(t) + \vec{v}'(t) \\
(iii) &\quad \frac{d}{dt}\left(f(t)\vec{u}(t)\right) = f'(t)\vec{u} + f(t)\vec{u}'(t) \\
(iv) &\quad \frac{d}{dt}\left(\vec{u}(f(t))\right) = \vec{u}'(f(t))f'(t) \\
(v) &\quad \frac{d}{dt}\left(\vec{u}(t) \cdot \vec{v}(t)\right) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t) \\
(vi) &\quad \frac{d}{dt}\left(\vec{u}(t) \times \vec{v}(t)\right) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)
\end{aligned}
$$

On thing that's important to realize is that if a vector has constant length (say c) then

$$\vec{u}(t) \cdot \vec{u}(t) = c^2 \tag{7}$$

and using the property (v) above then

$$\vec{u}'(t) \cdot \vec{u}(t) + \vec{u}(t) \cdot \vec{u}'(t) = 0 \tag{8}$$

so that

$$\vec{u}(t) \cdot \vec{u}'(t) = 0 \tag{9}$$

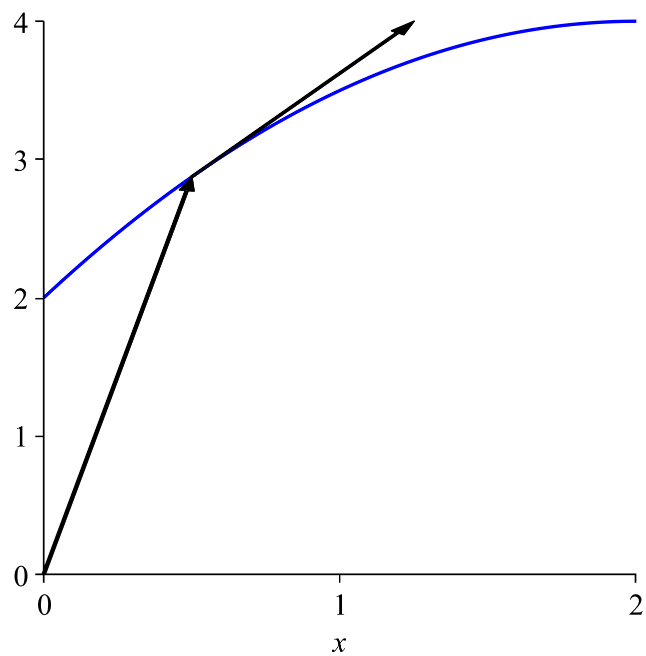
meaning that $\vec{u}(t)$ and $\vec{u}'(t)$ are perpendicular to one another. This is important in what follows.

Unit Tangent Vector

As we saw when we first introduced derivatives, that $\vec{r}'(t)$ is tangent to the space curve given by

$$x = f(t), \quad y = g(t). \tag{10}$$

so we define the unit tangent vector as



$$\vec{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \tag{11}$$

So, for example, if $\vec{r} =< t, \frac{1}{2}t^2 >$ then $\vec{r}' =< 1, t >$ and the unit tangent vector is

$$\vec{T} = \frac{< 1, t >}{\sqrt{1+t^2}}, \tag{12}$$

if $\vec{r} =< \cos t, \sin t, t >$ then $\vec{r}' =< -\sin t, \cos t, 1 >$ and the unit tangent vector is

$$\vec{T} = \frac{< -\sin t, \cos t, 1 >}{\sqrt{2}} \tag{13}$$

since the magnitude of $\vec{r}'(t)$ is $\sqrt{2}$. Since the tangent vector is a unit vector, then the derivative of this would give another vector that is perpendicular to $\vec{T}'$.

Unit Normal Vector

We define the unit normal vector as

$$\vec{N} = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \tag{14}$$

If we consider the examples above then

$$\vec{T}' = \left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle \tag{15}$$

then

$$\|\vec{T}'\| = \sqrt{\left(\frac{-t}{(1+t^2)^{3/2}}\right)^2 + \left(\frac{1}{(1+t^2)^{3/2}}\right)^2} = \frac{1}{1+t^2} \tag{16}$$

and we obtain

$$\vec{N} = \frac{\left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle}{\frac{1}{1+t^2}} = \left\langle \frac{-t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}} \right\rangle \tag{17}$$

and the reader can verify that $|\vec{T}| = 1$, $|\vec{N}| = 1$ and $\vec{T} \cdot \vec{N} = 0$.

In the second example, where

$$\vec{T} = \frac{< -\sin t, \cos t, 1 >}{\sqrt{2}} \tag{18}$$

then

$$\vec{T}' = \frac{< -\cos t, -\sin t, 0 >}{\sqrt{2}} \tag{19}$$

$|\vec{T}'| = \frac{1}{\sqrt{2}}$ and $\vec{N}$ is given by

$$\vec{N} = < -\cos t, -\sin t, 0 > \tag{20}$$

and the reader can also verify that $\|\vec{T}\| = 1$, $\|\vec{N}\| = 1$ and $\vec{T} \cdot \vec{N} = 0$.

Unit Binormal

If we are in 3D, we now have two vectors, the unit tangent vector and the unit normal vector. We can create a third vector which is perpendicular to both. We define the unit Binormal as

$$\vec{B} = \vec{T} \times \vec{N} \tag{21}$$

For the example above where $\vec{r} = < \cos t, \sin t, t >$ then

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{-\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = < \frac{\sin t}{\sqrt{2}}, \frac{-\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$$