Math 1497 - Calc 2

L'Hôpital's

consider $\lim\limits_{x\to 0} \frac{x^2-1}{x-1} = \lim\limits_{x\to 0} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = 2$

but what about more complicated limits

For example

$\lim\limits_{x\to 0} \frac{e^x-1}{x}$, $\lim\limits_{x\to 0^+} x\ln x$

$\lim\limits_{x\to 1} \frac{\ln x}{x-1}$, $\lim\limits_{x\to 0^+} x^x$ etc

How does one evaluate these limits

analytically?

L'Hôpital's Rule (LH)

Suppose $f(x)$ & $g(x)$ are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at $x = a$)

& suppose

$$\lim_{x \to a} f(x) = 0 \quad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \lim_{x \to a} g(x) = \pm\infty$$

ie. form "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ provided the limit exists!

<u>Note</u> a could be ∞

Ex 1 $\lim_{x \to 0} \frac{e^x - 1}{x} = \text{"}\frac{0}{0}\text{"}$ so L'H

$\lim_{x \to 0} \frac{e^x}{1} = e^0 = 1$ ✓

Ex 2 $\lim_{x \to \infty} \frac{2x^3 + 4x}{x^3 + 1} = \text{"}\frac{\infty}{\infty}\text{"}$ so L'H

$\lim_{x \to \infty} \frac{6x^2 + 4}{3x^2} = \text{"}\frac{\infty}{\infty}\text{"}$ so L'H

$\lim_{x \to \infty} \frac{12\cancel{x}}{6\cancel{x}} = \frac{12}{6} = 2$

Could have done

$\lim_{x \to \infty} \frac{2 + \frac{4}{x^2}}{1 + \frac{1}{x^3}} = \frac{2+0}{1+0} = 2$ ✓

Ex3 $\lim_{x \to 1} \frac{\ln x}{x-1} = \text{"}\frac{0}{0}\text{"}$ so LH

$$\lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

Ex4 $\lim_{x \to \infty} \frac{e^x}{x} = \text{"}\frac{\infty}{\infty}\text{"}$ LH

$\lim_{x \to \infty} \frac{e^x}{1} \Rightarrow \infty$ so limit DNE

Ex5 $\lim_{x \to \infty} \frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"}$ so LH

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

Now sometimes the form is

"$\infty - \infty$", "$0 \cdot \infty$", "0^0" "∞^0" "1^∞"

if so we need to recast into the form

$$\text{"}\frac{0}{0}\text{"} \quad \text{or} \quad \text{"}\frac{\infty}{\infty}\text{"}$$

Ex "$\infty - \infty$"

$$\lim_{x \to \pi/2^-} \sec x - \tan x = \text{"}\infty - \infty\text{"}$$

$$\lim_{x \to \pi/2^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \lim_{x \to \pi/2^-} \frac{1 - \sin x}{\cos x} = \text{"}\frac{1-1}{0}\text{"} ✓$$

$$\lim_{x \to \pi/2^-} \frac{-\cos x}{-\sin x} = +\frac{0}{1} = 0$$

Ex "$0 \cdot \infty$"

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \text{"}\frac{-\infty}{\infty}\text{"} \text{ LH}$$

$$\lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -\frac{1}{x} x^2 = -\lim_{x \to 0^+} x = 0$$

Ex "0^0"

$$\lim_{x\to 0^+} x^x$$

Note

$$e^{\ln f} = f$$

$$e^{\ln f^g} = e^{g\ln f}$$

1st

$$e^{\ln x^x}$$

move power

$$e^{x\ln x}$$

previous problem

$$\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1$$

Ex "∞^0"

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

$$\lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x}$$

$$= e^{\lim_{x\to\infty} \frac{\ln x}{x}}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"} \text{ so LH} \quad \lim_{x\to\infty} \frac{1}{x} = 0$$

$$= e^{0} = 1$$

$$\text{"} \frac{\infty}{1} \text{"}$$

<u>Ex</u> $\lim_{x\to\infty^+} \left(1+\frac{1}{x}\right)^x$

$$e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{x \ln\left(1+\frac{1}{x}\right)}$$

so consider $\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) = \text{"}\infty \cdot 0\text{"}$

$$\lim_{x\to\infty} \frac{\ln 1+\frac{1}{x}}{\frac{1}{x}} = \text{“}\frac{0}{0}\text{”} \quad \text{LH}$$

$$\lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}}$$

$$\lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1$$

Note $\lim_{x\to\infty} \frac{1}{x} = 0$

so $e^{\lim x \ln\left(1+\frac{1}{x}\right)} = e^{1}$

so $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x} = e$