Math 4315 - PDEs

Sample Test 1 - Solutions

1. Solve the following first order PDEs by introducing an alternate coordinate system (*i.e.* $(x,y) \to (r,s)$)

$$\begin{aligned} (i) \quad xu_x - yu_y &= 2u, \\ (ii) \quad yu_x - u_y &= 1, \end{aligned}$$

Solutions
(i) If $u_s = u_x\, x_s + u_y\, y_s$ then choosing

$$x_s = x, \quad y_s = -y, \quad \text{gives} \quad u_s = 2u.$$

Solving gives

$$x = a(r)e^s, \quad y = b(r)e^{-s}, \quad u = c(r)e^{2s},$$

where $a(r)$, $b(r)$ and $c(r)$ are arbitrary functions of r. Eliminating s from the first two and first and third gives

$$xy = a(r)b(r) = A(r), \quad \frac{u}{x^2} = c(r)/a(r)^2 = B(r).$$

Noting again that $A(r)$ and $B(r)$ are arbitrary. Further, elimination of r gives

$$\frac{u}{x^2} = f(xy) \quad \text{or} \quad u = x^2\, f(xy).$$

(ii) If $u_s = u_x\, x_s + u_y\, y_s$ then choosing

$$x_s = y, \quad y_s = -1, \quad \text{gives} \quad u_s = 1.$$

Solving the second and third gives

$$y = -s + b(r), \quad u = s + c(r),$$

giving the first as

$$x_s = y = -s + b(r).$$

Integrating gives

$$x = -\frac{(b(r) - s)^2}{2} + a(r),$$

where $a(r)$, $b(r)$ and $c(r)$ are arbitrary functions of r. Eliminating s from x and y gives

$$x + \frac{y^2}{2} = a(r), \quad \text{or} \quad 2x + y^2 = A(r).$$

Eliminating s from y and u gives

$$u + y = c(r) + b(r) = B(r).$$

Noting again that $A(r)$ and $B(r)$ are arbitrary. Further, elimination of r gives

$$u + y = f(2x + y^2) \quad \text{or} \quad u = -y + f(2x + y^2).$$

2. Solve the following using the method of characteristics

$$\begin{aligned} (i) \quad xu_x + (x + 2y)u_y &= u, \quad u(x, 0) = x^2 \\ (ii) \quad xu_x + 2uu_y &= u, \quad u(x, 0) = x^2 \end{aligned}$$

Solutions

(i) The characteristic equations are

$$\frac{dx}{x} = \frac{dy}{x + 2y} = \frac{du}{u}.$$

Solving the first pair, *i.e.*

$$\frac{dx}{x} = \frac{dy}{x + 2y} \quad \text{gives} \quad c_1 = \frac{y}{x^2} + \frac{1}{x}.$$

Solving the first and third, *i.e.*

$$\frac{dx}{x} = \frac{du}{u} \quad \text{gives} \quad c_2 = \frac{u}{x}.$$

The solution is therefore given by

$$\frac{u}{x} = f\left(\frac{y}{x^2} + \frac{1}{x}\right) \quad \text{or} \quad u = x f\left(\frac{y}{x^2} + \frac{1}{x}\right).$$

Imposing the initial condition $u(x,0) = x^2$ gives

$$u(x,0) = x\, f\left(\frac{1}{x}\right) = x^2 \qquad \Longrightarrow \qquad f(x) = \frac{1}{x}.$$

This gives the solution as

$$u = x\, \frac{1}{\dfrac{y}{x^2} + \dfrac{1}{x}} = \frac{x^3}{x+y}.$$

(ii) The characteristic equations are

$$\frac{dx}{x} = \frac{dy}{2u} = \frac{du}{u}.$$

Solving the first and third, *i.e.*

$$\frac{dx}{x} = \frac{du}{u} \qquad \text{gives} \qquad \ln|x| = \ln|u| - ln|c_2| \qquad \text{or} \qquad \frac{u}{x} = c_2.$$

Solving the second and third, *i.e.*

$$\frac{dy}{2u} = \frac{du}{u} \qquad \text{gives} \qquad \frac{y}{2} = u - c_1 \qquad \text{or} \qquad c_1 = u - \frac{y}{2}.$$

The solution is therefore given by

$$\frac{u}{x} = f(u - \frac{y}{2}) \qquad \text{or} \qquad u = x\, f(u - \frac{y}{2}).$$

Imposing the initial condition $u(x,0) = x^2$ gives

$$x^2 = x\, f(x^2 - 0) \quad \Longrightarrow \quad f(x^2) = x \quad \Longrightarrow \quad f(x) = \sqrt{x}.$$

This gives the solution as

$$u = x\sqrt{u - \frac{y}{2}}.$$

3. Solve the following nonlinear PDE

$$\begin{aligned} (i) \quad xu_x^2 + u_y &= 1, \quad u(x,1) = x+1. \\ (ii) \quad u_x u_y - 2xu_x - 2yu_y &= 0, \quad u(x,0) = x^2 \end{aligned}$$

Solution
(i) If $F = xp^2 + q - 1$ then the characteristic equations are

$$x_s = F_p = 2xp \tag{1.4a}$$
$$y_s = F_q = 1 \tag{1.4b}$$
$$u_s = pF_p + qF_q = 2xp^2 + q \tag{1.4c}$$
$$p_s = -(F_x + pF_u) = -p^2 \tag{1.4d}$$
$$q_s = -(F_y + qF_u) = 0. \tag{1.4e}$$

To these we associate the following initial condition. When $s = 0$, then

$$y = 1, \quad x = r, \quad u = r + 1, \quad p = 1, \quad q = 1 - r, \tag{1.5}$$

where p is obtained from differentiating the initial condition and q from the original equation.

From (1.4d) and (1.4e) we find that

$$\frac{1}{p} = s + A(r), \quad q = B(r).$$

From the boundary conditions (1.5) we find that $A = 1$ and $B = 1 - 4$ so

$$\frac{1}{p} = s + 1, \quad q = 1 - r.$$

From (1.4a) and (1.4a) we integrate giving

$$x = C(r)(s + 1)^2, \quad y = s + D(r).$$

The boundary conditions (1.5) gives $A = r$ and $D = 1$. Thus,

$$x = r(s + 1)^2, \quad y = s + 1. \tag{1.6}$$

From (1.4c) we have

$$u_s = 2xp^2 + q = r + 1 \quad \Rightarrow$$
$$u = (r + 1)s + E(r) = (r + 1)s + r + 1. \tag{1.7}$$

Eliminate r and s from (1.6) and (1.7) gives the solution

$$u = y + \frac{x}{y}.$$

(ii) If $F = pq - 2xp - 2yq$ then the characteristic equations are

$$\begin{aligned} x_s &= F_p = q - 2x && (1.8a) \\ y_s &= F_q = p - 2y && (1.8b) \\ u_s &= pF_p + qF_q = 2pq - 2xp - 2yq = pq && (1.8c) \\ p_s &= -(F_x + pF_u) = 2p && (1.8d) \\ q_s &= -(F_y + qF_u) = 2q. && (1.8e) \end{aligned}$$

To these we associate the following initial condition. When $s = 0$, then

$$y = 0, \quad x = r, \quad u = r^2, \quad p = 2r, \quad q = 2r, \tag{1.9}$$

where p is obtained from differentiating the initial condition and q from the original equation. As we need p and q to find x, y and u, we focus on these first. Solving (1.8d) and (1.8e) for p and q gives

$$p = a(r)e^{2s}, \quad q = b(r)e^{2s},$$

and imposing the initial condition gives $a(r) = 2r$ and $b(r) = 2r$. Thus,

$$p = 2re^{2s}, \quad q = 2re^{2s}. \tag{1.10}$$

From (1.8c) we see that

$$u_s = 4r^2e^{4s},$$

which integrates giving

$$u = r^2e^{4s} + c(r),$$

and the initial condition here gives $c(r) = 0$. Substituting (1.10) into (1.8a) and (1.8b) gives

$$x_s + 2x - 2re^{2s} = 0, \quad y_s + 2y - 2re^{2s} = 0.$$

Solving yields

$$x = \frac{r}{2}\, e^{2s} + d(r)\, e^{-2s}, \quad y = \frac{r}{2}\, e^{2s} + e(r)\, e^{-2s}.$$

Applying the remaining initial conditions gives

$$r = \frac{r}{2}\, e^{0} + d(r)\, e^{0}, \quad 0 = \frac{r}{2}\, e^{0} + e(r)\, e^{0},$$

showing that $d(r) = \frac{r}{2}$ and $e(r) = -\frac{r}{2}$. Thus, we have the following parametric solutions

$$x = \frac{r}{2}\left(e^{2s} + e^{-2s}\right), \quad y = \frac{r}{2}\left(e^{2s} - e^{-2s}\right), \quad u = r^2e^{4s}.$$

Eliminating r and s gives

$$u = (x + y)^2.$$