

Infinite Series

Special Series

#1 Geometric

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots \quad (1)$$

converges if $|r| < 1$ and if so, $S_{\infty} = \frac{a}{1-r}$.

#2 Harmonic

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \quad (2)$$

The series diverges.

#3 Telescopic

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - b_{n+1} \quad (3)$$

- (i) stop the series at N ,
- (ii) write out the terms and cancel, and
- (iii) $N \rightarrow \infty$

#4 p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{ll} p < 1 & \text{diverges} \\ p > 1 & \text{converges} \end{array} \quad (4)$$

Tests

#1 n^{th} Term Test

$$\text{If } \lim_{n \rightarrow \infty} a_n = \# \text{ (not zero)} \quad (5)$$

the series diverges.

#2 Integral Test $a_n = f(n)$

First check the conditions:

- (i) $f(x) \geq 0$,
- (ii) $f(x)$ is continuous, and
- (iii) $f(x)$ is decreasing

If

$$\int_1^{\infty} f(x) dx \text{ converges (diverges)} \quad \text{then} \quad \sum_1^{\infty} a_n \text{ converges (diverges)}. \quad (6)$$

#3 Limit Term Test (LCT) If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \# \text{ (not zero)} \quad (7)$$

both series do the same, either converge or diverge.

#4 Direct Comparison Term Test (DCT)

Suppose $0 \leq a_n \leq b_n$. If

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \text{ diverges} & \text{ then } \sum_{n=1}^{\infty} b_n \text{ diverges} \\ \sum_{n=1}^{\infty} b_n \text{ converges} & \text{ then } \sum_{n=1}^{\infty} a_n \text{ converges} \end{aligned} \quad (8)$$

#5 Ratio Test

$$\text{If } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L. \quad \text{If } \begin{array}{ll} L > 1 & \text{diverges} \\ L < 1 & \text{converges} \\ L = 1 & \text{no conclusion} \end{array} \quad (9)$$

#6 Root Test

$$\text{If } \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L. \quad \text{If } \begin{array}{ll} L > 1 & \text{diverges} \\ L < 1 & \text{converges} \\ L = 1 & \text{no conclusion} \end{array} \quad (10)$$

#7 Alternating Series Term Test (AST)

For series of the form

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n \quad a_n \geq 0 \quad (11)$$

If

$$\begin{aligned} (i) \quad & a_{n+1} < a_n \\ (ii) \quad & \lim_{n \rightarrow \infty} a_n = 0 \end{aligned} \quad (12)$$

the series converges.

Absolute and Conditional Convergence ($a_n \geq 0$)

If $\sum_{n=1}^{\infty} (-1)^n a_n$ converges and $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges conditionally.

If $\sum_{n=1}^{\infty} (-1)^n a_n$ converges and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely.