Research Article

Norms of Normally Represented Elementary Operators

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Abstract

The norm problem involves finding the formula describing the norm from the coefficients of the elementary operators. Upper estimate of the norm has been easy to find but estimating the norm from below has been proven difficult in general. In this study, we considered a special type of elementary operators called normally represented elementary operators. Some of our results show that the norm of an elementary operator is equal to the largest singular value of the operator itself i.e. $S_M = \|M\|$ and also if $U_{A,B} = A \otimes h B + B \otimes h A$ is normally represented, then $\|U_{A,B}\|_{lin} \geq 2(\sqrt{2} - 1)\|A\|\|B\|$.

Keywords: Norms; Elementary operator; Normally represented elementary operator; Norm-attainable operators.

Introduction

The structural properties of the elementary operators have been of great concern in analysis mathematics [1]. Several of them have been studied and of the most interesting concern is the norm property. The term elementary operator came as a result of the knowledge of the basic elementary operators from an algebra [2]. If $A$ is an algebra, then given $a, b \in A$, we define the basic elementary operator (implemented by $A$, $B$) by: $M_{A,B}(H) = AXB$, $\forall X \in B(H)$. This led to the form describing the elementary operators as the sum of basic elementary operators.

For the lower estimates, Mathieu proved that for $\|M\|$, $M_{a,b}$ is the operator on $L^r(E)$ defined by $MA(B(T) = ATB$, then $\|M_{a,b}\| \geq \|A\| \|B\| > 0$ but that there is no real $\alpha > 0$ such that $\|M_{a,b}\| \geq \alpha \|A\| \|B\|$. The objectives of this study were to determine the norm inequalities for normally represented elementary operators. Both the lower norm bounds and upper norm bounds have been established for normally represented elementary operators. The results show that the norm of an elementary operator is equal to the largest singular value of the operator itself i.e. $S(M) = \|M\|$ and also if $U_{A,B} = A \otimes h B + B \otimes h A$ is normally represented, then $\|U_{A,B}\|_{lin} \geq 2(\sqrt{2} - 1)\|A\|\|B\|$.

Research Methodology

In the present study, some definitions and known results used are shown below.

Definition 1.1. ([8], Definition 1.2.1) Field. A field $F$ is a set closed under two binary operations $+$ and $\cdot$.
operations of addition and scalar multiplication satisfying the following properties:
(i). Closure under addition and multiplication. \( a + b \in F \) and \( a.b \in F \), \( \forall \ a, b \in F \).
(ii). Associativity: \( a + (b + c) = (a + b) + c \), \( \forall \ a, b, c \in F \).
(iii). Commutativity: \( a + b = b + a \) and \( (a.b).c = (b.c).a \), \( \forall \ a, b, c \in F \).
(iv). Additive and multiplicative identities: \( \exists \ a \in F : a + (-a) = 0 \) and \( \exists \ a^{-1} \in F : a.a^{-1} = 1 \).
(v). Distributivity: \( (a + b).c = (a.b) + (b.c) \forall \ a, b, c \in F \).
(vi). Existence of additive inverse: \( \exists \ a \in F \exists \ x \in K : a + x = 0 \), and \( x + a = 0 \) then \( a = -x \forall a, x \in F \).
(vii). Existence of multiplicative inverses: For each \( a \in F \) with \( 0 < a > 0 \) the equations \( a.x = 1 \) and \( x.a = 1 \) have a solution \( x \in F \), called the multiplicative inverse of \( a \) and denoted by \( a^{-1} \).

Definition 1.2. ([9], Definition 1.2.2) Vector space. Let \( F \) be a field and \( V \) a collection of objects called vectors, then \( V \) is a vector space over a field \( F \), if \( V \) is closed under vector addition and scalar multiplication, i.e. \( \forall v_1, v_2 \in V, v_1 + v_2 \in V \) and \( \forall a \in F, a.v \in V \), and satisfies the following properties:
(i). Commutativity. \( v_1 + v_2 = v_2 + v_1 \forall v_1, v_2 \in V \).
(ii). Associativity. \( v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3, \forall v_1, v_2, v_3 \in V \).
(iii). Additive inverse. \( \forall v \in V, \exists -v \in V : v + -v = 0 \forall v_1, -v \in V \).
(iv). Additive Identity. \( \forall v \in V, \exists 0 \in V : v + 0 = v, \forall v \in V \).
(v). Multiplicative Identity. \( 1.v = v \forall v \in V \).
(vi). Distributive property. \( \forall a \in F, a(v_1 + v_2) = (av_1 + av_2) \forall v_1, v_2 \in V \).
(vii). Unitary law. \( \forall v \in V, 1.v = v \).

Definition 1.3. ([10], Definition 2.1.8) Banach space. This is a complete normed space.

Definition 1.4. ([11], Definition 2.1.7) Hilbert space. A Hilbert space is a complete inner product space.

Definition 1.5 ([12], Definition 2.1.10) \( C^* \) algebra.

A complex Banach \( * \) algebra \( A \) is called a \( C^* \) algebra if \( \|xx^*\| = \|x\|^2 \forall x \in A \).

Definition 1.6. ([13], Definition 1.19) Spectrum. Let \( A \) be a unital Banach algebra, then the spectrum of \( a \in A \) is given by \( \sigma(a) = \{ \lambda \in C : a - \lambda I \text{ is not invertible} \} \).

Definition 1.7. ([14], Definition 2.1.1) Norm. A norm is a non-negative real valued function that takes the elements of a vector space to a field of real numbers denoted by \( \|\| : V \to R \) satisfying the following conditions:
(i.) Non-negativity: \( \|x\| \geq 0, \forall x \in V \).
(ii.) Zero property: \( \|x\| = 0 \), if and only if \( x = 0 \), for all \( x \in V \).
(iii.) Homogeneity: \( \|a.x\| = |a|\|x\|, \forall x \in V \) and \( a \in F \).
(iv.) Triangle inequality: \( \|x + y\| \leq \|x\| + \|y\| \forall x, y \in V \).
The pair \( (V, \|\|) \) is called a normed linear space.

Definition 1.8. ([15]). Elementary Operator. Let \( H \) be an infinite dimensional complex Hilbert space and \( B(H) \) be an algebra of all bounded linear operators on the \( H \). We define an elementary operator \( T : B(H) \to B(H) \) by \( T_a, b(X) = \sum_{i=1}^{n} A_i \times B_i \), \( \forall X \in B(H) \) and \( \forall A_i, B_i \) fixed in \( B(H) \) where \( i = 1, \ldots, n \). Examples of elementary operators include:
(i). The left multiplication operator \( L_A : B(H) \) by: \( L_A(x) = A.x, \forall x \in B(H) \).
(ii). The right multiplication operator \( R_B : B(H) \) by: \( R_B(X) = B.X, \forall X \in B(H) \).
(iii). The Basic elementary operator (implemented by \( A, B \)) by: \( M_{A, B}(X) = AXB, \forall X \in B(H) \).
(iv). The Jordan elementary operator (implemented by \( A, B \)) by: \( U_{A, B}(X) = AXB + BXA, \forall X \in B(H) \).
(v). The Generalized derivation (implemented by \( A, B \)) by: \( \delta_{A, B} = L_A - R_B \).
(vi). The inner derivation (implemented by \( A, B \)) by: \( \delta_A = AX - XA \).

Definition 1.9. ([16]). Normally represented elementary operator. Let \( H \) be an infinite dimensional complex Hilbert space and \( B(H) \) be the algebra of all bounded linear operators on \( H \). We define an elementary operator, \( T : B(H) \to B(H) \) by \( T_{A_i, B_i}(X) = \sum_{i=1}^{n} A_i \times B_i \), \( \forall X \in B(H) \) and \( \forall A_i, B_i \) fixed in \( B(H) \) where \( i = 1, \ldots, n \). From this operator, we can define the generalized adjoint by \( T_{A_i, B_i}(X) = \sum_{i=1}^{n} A_i^* \times B_i^* \) and we say that \( T \) is normal if and only if \( T^*T = T^*T \). Now \( AC = CA, BD = DB, \) together with \( AA^* = A*A, BB^* = B*B, CC^* = C*C \) and \( DD^* = D*D \) ensures that the operator \( T_{A_i, B_i}(X) = AXC + BXD \) is normal. Therefore, the
elementary operator of the form: \( T_{A_i,B_i}(X) =\sum_{i=1}^{n} A_i X B_i \) where \( A_i \) and \( B_i \) are commuting families of normal operators are called normally represented elementary operator.

**Results and Discussions**

The norms of normally represented elementary operators are determined and discussed.

**Proposition 4.13.** Let \( H \) be a complex Hilbert space and \( M: B(H) \rightarrow B(H) \) be a basic elementary operator. Then \( S(M) = \| M \| \). Such that \( S(M) \) are the singular values of \( M \).

**Proof.** Since \( M \) is an operator, we represent the spectral norm on \( M \) as \( \| M \| \) i.e. \( \| M \|_2 \) then by definition of a singular value, is known that the spectral norm is equal to the largest singular value of the operator. i.e. Since \( M \) is an operator, then \( \| M \|_2 = \sqrt{\lambda_{\max}(M^*M)} = \sigma_{\max}(M) \). Indeed the spectral norm \( \| M \|_2 \) of the complex matrix of \( M \) is defined by \( \max\{ \| M \|_2 : \| x \| = 1 \} \) then also let there be a linear transformation of the Euclidean vector space \( E \). \( E \) is hermite if there exist an orthonormal basis of \( E \) consisting of all eigenvectors of \( E \) [17]. So, suppose that \( E = M^*M \) i.e Hermitian matrix, Let \( \lambda_1, \ldots, \lambda_n \) be eigenvalues of \( E \) and \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( E \) then \( x = a_1 e_1 + \ldots + a_n e_n \), we have that:

\[
\| x \| = \sqrt{\sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} a_i^2 e_i}\]
and
\[
\| E \| = \| x \| (x, M^*Mx) = \langle x, E x \rangle = \sum_{i=1}^{n} a_i \lambda_i e_i = \sum_{i=1}^{n} \lambda_i \| e_i \|.
\]

We write \( \lambda_0 \) to be the largest eigenvalue of \( E \). Therefore:

\[
\| M \| = \langle Mx, Mx \rangle = \langle x, M^*Mx \rangle = \langle x, E x \rangle = \sum_{i=1}^{n} a_i \lambda_i e_i = \sum_{i=1}^{n} \lambda_i a_i e_i = \sqrt{\sum_{i=1}^{n} a_i^2 e_i} = \sqrt{\lambda_{\max}(M^*M)} = \| M \|_2
\]

\[
\leq \max_{1 \leq j \leq n} \sqrt{\lambda_j} |x| = \| M \|_2 \text{ (4.3.2)}.
\]

Consider also, \( x_0 = e_0 \Rightarrow \| x_0 \| = 0 \) so that \( \| M \| \leq \| e_0 \| \). According to [4.3.1] and [4.3.2] gives

\[
\| M \| = \max_{1 \leq j \leq n} \sqrt{\lambda_j} | \lambda_j \| \quad \text{where } \lambda_j \text{ is the eigenvalue of } E = M^*M \text{. By [18], we therefore conclude that } \lambda_{\max}(M^*M) = \sigma(M). \text{ Since } \lambda_j = \max_{1 \leq j \leq n} \sqrt{\lambda_j} \text{ where } \lambda_j \text{ is the eigenvalue i.e. } \lambda_j = S(M) \text{ then we have that } S(M) = \| M \|.
\]

**Theorem 4.14.** Let \( U_{A,B}(X) = AXB + BXA \) be normally represented then, \( \| U_{A,B} \|_{CB} \geq \| A \| \| B \| \) for \( A, B \in B(H) \).

**Proof.** Let \( \text{diam } H = 2 \) and choose \( W_2 \in B(H) \). Let also \( \_A = [A, B], \_B = [B, A] \), then we shall use the notation \( \_A \circ \_B = A \otimes B + B \otimes A \). It is necessary to show that the Haangerup norm of \( \_A \circ \_B \) holds for \( \| \_A \circ \_B \|_H \geq \| A \| \| B \| \). Multiplying \( A \) by a scalar \( \lambda \) and \( B \) by \( \lambda \), let \( \| \_A \| = \| B \| \) and then \( \| A \| = 1 = \| B \| \). Is known that for each invertible matrix \( D \in W_2 \); it follows that \( \_A D^-1 \_B = \_A \circ \_B \) then we have that \( \| \_A \| = \| \_B \| \) and similarly for columns, by using the polar decomposition, it suffices that the infimum over all positive matrices \( A \) only and clearly we may also let that \( \det D = 1 \). We therefore show that \( \| \_A D^-1 \| \| \_B \| \geq 1 \) for all positive \( D \in W_2 \) with \( \det D = 1 \). We let \( D = \begin{bmatrix} \alpha & \beta \\ -\beta & \gamma \end{bmatrix}, \alpha, \gamma \geq 0 \) then

\[
\text{AD}^-1 \| \_B DB = (\gamma A - \beta B) \otimes (\beta A + \alpha B) \otimes (\gamma A + \beta B).
\]

This can be reduced by putting \( X = \| A \|^2 + \gamma^2, \beta = \gamma A + \beta B \), so we can write \( \| XX A^* - 2Re(Y AB^*) + ZBB^* = 2Re(Y AA^* + 2Re(Y AB^*) + ZBB^* \geq 1 \) assuming that \( \chi \geq 2 \), then noting that \( \| U_{A,B} \| = \| U_{A,B} \|_{CB} \) for all unitaries \( u, v \in W_2 \) we can write \( A \) by \( |A| \) and \( B \) by \( vB \), where \( A = u|A| \) is a polar decomposition of \( A \) i.e. we let \( A \) to be positive, so that \( A \) and \( B \) are of the same form \( [1, 0] \) and \( B = [\beta_1, \beta_2, \beta_3, \beta_4] \) where \( h = [0, 1] \) and \( \beta_1, \beta_2, \beta_3, \beta_4 \in C \) then

\[
\text{XAX}^* - 2Re(Y AB^*) + ZBB^* = \begin{cases} A & \text{if } \det D = 1 \\
A - 2Re(Y \beta_1) + Z (\| \beta_1 \| + | \| \beta_2 \| |) & \text{otherwise}
\end{cases}
\]

\[
\| XX A^* - 2Re(Y AB^*) + ZBB^* = \begin{cases} A & \text{if } \det D = 1 \\
A + 2Re(Y \beta_1) + Z (| \| \beta_1 \| | + | \| \beta_2 \| |) & \text{otherwise}
\end{cases}
\]

\[
\| XX A^* - 2Re(Y AB^*) + ZBB^* = \begin{cases} A & \text{if } \det D = 1 \\
A + 2Re(Y \beta_1) + Z (| \| \beta_1 \| | + | \| \beta_2 \| |) & \text{otherwise}
\end{cases}
\]

\[
\| XX A^* - 2Re(Y AB^*) + ZBB^* = \begin{cases} A & \text{if } \det D = 1 \\
A + 2Re(Y \beta_1) + Z (| \| \beta_1 \| | + | \| \beta_2 \| |) & \text{otherwise}
\end{cases}
\]
Suppose that $\dim H > 2$, then it follows that, by choosing some $\delta > 0$, and a unit vector $\eta$ and $\xi \in H$ such that $\lVert A\xi \rVert \geq \lVert A \rVert - \delta$ and $\lVert B\eta \rVert \geq \lVert B \rVert - \delta$. Let $N$ be a two dimensional space containing $\xi$ and $\eta$ and let $p \in B(H)$ be partial symmetry with final space $N_1$ and initial space $N_2$, then $\lVert pAq \rVert \geq \lVert A \rVert - \delta$ and $\lVert pBq \rVert \geq \lVert B \rVert - \delta$. It becomes to verify that $\lVert U_{A,B} \rVert \geq \lVert pAq \rVert \geq \lVert A \rVert \geq \lVert B \rVert \Rightarrow \lVert U_{A,B} \rVert \geq \lVert A \rVert \lVert B \rVert$.

**Theorem 4.15.** Let $A, B \in B(H)$ and $U_{A,B} = A \otimes h B + B \otimes h A$ be normally represented then $\lVert U_{A,B} \rVert \geq 2(\sqrt{2} - 1) \lVert A \rVert \lVert B \rVert$.

**Proof.** Let $\lVert A \rVert = \lVert B \rVert = 1$ and $A, B$ be functions on $D := (B(H)^*)$, and $U_{A,B}$ as a function on $D \times D$. Taking dot products of $A$ and $B$ using a suitable scalars of modulus 1, we let $A(x_0) = 1$ and $B(y_0) \forall x_0, y_0 \in D$. Putting $A_1 = A(x_0)$, and $B_1 = B(y_0)$. Then it gives $U_{A,B}(x_0, y_0) = 2B_1, U_{A,B}(y_0, x_0) = 2A, U_{A,B}(x_0, y_0) = 1 + A_1B_1$ if $\lVert A \rVert \ lA \rVert \lVert B \rVert \geq \sqrt{2} - 1$. This completes the proof. On the other hand, if suppose that $\lVert A_1 \rVert < \sqrt{2} - 1$ and $\lVert B \rVert < \sqrt{2} - 1$ then,

$$\lVert A_1B_1 \rVert > (\sqrt{2} - 1) = 2(\sqrt{2} - 1) \lVert A \rVert \lVert B \rVert.$$ 

**Corollary 4.16.** Let $R = \sum_{i=1}^{n} A_i \otimes B_i \in B(H)$ then we have $\lVert R \rVert_{\sup} = \sup\lVert A_i \rVert \rVert B_i \rVert : X \in \lVert B(H) \rVert \lVert X \rVert = 1$ if and only if $X$ is rank one Operator.

**Proof.** Let $R(X) = \sum_{i=1}^{n} A_i \times B_i$ and $\lVert R \rVert = \sup\{\lVert R x \rVert : X \in \lVert B(H) \rVert \lVert X \rVert = 1, \text{ and, rank } X \rVert = 1\}$. It is known [70] that every rank one operator $X \in B(H)$ is of the form $x = v \otimes \bar{v} \eta$ for all $v, \eta \in \lVert H \rVert$ then this gives $\lVert R \rVert = \sup\{\sum_{i=1}^{n} \langle A_i \times B_i \rangle \eta \} : \lVert x \rVert = 1, \text{ rank } (X \rVert = 1, \lVert x \rVert = 1 \Rightarrow \lVert R \rVert \leq 1\}$

$$= \sup\{\sum_{i=1}^{n} \langle A_i \rangle \eta \} \lVert x \rVert = \sum_{i=1}^{n} \langle A_i \rangle \eta$$

Taking the last supremum all over all functionals of the form $f = V \otimes \bar{v} \eta$, $g = \xi \otimes \bar{\xi} \eta$. For all elements in the product $U(H)$ of $B(H)$ is a norm limit of convex combinations of elements of the form $V \otimes \bar{v} \eta$ and the unit ball $U(H)$ is a weak dense in the unit ball of dual of $B(H)$. This implies that $\lVert R \rVert = \lVert R \rVert_{\sup}$.

**Proposition 4.17.** Let $A, B \in B(H)$ where $H$ is infinite dimensional, then $r(A^*A + B^*B) = \inf_{x \in \partial \sigma(A^*A + B^*B)}$ if and only if there exists $\eta \in H$, $\lVert \eta \rVert = 1$ such that $\lVert A\eta \rVert = \lVert B\eta \rVert = 2\|B\eta\|_2 = 2r(A + B)$.

**Proof.** By letting $r(A^*A + B^*B) = 1$, if the second part is satisfied, then $(\lVert A \rVert + \lVert B \rVert)\eta \in \lVert \eta \rVert = \lVert B\eta \rVert^2 = 12(1 + 11) \geq 1$. On the other hand, if the first part holds, put $k = Ker(A^*A + B^*B) - 1$ and $\rho(B) \lVert A \rVert \lVert B \rVert$ be orthogonal projection onto $k$. Put $y = 1$, hence such that $n > 2$ then by the first part, there exists a sequence $(x_n)$ of the unit vector $x$ in $H$ such that $\langle (1 - y_n)A + A + 11y_n(B^*B)x_n, x_n \rangle > r(A^*A + B^*B)$. For all $n \geq 2$, let $q = (A^*A + B^*B)$ and $z = (B^*B - A^*A)$ then, we have $\langle q x_n, x_n > \rangle y_n(1 - y_n)B^*Bx_n, x_n \rangle \geq 0$. Dividing through by $y_n$, we have that $\langle x_n, y_n > \rangle + y_n(1 - y_n)B^*Bx_n, x_n \rangle \geq 0$. If $n \rightarrow \infty$, we conclude that $\langle x_n, y_n > \rangle \geq 0$. Likewise, from the sequence $\langle x_n, y_n > \rangle \geq 0$. From (iv), so we have that $\langle q x_n, x_n \rangle \geq 0$. Hence $q x = x$ since $\lVert q \rVert = 1$ and for $x \in k$. From (iv), we have that $\lVert B\eta \rVert^2 = 2\|B\eta\|_2^2 = \frac{1}{2}r(A + B)$. 

**Theorem 4.18.** Let $A_{U,B} = B(H) \rightarrow B(H)$ be defined by $A_{U,B} = A + B^*B$ be normally represented, then $\lVert A_{U,B} \rVert = \lVert A \rVert + \|B^*B\|$. 

**Proof.** Let $\|A^* + B^*B\| = 1$ then since $U_{A,B} = U 1\|A\| + \|B^*B\| = 0$ then we have that $\|A^* + B^*B\| = 1$ if $H$ is infinite dimensional, then by the proposition [4.13], there exist a unit vector $\eta$ satisfying $\lVert A\eta \rVert^2 = \|B\eta\|^2 = \frac{1}{2}$ meaning that the linear span $L$ of $\{A\eta, B\eta\}$ can be defined by the adjoint of $X$ by $AX\eta = B\eta$ and $B\eta = A\eta$ then the adjoint $X^*$, we can extend $X$ to $X^*$ as an operator on $H$ such that $X + X^*$ and $X^2 = 1$ then $\|A^*A + B^*B\eta\| = 1$ hence $U_{A,B} \geq 1$. But $\|U_{A,B} \| \geq \|U_{A,B} \rVert \rVert \leq \|A^* + B^*B\| = 1$. This completes the proof when $H$ is finite dimensional. Suppose that $H$ is infinite.
dimensional, then let \( \{ s_n \} \) be a net of finite rank orthogonal projections increasing to identity, we denote by the restrictions of \( s_n A \) to the range of \( s_n \) and analogously for \( B \) for each \( n \), let \( \text{In} \) be such that
\[
\inf_{l>0} \| l_n A^* s_n A + \frac{1}{l} B^* s_n B \| = \| l_n A^* s_n A + \frac{1}{l} B^* s_n B \| \text{ so we have; }
\]
\[
\| U_{A,B} \| = \sup_{l>0} \| l_n A^* s_n A + \frac{1}{l} B^* s_n B \| \geq \sup_{l>0} \| l_n A^* s_n A + \frac{1}{l} B^* s_n B \| \rightarrow l_n \rightarrow 0, \text{then}
\]
\[
\lim_n \| l_n A^* s_n A + \frac{1}{l} B^* s_n B \| = \| l_0 A^* A + \frac{1}{l} B^* B \| \geq \| l_0 A^* A \| + \frac{1}{l} \| B^* B \| . \]

Hence \( \| U_{A,B} \| \geq \inf_{l>0} \| l_0 A^* A \| + \frac{1}{l} \| B^* B \| . \)

**Proposition 4.19.** Let \( U_{A,B} : B(H) \rightarrow B(H) \) be defined by \( U_{A,B} = A^* X B + X B^* X A \) be normally represented, and that \( U_{A,B} \) is real and linear, then \( \| U_{A,B} \| \geq 1/2 \| l>0 \| \| A^* A \| + \frac{1}{l} \| B^* B \| . \)

**Proof.** From the theorem above, let \( \eta \) be a unit vector and \( X \) a unitary operator such that \( X B \eta = \eta \), then \( \| U_{A,B} \| \geq \| ( l_0 A^* X B + \frac{1}{l} B^* X A ) \eta \| \). \( \eta \) is an eigenvector of the real symmetric matrix \( A^2 + B^2 \eta = \eta \) and \( \| A \| ^2 \| B \| ^2 \geq \frac{1}{2} l \). Since \( A^2 + B^2 \) is real, then \( \| A \| ^2 \| B \| ^2 \geq \frac{1}{2} l \) and \( \| A \| ^2 \| B \| ^2 = \frac{1}{2} ( l \) if \( A \) and \( B \) are normal operators.

**Corollary 4.20.** Let \( A, B \in B(H) \) be self adjoint and normal, then for \( U_{A,B} = A X B + B X A \) we have that \( \| U_{A,B} \| \geq \| A \| \| B \| \).

**Example 4.23.** ([4] Example 4.5) Put \( A = e - iu \) and \( B = (e + iu)/2 \) where \( e = (\frac{1}{0}, \frac{0}{1}) \). We have \( \| U_{A,B} \| = \| A \| \| B \| \) so that \( \| U_{A,B} \| = 1 \) we need to show that \( \| U_{A,B} \| = \sqrt{2} \). We note that \( U_{A,B}(X) = AXB + BXA = eXe + uXu \). Expressing \( w \) by Hangean norm, \( w = \| eXe + uXu \| \). It suffices to consider the representation of \( w \) of the form \( w = \| e - \beta u \| \), \( \beta \) a real number.

**Theorem 4.24.** Let \( A_i \) and \( B_i \) be commuting operators in \( B(H) \), then
\[
\inf_{l>0} \inf_{l>0} \| A_i \| \| B_i \| \geq \| \sigma(A_i) \circ \sigma(B_i) \| \text{ if and only if } A_i \text{ and } B_i \text{ are normal operators.}
Proof. Let \((\varphi, \psi) \in \Gamma A \times \Gamma B\), where \(\Gamma\) is the set of all multiplicative linear functionals on commutative algebras in \(B(H)\) then is clear that we can extend \(\varphi\) and \(\psi\) to unit functionals \(f\) and \(g\) on \(B(H)\) respectively using the Hahn-Banach theorem. This follows that

\[
\|U_{A,A^{-1}}\| \geq \sum_{i=1}^{n} f(A_i)g(B_i) = |\sum_{i=1}^{n} \varphi(A_i)\psi(B_i)|.
\]

Therefore \(\sum_{i=1}^{n} A_i \otimes B_i \|_{\text{Inj}} \geq |\sigma(A_i) \circ \sigma(B_i)|\). Since \(A_i\) and \(B_i\) are commuting operators in \(B(H)\), it suffices that \(|\sigma(A_i) \circ \sigma(B_i)| \geq \|U_{A,B}\|\). Because \(|\sigma(A_i) \circ \sigma(B_i)| \geq |\varphi(A_i)\psi(B_i)|\) and \(\phi(\sum_{i=1}^{n} \varphi(A_i)B_i)\) is normal for all \((\varphi, \psi) \in \Gamma A \times \Gamma B\) then \(|\sigma(A_i) \circ \sigma(B_i)| \geq \sup_{\psi \in \Gamma} \|\varphi(\sum_{i=1}^{n} \psi(A_i)B_i)\|\). This implies that \(|\sigma(A_i) \circ \sigma(B_i)| \geq |\varphi(\sum_{i=1}^{n} \varphi(A_i)f(B_i))| = \|\varphi(\sum_{i=1}^{n} f(A_i)B_i)\|\) for all \((\varphi, \psi) \in \Gamma A \times \Gamma B\) and \(f \in S1\) then we have that \(|\sigma(A_i) \circ \sigma(B_i)| \geq \|U_{A,B}\|\).

Theorem 4.25. Let \(A\) be invertible posinormal in \(B(H)\) and normally represented, then for \(U_{A,A^{-1}} = A \otimes A^{-1} + A^{-1} \otimes A\) we have

(i) \(\|A \otimes A^{-1} + A^{-1} \otimes A\|_{\text{Inj}} = \|A\|\|A^{-1}\| + \frac{1}{\|A\|\|A^{-1}\|}\)

(ii) \(\|A^* \otimes A^{-1} + A^{-1} \otimes A^*\|_{\text{Inj}} = \|A\|\|A^{-1}\| + \frac{1}{\|A\|\|A^{-1}\|}\) if and only if \(A \in J(H)\) where \(J(H)\) is the set of all invertible operators in \(B(H)\).

Proof. From ([9]), it follows clearly that \(\|\Phi A\| = \sup_{p \in S} \epsilon(M_A)(p^{1/2}p).\) Also that \(\min \|\Phi A\| = 1\|A^{-1}\|\) and \(\|\Phi A\| = \|A\|\), because \(\delta(M_A) = 0|A^{-1}|\) then \(\min \|\Phi A\| = 1\|A^{-1}\| = A \otimes A^{-1}\) and \(\|\Phi A\| = 1\|A\|\|A^{-1}\|\) then have that Max \(\{p \leq 1\} = A + \frac{1}{A}\). This maximum is attainable in \(A\) and \(\frac{1}{A}\) thus the result follows clearly since it is true that \(A \in S\) for \((\varphi, \psi) \in \Gamma A \times \Gamma B\). Since \(A \in J(H)\), there exist \(V \in \Gamma (H)\) such that \(A = V p|p = |A|\). By ([72] corollary 1), we have that \(\{x \in B(H) : \text{rank } x = 1\} = \{V^*x : x \in B(H)\}, \text{rank } x = 1\}\) and \(\|A\| = \|P\|\), \(\|A^{-1}\| = \|P^{-1}\|\) this implies that;

\[
\|A^* \otimes A^{-1} + A^{-1} \otimes A\|_{\text{Inj}} = \sup_{\|x\| = \text{rank } x} \|A^{*}X^{A^{-1} + A^{-1}X^*A}|X\| = \sup_{\|x\| = \text{rank } x} \|P V^{*}X V^{*} + \frac{1}{P} V^{*} X^*V^{*}\| = \sup_{\|x\| = \text{rank } x} \|P^{-1} X P + \frac{1}{P} V^{*} X^*V^{*}\| = \|P \otimes P^{-1} \| + \|P\|\|P^{-1}\| + \frac{1}{\|A\|\|A^{-1}\|} = \frac{\|P\|\|P^{-1}\|}{\|A\|\|A^{-1}\|} + \frac{1}{\|A\|\|A^{-1}\|}.
\]

Conclusions

The field of elementary operators has been so interesting over the past decades and much have been done. The norm property in particular has attracted many scholars but a lot can be done further. In our study, we considered the normally represented elementary operators. We recommend that other properties of the normally represented elementary operators can be studied like numerical ranges, positivity and spectrum. For the Jordan elementary operator, we conjecture that the norm \(\|U\| = \frac{1}{4}\|A\|\|B\| + \epsilon\) where the \(\epsilon > 0\) is arbitrary.

Conflicts of Interest

The authors hereby declare that they have no conflict of interest.

References


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