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Research Article

Norms of Normally Represented Elementary Operators

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Abstract

The norm problem involves finding the formula describing the norm from the coefficients of the elementary operators. Upper estimate of the norm has been easy to find but estimating the norm from below has been proven difficult in general. In this study, we considered a special type of elementary operators called normally represented elementary operators. Some of our results show that the norm of an elementary operator is equal to the largest singular value of the operator itself i.e. $S_i(M) = \|M\|$ and also if $U_{A,B} = A \otimes h\ B + B \otimes h\ A$ is normally represented, then $\|U_{A,B}\|_{Inj} \geq 2(\sqrt{(2-1)})\|A\|\|B\|$.

Keywords: Norms; Elementary operator; Normally represented elementary operator; Norm-attainable operators.

Introduction

The structural properties of the elementary operators have been of great concern in analysis mathematics [1]. Several of them have been studied and of the most interesting concern is the norm property. The term elementary operator came as a result of the knowledge of the basic elementary operators from an algebra [2]. If A is an algebra, then given a, b ∈ A, we define the basic elementary operator (implemented by A, B) by: $M_{A,B}(H) = AXB$, ∀ X ∈ B(H). This led to the form describing the elementary operators as the sum of basic elementary operators.

$T : B(H) \rightarrow B(H)$ by $T_{A_i,B_i}(X) = \sum_{i=1}^{n} A_i X B_i$ ∀ X ∈ B(H) and ∀ A_i , B_i fixed in B(H). The coefficients of the norm property have been studied by several scholars and their notations noted down [3]. For example, the basic elementary operator $\|M_{a,b}\| \leq 2\|a\|\|b\|$. For Jordan elementary, $U = \| M_{a,b} + M_{a,b} \|$, $\| M_{a,b} + M_{a,b} \| \leq 2\|a\|\|b\|$ for the upper estimates. For the lower estimates, Mathieu proved that for the prime C*-algebra, $\| M_{a,b} + M_{a,b} \| \geq \frac{2}{3}\|a\|\|b\|$. In [4] the authors showed that for a JB∗-algebra, $\|M_{b,a} + M_{b,a}\| \geq \frac{1}{20412}\|a\|\|b\|$ while [5] proved that for standard algebra operator on H $\|M_{b,a} + M_{b,a}\| \geq 2(\sqrt{2} - 1)\|a\|\|b\|$. In [6], the authors also showed that $\|M_{b,a} + M_{b,a}\| \geq \|a\|\|b\|$ and further described the formula for norm of the general elementary operators with tracial Geometric mean.

In [7], they described the norm of basic elementary operators on algebra of regular norm. The study gave a description by showing that E is an atomic Banach Lattice with an order continuous norm A, B ∈ $L^r(E)$ and $M_{a,b}$ is the operator on L r(E)defined by MA,B(T) = AT B, then $\|M_{A,B}\|_r = \|A\|_r \|B\|_r$ but that there is no real α > 0 such that $\|M_{A,B}\|_r \geq \alpha\|A\|_r \|B\|_r$. The objectives of this study were to determine the norm inequalities for normally represented elementary operators. Both the lower norm bounds and upper norm bounds have been established for normally reperesented elementary operators. The results show that the norm of an elementary operator is equal to the largest singular value of the operator itself i.e. $S_i(M) = \|M\|$ and also if $U_{A,B} = A \otimes h\ B + B \otimes h\ A$ is normally represented, then $\|U_{A,B}\|_{Inj} \geq 2(\sqrt{(2-1)})\|A\|\|B\|$.

Research Methodology

In the present study, some definitions and known results used are shown below.

Definition 1.1. ([8], Definition 1.2.1) Field. A field F is a set closed under two binary

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operations of addition and scalar multiplication satisfying the following properties:
(i). Closure under addition and multiplication. $a + b \in F$ and $a.b \in F$, $\forall$ a, b $\in$ F,
(ii). Associativity: $a + (b + c) = (a + b) + c$, $\forall$ a, b, c $\in$ F,
(iii). Commutativity: $a + b = b + a$ and $(a.b).c = (b.c).a$, $\forall$ a, b, c $\in$ F,
(iv). Additive and multiplicative identities: $\forall a \in F$, $\exists -a \in F$: $a + -a = 0$. And $\exists$ $a^{-1} \in F$: $a.a^{-1}=1$
(v). Distributivity: $a(b + c) = (ab + ac)$ $\forall$ a, b, c $\in$ F,
(vi). Existence of additive inverse: $\forall a \in F$ $\exists x \in K$: $a + x = 0$, and $x + a = 0$ then $a = -x$ $\forall$a, x $\in$ F,
(vii). Existence of a multiplicative inverses: For each $a \in F$ with 0<a> 0 the equations $a.x = 1$ and $x.a = 1$ have a solution $x \in F$, called the multiplicative inverse of a and denoted by a^{-1}.

Definition 1.2. ([9], Definition 1.2.2) Vector space. Let F be a field and V a collection of objects called vectors, then V is a vector space over a field F if V is closed under vector addition and scalar multiplication. i.e. $\forall$ $v_1, v_2 \in V$, $v_1 + v_2 \in V$ and $\forall v \in V$, and $\forall a \in F$, $a.v \in V$, and satisfies the following properties:

(i). Commutativity. $v_1 + v_2 = v_2 + v_1$, $\forall v_1, v_2 \in V$,
(ii). Associativity. $v_1+ (v_2 + v_3) = (v_1 + v_2) + v_3$. $\forall$ $v_1, v_2, v_3 \in V$,
(iii). Additive inverse. $\forall v \in v$, $\exists -v \in V$: $v + -v = 0$ $\forall$ v_1, $-v \in V$
(iv). Additive Identity. $\forall v \in V$, $\exists 0 \in V$: $v + 0 = v$. $\forall v \in V$
(v). Multiplicative Identity. $1.v = v$ $\forall v \in V$
(vi). Distributive property. $\forall a \in F$, and $\forall$v1, v2 $\in$ V, $a(v_1 + v_2) = (av_1 + av_2)$ and the space (V, ‖.‖) is called a normed vector space.
(vii). Unitary law. $\forall v \in V$, $1.v = v$.

Definition 1.3. ([10], Definition 2.1.8) Banach space. This is a complete normed linear space.

Definition 1.4. ([11], Definition 12.7) Hilbert space. A Hilbert space is a complete inner product space.

Definition 1.5 ([12], Definition 2.1.10) C* - algebra.

A complex Banach *algebra A is called a C∗-algebra if $\|xx^*\|=\|x\|^2$ $\forall x \in A$.

Definition 1.6. ([13], Definition 1.19) Spectrum. Let A be a unital Banach algebra, then the spectrum of $a \in A$ is given by σ(a); = $\{\lambda \in C : a - \lambda I$ is not invertible$\}$.

Definition 1.7. ([14], Definition 2.1.1) Norm. A norm is a non-negative real valued function that takes the elements of a vector space to a field of real numbers denoted by $\|.\|: V \to R$ satisfying the following conditions:

(i.) Non-negativity: $\|x\| \geq 0$, $\forall x \in V$.
(ii.) Zero property: $\|x\| = 0$, if and only if x=0, for all x $\in$.
(iii.) Homogeneity: $\|\alpha x\| \leq |\alpha|\|x\|$, $\forall x \in V$ and $\alpha \in F$
(iv.) Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$, $\forall$ x and y $\in$ V
The pair (V, ‖.‖) is called a normed linear space.

Definition 1.8. [15]. Elementary Operator. Let H be an infinite dimensional complex Hilbert space and B(H) be an algebra of all bounded linear operators on the H . We define an elementary operator $T : B(H) \to B(H)$ by $T_{Ai,Bi}(X) = \sum_{i=1}^{n} A_i X B_i$ $\forall X \in B(H)$ and $\forall A_i$, B_i fixed in B(H) where i = 1, . . . , n. Examples of elementary operators include:

(i). The left multiplication operator L_A: B(H) by: $L_A(X) = AX$, $\forall X \in B(H)$.
(ii). The right multiplication operator R_B: B(H) by: $R_B (X)=BX$, $\forall X \in B(H)$.
(iii). The Basic elementary operator (implemented by A, B) by: $M_{A,B}$ (H) = AXB, $\forall X \in B(H)$.
(iv). The Jordan elementary operator (implemented by A, B) by: $U_{A,B}$ (X)=AXB + BXA, $\forall X \in B(H)$.
(v). The Generalized derivation (implemented by A, B) by: $\delta_{A,B} = L_A - R_B$.
(vi).The inner derivation (implemented by A, B) by: $\delta_A = AX - XA$.

Definition 1.9. [16], Normally represented elementary operator. Let H be an infinite dimensional complex Hilbert space and B(H) be the algebra of all bounded linear operators on H. We define an elementary operator, $T : B(H) \to B(H)$ by $T_{Ai,Bi}(X) = \sum_{i=1}^{n} A_i X B_i$ $\forall X \in B(H)$ and $\forall A_i$, B_i fixed in B(H) where i = 1, . . . , n. From this operator, we can define the generalized adjoint by $T_{Ai,Bi}(X) = \sum_{i=1}^{n} A_i^* X B_i^*$ and we say that T is normal if and only if T T*= T*T. Now AC = CA, BD = DB, together with AA*= A*A, BB*= B*B, CC*= C*C and DD*= D*D ensures that the operator $T_{Ai,Bi}(X)$ = AXC + BXD is normal. Therefore, the elementary operator of the form: $T_{A_i, B_i}(X) = \sum_{i=1}^{n} A_i X B_i$ where A_i and B_i are commuting families of normal operators are called normally represented elementary operator.

Results and Discussions

The norms of normally represented elementary operators are determined and discussed.

Proposition 4.13. Let H be a complex Hilbert space and M: B(H) →B(H) be a basic elementary operator. Then $S_i(M) = \|M\|$. Such that $S_i(M)$ are the singular values of M.

Proof. Since M is an operator, we represent the spectral norm on M as ‖M‖ i.e. $\|M\|_2$ then by definition of a singular value, is known that the spectral norm is equal to the largest singular value of the operator. I.e. Since M is an operator, then $\|M\|_2 = \sqrt{\lambda_i(M*M)} = \sigma_{max}(M)$. Indeed the spectral norm ‖M‖2 of the complex matrix of M is defined by max{‖M‖2: ‖x‖ = 1} then also let there be a linear transformation of the Euclidean vector space E. E is hermite if there exist an orthonormal basis of E consisting of all eigenvectors of E [17]. So, suppose that E = M*M i.e Hermitian matrix, Let $\lambda'_i. . . , \lambda_n$ be eigenvalues of E and $\{e_i, . . . , e_n\}$ be an orthonormal basis of E then $x = a_ie_i +, . . . , + a_ne_n$, we have that:

$$\|x\| = \sqrt{\langle \sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} a_i e_n \rangle} = \sqrt{\sum_{i=1}^{n} a_i^{i}}$$

and $$Ex = B(\sum_{i=1}^{n} a_i e_i) = \sum_{i=1}^{n} a_i B(e_i) = \sum_{i=1}^{n} \lambda_i a_i e_i$$

We write λ_{j0} to be the largest eigenvalue of E. Therefore;

$\|M\| = \langle Mx, Mx\rangle = \langle x, M*Mx\rangle = \langle x, Ex\rangle = \langle \sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} \lambda_i a_i e_i \rangle = \sqrt{\sum_{i=1}^{n} a_i \lambda_i e_i}$ $\leq Max_{1=j<n}\sqrt{|\lambda_j|} \times (\|x\|)$. So if $\|M\|_2 = \max\{\|M\|: \|x\| = 1\}$ then

$\|M\|_2 = \max_{1=j<n}\sqrt{|\lambda_j|}$……………..(1)

Consider also, $x_0 = e_{jo} \Rightarrow \|x\| = 0$ so that $\|M\| \leq \langle e_{jo}, E(e_{jo})\rangle$…………………….. ……..(4.3.2). Adding inequalities [4.3.1] and [4.3.2] gives $\|M\| = Max_{1=j<n}\sqrt{|\lambda_j|}$ where λ_j is the eigenvalue of E = M*M. By [18], we therefore conclude that $\sqrt{\lambda_{max}}(M*M) = \sigma(M)$. Since $\lambda_j = \max_{1=j<n}\sqrt{|\lambda_j|}$ where λ_j is the eigenvalue i.e. $\lambda_j = S_i(M)$ then we have that $S_i(M) = \|M\|$.

Theorem 4.14. Let $U_{A,B}(X) = AXB + BXA$ be normally represented then, $\|U_{A,B}\|_{CB} \geq \|A\|\|B\|$ for A, B ∈ B(H).

Proof. Let diam H = 2 and choose W_2 ∈ B(H). Let also _A = [A, B], _B =[B, A][1], then we shall use the notation _A ⊙ _B = A ⊗ B + B ⊗ A. It is necessary to show that the Haangerup norm of _A ⊙ _B holds for $\|\_A \odot \_B\|_H \geq \|A\|\|B\|$. Multiplying A by a scalar ***l*** and B by $\frac{1}{l}$, let ‖_A‖ = ‖B‖ and then ‖A‖ = 1 = ‖B‖. Is known that for each invertible matrix D ∈ W_2; it follows that $\_AD^{-1} \odot D\_B = \_A \odot \_B \Rightarrow \|\_A \odot \_B\|H = \inf \|\_AA^{-1}\|\|D\_B\|\ \forall\ D \in W_2$. Since for each unitary 2 × 2 matrix v, we have that ‖_Av‖ = ‖_B‖ and similarly for columns, by using the polar decomposition, it suffices that the infimum over all positive matrices A only and clearly we may also let that *det* D = 1. We therefore show that $\|\_AD^{-1}\|^2\|D\_B\_\|^2 \geq 1$ for all positive D ∈ W_2 with *det* D = 1. We let $D = \begin{bmatrix} \alpha & \beta \\ \beta^- & \gamma \end{bmatrix}$, α, γ ≥ 0 then

$AD{-}1\|2\|DB = (\gamma A - \beta B^-) \otimes (\beta A + \alpha B) + (-\beta A + \alpha B) \otimes (\gamma A + \beta B^-)$. This can be reduced by putting $X = |\beta|^2 + \gamma^2$, $Y = \beta(\alpha + \gamma)$, $Z = \alpha^2 + |\beta|^2$, so we can write ‖XAA*− 2Re(Y AB*) + ZBB*‖ · ‖XA*A+ 2Re(Y B*A) + ZB*B‖ ≥ 1 Assuming that X ≥ Z , then noting that $\|UA,B\|CB = \|U_{uAv,u}B_v\|_{CB}$ for all unitaries u, v ∈ W_2 we can write A by |A| and B by v*B, where A = u|A| is a polar decomposition of A i.e. we let A be positive, so that A and B are of the same form $A = \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix}$ and $B = \begin{bmatrix} \beta 1 & \beta 2 \\ \beta 3 & \beta 4 \end{bmatrix}$ where h=[0,1] and β1, β2, β3, β4 ∈ C then,

XAA*−2Re(YAB*)+ZBB*= $\begin{bmatrix} A - 2Re(Y\beta 1) + Z(|\beta|2 + |\beta 2|2) & * \\ * & * \end{bmatrix}$, and

XAA* + 2Re(YAB*)+ZBB*= $\begin{bmatrix} A + 2Re(Y\beta 1) + Z(|\beta|2 + |\beta 2|2) & * \\ * & * \end{bmatrix} \Rightarrow$

det D = 1 and that $|Y|^2 - XZ = -1$ because Z ≥ 0 we have that $X \pm 2|Y||\beta_1 + Z|\beta_1|^2 \geq 0$ since X ≥ 2 and $XZ = 1 + |Y|^2$, also we have that X ≥ 1 and thus

$(X - 2Re(Y\beta^-) + Z(|\beta_1|^2 + |\beta_2|^2))(A + 2Re(Y\beta^-_1) + Z(|\beta_1|^2 + |\beta_3|^2)$

$\geq (X + Z|\beta_1|^2 - 2Re(Y\beta 1)(X + Z|\beta_1|^2 + 2Re(Y\beta 1))$

$= (X + Z|\beta_1|^2)^2 - 4(Re(Y\beta 1))^2 \geq (X + Z|\beta_1|^2)^2 - 4|Y|^2|\beta_1|^2$

$= (X + Z|\beta_1|^2)^2 - 4(XZ - 1)|\beta_1|^2 = (X - Z|\beta_1|^2)^2 + 4|\beta_1|^2$

$\geq X^2(1 - |\beta_1|^2)^2 + 4|\beta_1|^2 \geq (1 - |\beta_1|^2) + 4|\beta_1|^2(1 + |\beta_1|^2)^2$

This completes the proof if *dim* H = 2.

Suppose that dim H > 2, then it follows that, by choosing some $\delta > 0$, and a unit vector η and $\xi \in H$ such that $\|A\xi\| \geq \|A\| - \delta$ and $\|B\eta\| \geq \|B\| - \delta$. Let N1be a two dimensional space containing ξ and η and let n2also be a two dimensional space containing $A\xi$ and $B\eta$, moreover, let $q \in B(H)$ be orthogonal projections onto N_1 and let $p \in B(H)$ be partial symmetry with final space N_1 and initial space N_2, then $\|pAq\| \geq \|A\| - \delta$ and $\|pBq\| \geq \|B\| - \delta$. It becomes trivial to verify that $\|U_{A,B}\|_{CB} \geq \|pAq, \ pBq\|_{CB}$ hence regarding pAq and pBq as operators on the two dimensional space n_1, it follows that $\|U_{A,B}\|_{CB} \geq \|pAq\|\|pBq\| \geq (\|A\| - \delta)(\|B\| - \delta)$. By letting $\delta > 0$, we have $\|U_{A,B}\|_{CB} \geq \|pAq\|\|pBq\| \geq \|A\| \geq \|B\| \Rightarrow \|U_{A,B}\|_{CB} \geq \|A\|\|B\|$.

Theorem 4.15. Let A, B ∈ B(H) and $U_{A,B} = A \otimes h\ B + B \otimes h\ A$ be normally represented then $\| U_{A,B} \|_{Inj} \geq 2(\sqrt{2} - 1)\|A\|\|B\|$.

Proof. Let $\|A\| = \|B\| = 1$ and A, B be functions on D := (B(H)*), and $U_{A,B}$ as a function on D × D. Taking dot products of A and B using a suitable scalars of modulus 1, we let $A(x_0) = 1$ and $B(y_0)$ $\forall\ x_0, y_0 \in D$. Putting $A_1 = A(x_0)$, and $B_1 = B(y_0)$. Then it gives $U_{A,B}\ (x_0, y_0) = 2B_1$, $U_{A,B}\ (y_0, y_0) = 2A$, $U_{A,B}(x_0, y_0) = 1 + A_1B_1$ if $|A_1|$ or$|B_1| \geq \sqrt{(2 - 1)}$. This completes the proof. On the other hand, if suppose that $|A_1| < \sqrt{(2 - 1)}$ and $|B_1| < \sqrt{(2 - 1)}$ then,

$|1+ A_1B_1| > |- (\sqrt{(2 - 1)})^2| = 2(\sqrt{(2 -1)})\ \|A\|\|B\|$.

Corollary 4.16. Let $R = \sum^n_{i=1} Ai \otimes Bi \in B(H) \otimes B(H)$. *Then we have* $\|R\|_{Inj} = \sup\{\|Ai \otimes Bi\| : X \in B(H)\}$, $\|X\| = 1$ *if and only if X is rank one Operator*.

Proof. Let $R(X\ ) = \sum^n_{i=0} A_i \times B_i$ and $\|R\|\beta = \sup\{\|Rx\| : X \in B(H), \|X\| = 1$, and, rank $(X\ ) = 1\}$. It is known [70] that every rank one operator $X \in B(H)$ is of the form $x = v \otimes \bar{\varsigma}$ for all $v, \varsigma \in H$ then this gives

$\|R\|\beta = \sup\{| \sum^n_{i=1}\langle A_i \times B_i\rangle\ \xi\eta| : \|X\| = 1$, Rank $(X\ ) = 1$, $\|\xi\| = \|\eta\| = 1\}$

$= \sup\{|n \sum i=1\langle A_i, v, \eta\ \rangle\langle B_1\xi, \zeta\ \rangle| : \|\zeta\| = \|v\| = \|\xi\| = \|\eta\| = 1\}$

$= \sup\{|\sum^n_{i=1} f\ (A_i)\ g(B_i)|\}$.

Taking the last supremum all over all functionals of the form $f = V \otimes \bar{\eta}$, $g = \xi \otimes \bar{\zeta}$. For all elements in the product U(H) of B(H) is a norm limit of convex combinations of elements of the form $V \otimes \bar{\ }\eta$ and the unit ball U(H) is a weak dense in the unit ball of dual of B(H). This implies that $\|R\|_\beta = \|R\|_{Inj.}$

Proposition 4.17. Let A, B ∈ B(H) where H is infinite dimensional, then $r(A^*A = B^*B) = \inf_{l>0} r(lA^*A + 1lB^*B)$ if and only if there exists $\eta \in H$, $\|\eta\| = 1$ such that $\|A\eta\|2 = \|B\eta\|2 = 12r(A*A + B*B)$.

Proof. By letting $r(A^*A + B^*B) = 1$, if the second part is satisfied, then $\langle r(A^*A + B^*B)\eta, \eta\rangle = \|A\eta\|^2 = \|B\eta\|^2 = 12\ (l\ +1l) \geq 1$ On the other hand, if the first part holds, put $k = Ker(A^*A + B^*B) - 1$ and ϱB (H)be orthogonal projection onto k. Put $y = 1_n$, such that $n \geq 2$ then by the first part, there exists a sequence $\{x_n\}$ of the unit vector x in H such that $\langle((1 - y_n)A*A + 11-y_nB^*B)x_n, x_n\rangle \geq r(A^*A + B^*B)$. For all $n \geq 2$, let $q = (A^*A + B^*B)$ and $z = (B^*B - A^*A)$ then, we have, $\langle qx_n, x_n\rangle + y_n\langle zx_n, x_n\rangle \geq 1$. Since H is finite dimensional, the unit ball of H is compact and there exists a convergent sequence of $\{x_n\}$. Letting $x = \lim_n x_n$, then from the above equation and that $\langle qx_n, x_n\rangle \leq 1$ then it follows that $\langle qx_n, x_n\rangle = 1$hence $qx = x$ since $\|q\| = 1$ and for $x \in k$. From (iv), so we have that

$yn\langle zx_n, x_n\rangle + y^2{}_n/(1 - yn)\langle B^*Bx_n, x_n\rangle \geq 0$. Dividing through by y_n, we have that $\langle zx_n, xn\rangle + y_n(1-y_n)\langle B^*B\ x_n, x_n\rangle \geq 0$. Letting $n \to \infty$, we conclude that $\langle zx_n, x_n\rangle \geq 0$. Likewise, from the sequence $ln = -(1_n)$ instead of $y_n = 1_n$, we obtain a unit vector $v \in H$ such that $\langle zv, v\rangle \leq 0$. So there exists a unit vector $\eta \in H$ such that $\langle(A*A - B*B)\eta, \eta\rangle > 0$. This together with $(A*A + B*B)\eta = \eta$ implies that $\|A\eta\|^2 = \|B\eta\|^2 = \frac{1}{2} r\ (A*A + B*B)$.

Theorem 4.18. Let $U_{A,B}$*:* $B(H) \to B(H)$ *be defined by* $U_{A,B} = A*XB + B*\ XA$ *be normally represented, then* $\|U_{A,B}\| = \|U_{A,B}|B^*(H\ )SA\| = \inf_{l>0}\|lA*A + 1lB*B\|$.

Proof. Let $\|A^*A + B^*B\| = 1$ then since $U_{A,B} = U\ lA(=\frac{1}{l})B$ for all $l /= 0$ then we have that $\inf_{l>0}\|lA*A + 1lB*B\| = \|*A + B*B\| = 1$. If H is infinite dimensional, then by the proposition [4.13], there exist a unit vector η satisfying $\|A\eta\|^2 = \|B\eta\|^2 == \frac{1}{2}$ meaning that the linear span L of $\{A\eta, B\eta\}$ can be defined by the adjoint of X by $XA\eta = B\eta$ and $XB\eta = A\eta$ then the adjoint X*, we can extend X to X* as an operator on H such that $X = X^*$ and $X^2 = 1$ then $|\langle(A^*XB + B^*XA)\eta, \eta\rangle| = 1$ hence $U_{A,B} \geq 1$. But $\|U_{A,B}\| \geq \|U_{A,B}\|_{CB} \leq \inf_{l>0}\|lA*A + \frac{1}{l}B*\ B\| = 1$. This completes the proof when H is finite dimensional. Suppose that H is infinite dimensional, then let $\{s_n\}$ be a net of finite rank orthogonal projections increasing to identity, we denote by the restrictions of s_nA to the range of s_n and analogously for B for each n, let ln be such that

$\inf_{l>0} \| l_nA^*_nA_n + \frac{1}{l}B^*_nB_n \| = \| l_nA^*_nA_n + \frac{1}{l}B^*_nB_n \|$ so we have;

$\|U_{A,B}\| = \sup_{\|x\|=1} \|A^*XB + B^*XA \geq A^*s_n X s_n B + B^* s_nXs_n A$

$\geq \sup_{\|x\|=1} (s_nA^*s_n)(s_nXs_n)(s_nB^*s_n)(s_nXs_n)(s_nAs_n)$

$= \|l_nA^*_nA_n + \frac{1}{ln}B^*_nB_n \| \; l_n \to l_0$, then

$\lim_n \| l_nA^*_nA_n + \frac{1}{ln}B^*_nB_n \| = \|l_0A^*A + \frac{1}{l} B^*B \| \geq \|lA^*A + \frac{1}{l} B^*B \|$.

Hence $\|U_{A,B}\| \geq \inf_{l>0} \|lA^*A + \frac{1}{l}B^*B \|$.

Proposition 4.19. Let $U_{A,B} : B(H) \to B(H)$ be defined by $U_{A,B} = A^*XB + B^*XA$ be normally represented, and that $U_{A,B}$ is real and linear, then, $\|U_{A,B}\| \geq \inf_{l>0} \|lA^*A + \frac{1}{l} B^*B\|$.

Proof. From the theorem above, let η be a unit vector and X a unitary operator such that $XB\eta = A\eta$, then $\|U_{A,B}\| \geq \langle (lA^*XB + \frac{1}{l} B^*XA)\eta, \eta\rangle = r(A^*A + B^*B) = \|A^*A + B^*B\|$. The reverse inequality becomes, $\|U_{A,B}(X)\| = \| \begin{bmatrix} A^* & B^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \begin{bmatrix} B & 1 \\ A & 0 \end{bmatrix} \| \geq \|A^*A + B^* B\| \|X\|$.

Corollary 4.20. Let A, B ∈ B(H) be self adjoint and normal, then for $U_{A,B} = AXB + BXA$ we have that $\|U_{A,B}|B(H)SA\| = \inf_{l>0} \|A2+1lB2\|$ If dim H= 2.

Proof. If H is real, then by the theorem above, it is trivial. If H is complex with dim H = 2, we shall use the orthogonal basis {ξ1, ξ2} of H relative to which A is diagonal. So because B is self adjoint, the diagonal entries of B are real and the two off-diagonal entries of B can be made real by replacing ξ_2 by $\theta\xi_2$ for a suitable scalar θ of modulus 1. I.e. we let A and B be real matrices. We may assume that $\inf_{l>0} \|lA^2 + \frac{1}{l} B^2\| = \inf_{l>0} \|A^2 + B^2\| = 1$ then we obtain a unit vector η such that $\|A\eta\|^2 = \|B\eta\|^2 = \frac{1}{2}$ furthermore, η is an eigenvector of the real symmetric matrix $(A^2 + B^2)\eta = \eta$ hence η is real. Then $\langle A\eta, B\eta\rangle \in R$ and we can also get a unitary self adjoint matrix X satisfying $XA\eta = B\eta$ and $XB\eta = A\eta$. Then the proof follows the theorem 4.12

Proposition 4.21. Let W2 denote the algebra of a complex square matrix of order 2 ×2 and Let A, B ∈ W2 be self adjoint, then $\|U_{A,B}\|_{CB} = \|U_{A,B}\|$.

Proof. By proposition 4.19, we can clearly see that

$\|U_{A,B}|B(H)SA\| = \inf_{l>0} \|lA^2 + \frac{1}{l}B^2\| \geq \|U_{A,B}\|_{CB} \geq \|U_{A,B}\| = \|U_{A,B}|B(H)SA\| = \inf_{l>0} \|lA^2 + \frac{1}{l}B^2\|$

Hence $\|U_{A,B}\|_{CB} = \|U_{A,B}\|$

Corollary 4.22. Let W_2 denote the algebra of a complex square matrix of order 2 × 2 and Let A, B ∈ W_2 be self adjoint, and normally represented, then $\|U_{A,B}\| = \|U_{A,B}|(W_2)SA\| \geq \|A\|^2 \|B\|^2$.

Proof. Let $\inf_{l>0} \|lA^2 + \frac{1}{l}B^2\| = \inf_{l>0} \|A^2 + B^2\|$. Put n $= \| A^2 + B^2\|$ then by the proposition above, there exist a unit vector η satisfying $(A^2 + B^2)\eta = \eta$ and $\|A\eta|^2 \|B\eta\|^2 = \frac{n}{2}$. Let $\eta^\perp$ be a unit vector orthogonal to η and we put $k = \|A\eta^\perp\|^2$. Since $A^2 + B^2 \leq n_1$ we have $\|B\eta\|^2 \leq w - k$. From $\|A\|^2_2 = \|A\eta\|^2 = \frac{1}{2}w + k$ and $\|B\|^2_2 = \|B\eta\|^2 = \frac{3}{2}w - k$ then it follows that

$\|A\|^2_2 \|B\|^2_2 \leq (\frac{1}{2}w + k)(\frac{3}{2}w - k) \leq w_2 = \|U_{A,B}|(W_2)SA\|^2$.

Example 4.23. ([4] Example 4.5) Put A = e − iu and B = (e + iu)/2 where $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $u = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Let $X = [X_{ij}]$ then $U_{A,B}(X) = \begin{bmatrix} X\_11 & X\_21 \\ 0 & 0 \end{bmatrix}$ so that ‖UA,B‖ = 1 we need to show that $\|U_{A,B}\|_{CB} = \sqrt{2}$. We note that $U_{A,B}(X) = AXB + BXA = eXe + uXu$. Expressing w by Hangearup norm, $w = e \otimes e + u \otimes u$. It suffices to consider the representation of w of the form $w = (\gamma e - \bar{\beta} u) \otimes (\alpha e + \beta u) + (-\beta e + \alpha u) \otimes (\bar{\beta} e + \gamma u)$, then by a short computation, $\|w\|_H = \inf\{\|(A + C)e\|^{\frac{1}{2}} \|Ae\perp + Ce + 2Re(Bu^*)\|^{\frac{1}{2}} : AC - |B|2 = 1\}$ where $e\perp = 1 - e$. More so, $Ae^\perp + Ce + 2\,Re(BU^*) = \begin{bmatrix} C & -B \\ B & A \end{bmatrix}$ and the norm of the last matrix is equal to$(A + C + \sqrt{(A + C)^2 + 4|B|^2})/2$. By symmetry, we assume that $A \geq C$, hence $\|Ae^\perp + Ce + 2Re(BU^*)\| \geq \frac{1}{2} (A + C + |A - C|) = A$ therefore, $\|U_{A,B}\|CB = \|W\|H \geq (A + C)12\ A12 \geq (2AC)12 = (2(1+ |B|2))12 \geq \sqrt{2}$ thus in fact $\|U_{A,B}\|_{CB} = \sqrt{2}$.

Theorem 4.24. Let A_i and B_i be commuting operators in B(H), then $\|\sum^n_{i=1} A_i \otimes B_i\|_{Inj} \geq |\sigma(A_i) \circ \sigma(B_i)|$ if and only if A_i and B_i are normal operators.

Proof. Let $(\varphi, \psi) \in \Gamma A \times \Gamma B$, where Γ is the set of all multiplicative linear functionals on commutative algebras in B(H) then is clear that we can extend φ and ψ to unit functionals f and g on B(H) respectively using the Hahn-Banach theorem. This follows that $\|(U_{A,B})\| \geq |\sum^{n}_{i=1} f(A_i)g(B_i)| = |\sum^{n}_{i=1}\varphi(A_i)\psi(B_i)|$. Therefore $\|\sum^{n}_{i=1} A_i \otimes B_i\|_{Inj} \geq |\sigma(A_i) \circ \sigma(B_i)|$. Since A_i and B_i are commuting operators in B(H), it suffices that $|\sigma(A_i) \circ \sigma(B_i)| \geq \|(U_{A,B})\|$. Because $|\sigma(A_i) \circ \sigma(B_i)| \geq |\varphi(\sum^{n}_{i=1}\psi(A_i)B_i)|$ and $\varphi(\sum^{n}_{i=1}\psi(A_i)B_i)$ is normal for all $(\varphi, \psi) \in \Gamma A \times \Gamma B$ then $|\sigma(A_i) \circ \sigma(B_i)| \geq \sup_{\psi\in\Gamma} |\varphi(\sum^{n}_{i=1}\psi(A_i)B_i)| = \|\varphi(\sum^{n}_{i=1}\psi(A_i)B_i)\|$ this implies that $|\sigma(A_i)\circ\sigma(B_i)| \geq |\varphi(\sum^{n}_{i=1}\varphi(A_i)f(B_i)| = \|\varphi(\sum^{n}_{i=1} f(A_i)B_i)\|$ for all $(\varphi, \psi) \in \Gamma A \times \Gamma B$ and $f \in S1$ then we have that $|\sigma(Ai) \circ \sigma(Bi)| \geq \|(U_{A,B})\|$.

Theorem 4.25. Let A be invertible posinormal in B(H) and normally represented, then for $U_{A,A-1} = A \otimes A-1 + A-1 \otimes A$ *we have*

(i) $\|A \otimes A^{-1} + A^{-1} \otimes A\|_{Inj} = \|A\|\|A^{-1}\| + \frac{1}{\|A\|\|A-1\|}$

(ii) $\|A^* \otimes A^{-1} + A^{-1} \otimes A^*\|_{Inj} = \|A\|\|A^{-1}\| + \frac{1}{\|A\|\|A-1\|}$ *if and only if* $A \in J(H)$*where* $J(H)$ *is the set of all invertible operators in B(H).*

Proof. From ([9]), it follows clearly that $\|(\Phi A)\| = \sup_{p\in\delta}(M_A)(p + \frac{1}{p})$. Also that min $\|(\Phi A)\| = 1\|A^{-1}\|$and max $\|(\Phi A)\| = \|A\|$, because $\delta(M_A) = \delta(A^{-1})$ then min $\|(M_A)\| = \frac{1}{\|A\|\|A-1\|}$ = A and max $\|(M_A)\| = \|A\|\|A^{-1}\| = \frac{1}{A}$ then have that Max $\{p + \frac{1}{p}: A \leq p \leq 1A\} = A + \frac{1}{A}$. This maximum is attainable in A and $\frac{1}{A}$ thus the result follows clearly since it is true that $A \in \delta(M_A)$ For (ii), since $A \in J$ (H), there exist $V \in (H)$ such that $A = V p|p = |A|$, By ([72] corollary 1), we have that $\{x \in B(H) : \text{rank } x = 1\} = \{V^*x : X \in B(H), \text{rank } X = 1\}$ and $\|A\| = \|P\|$, $\|A^{-1}\| = \|P^{-1}\|$ this implies that;

$$\|A^* \otimes A^{-1} + A^{-1} \otimes A\|_{Inj} = \sup_{\|x\|=1=\text{rank } x} \|A^*X^{A-1} + {}^{A-1}XA^*\|$$
$$= \sup_{\|x\|=1=\text{rank } x} \|PV^*XP^{-1}V^* + P^{-1}V^*XP V^*\|$$
$$= \sup_{\|x\|=1=\text{rank } x} \|P (V^*X) P^{-1} + P^{-1} (V^*X) P\|$$
$$= \sup_{\|x\|=1=\text{rank } x} \|P XP^{-1} + P^{-1}XP\|$$
$$= \|P \otimes P-1 + P-1 \otimes P\|_{Inj}$$
$$= \|P\|\|P^{-1}\| + \frac{1}{\|P\|\|P-1\|}$$
$$= \|A\|\|A^{-1}\| + \frac{1}{\|A\|\|A-1\|}.$$

Conclusions

The field of elementary operators has been so interesting over the past decades and much have been done. The norm property in particular has attracted many scholars but a lot can be done further. In our study, we considered the normally represented elementary operators. We recommend that other properties of the normally represented elementary operators can be studied like numerical ranges, positivity and spectrum. The norm property is also not exhausted. For the Jordan elementary operator, we conjecture that the norm $\|U\| = \frac{1}{4}\|A\|\|B\| + \epsilon$ where the $\epsilon > 0$ is arbitrary.

Conflicts of Interest

The authors hereby declare that they have no conflict of interest.

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