Math 1496 Calc I

3.5 Implicit Differentiation

In the previous class - chain rule we saw

$$\frac{d}{dx}(x^2+1)^2 = 2(x^2+1)\cdot 2x$$

$$\frac{d}{dx}(\sin x)^2 = 2\sin x \cdot \cos x$$

and mentioned that

$$\frac{d}{dx} g^2(x) = 2g(x)\frac{dg}{dx}$$ if $g = x^2+1$ or $g = \sin x$ gives the 2 above

consider

$x^2+y^2=1$ what is y'?

well $y^2 = 1-x^2$ $y = \pm\sqrt{1-x^2}$

consider the case

$$y = \sqrt{1-x^2} \qquad u = 1-x^2, \quad y = u^{1/2}$$

$$\frac{du}{dx} = -2x \quad \frac{dy}{du} = \frac{1}{2}u^{-1/2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{2}u^{-1/2}\cdot(-2x) = \frac{-x}{\sqrt{u}}$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

or $\because y = \sqrt{1-x^2}$ $\boxed{\frac{dy}{dx} = -\frac{x}{y}}$ ← has x & y

Consider $x^2 + y^3 + y = 1$

could you solve for y (it's complicated)

but could we find y' w/o doing so?

Let return to

$x^2+y^2=1$ if $y=g(x)$ some $g(x)$

$\frac{d}{dx}\left(x^2+g^2(x)\right)=\frac{d(1)}{dx}=0$

$2x+2g(x)\frac{dg}{dx}=0$ by $g=y$

$2x+2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{\cancel{2}x}{\cancel{2}y}=-\frac{x}{y}$ (like we saw earlier)

This is when y is given implicitly and the derivative is "implicit diff^tion"

Ex $x^2+y^3+y=1$

$\frac{d}{dx}$: $2x+3y^2y'+y'=0$

$(3y^2+1)y'=-2x \Rightarrow y'=\frac{-2x}{3y^2+1}$

Ex Find the eq^n of the tangent to

$$x\cos y = 1 \quad \text{at} \quad (2, \pi/3)$$

Sol^n $\frac{d}{dx}(x\cos y) = 0$

$1 \cdot \cos y + x(-\sin y\, y') = 0$ ← solve for y'

$$y' = \frac{\cos y}{x\sin y}$$

Sub in pt $y'\Big|_p = \frac{\cos \pi/3}{2\sin \pi/3} = \frac{\frac{1}{2}}{2\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{2\sqrt{3}}$

tangent $y - \pi/3 = \frac{1}{2\sqrt{3}}(x-2)$

If $x^2+y^2=1$ find y''

well we already saw $\frac{dy}{dx} = -\frac{x}{y}$

take 1 more derv.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x}{y}\right)$$

$$y'' = -\left[\frac{1\cdot y - x\cdot y'}{y^2}\right]$$ but we know $y' = -\frac{x}{y}$

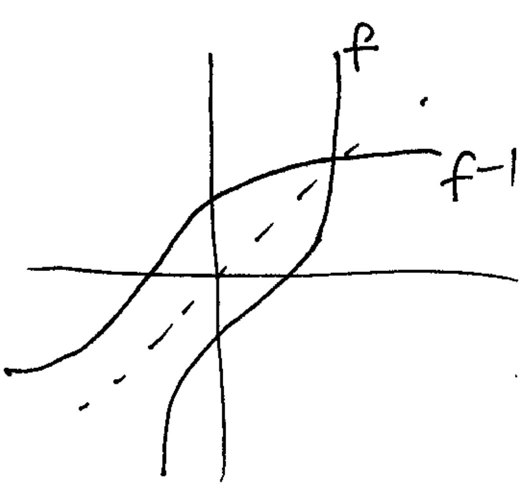
$$= -\left[\frac{y - x\cdot\left(-\frac{x}{y}\right)}{y^2}\right] = -\left(\frac{y + \frac{x^2}{y}}{y^2}\right)$$

$$= -\frac{x^2+y^2}{y^3}$$ (but $x^2+y^2=1$)

$$= -\frac{1}{y^3}$$

Derivative of inverse functions

If $f(x) = \frac{1}{4}x^3 + x - 1$ find $(f^{-1})'(3)$



I will not follow the book

For the universe fct switch x & y

So $x = \frac{1}{4}y^3 + y - 1$

if $x=3$ what is y

so $\frac{1}{4}y^3 + y - 1 = 3$ $\quad y^3 + 4y - 16 = 0$

$(y-2)(y^2+2y+8) = 0$

always positive so

$y=2$

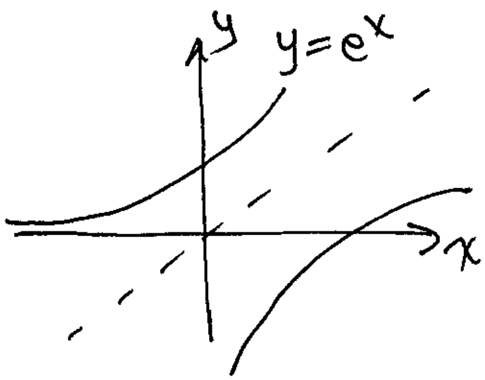
so find y' at $P(3,2)$

$x = \frac{1}{4}y^3 + y - 1$

$1 = \frac{3}{4}y^2y' + y' = \left(\frac{3}{4}y^2+1\right)y'$

$y' = \dfrac{1}{\frac{3}{4}y^2+1}$ at P $y' = \dfrac{1}{\frac{3}{4}\cdot 2^2+1} = \frac{1}{4}$

Derivative $y = \ln x$



$x = e^y \Rightarrow y = \ln x$

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(e^y\right) \Rightarrow 1 = e^y y' \Rightarrow y' = \frac{1}{e^y} = \frac{1}{x}$$

so $\dfrac{d}{dx}\ln x = \dfrac{1}{x}, \quad x > 0$

suppose $x < 0$ $\quad \dfrac{d}{dx}\ln -x = \dfrac{1}{-x}\cdot -1 = \dfrac{1}{x}$

so together $\boxed{\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}}$

Log's other Bases

Let $y = \log_a x$

so $a^y = a^{\log_a x} = x$

Now $\ln a^y = \ln x \Rightarrow y \ln a = \ln x$

$$y = \frac{\ln x}{\ln a}$$

so $\log_a x = \dfrac{\ln x}{\ln a}$ (note $\ln a$ #)

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{\ln a}\cdot\frac{1}{x}}$$

$$\boxed{\frac{d}{dx}a^x = a^x \ln a}$$

Proof let $y = a^x$ so $\ln y = x \ln a$

$\frac{d}{dx}\ln y = \frac{d}{dx}x\ln a \Rightarrow \frac{1}{y}y' = \ln a$

$y' = \ln a \cdot y = a^x \ln a$

so $\frac{d}{dx}a^x = a^x \ln a$