Math 3331 - ODE

2nd order linear ODE's (Homog)

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Ex $\quad y'' - 3y' + 2y = 0 \qquad (*)$

If we know 1 sol^n, we can find a second linearly indep. sol^n by using what is called reduction of order

let $y = u y_1$, $\quad y_1$ - known sol^n

we will then obtain an ODE for u. The following examples illustrate

1 sol^n of $\quad y'' - 3y' + 2y = 0$

is $\quad y = e^x$

let $\quad y = e^x u$, so $\quad y' = e^x u' + e^x u$

$$y'' = e^x u'' + e^x u' + e^x u' + e^x u$$

sub

$$e^x u'' + 2\cancel{e^x}u' + \cancel{e^x}u$$
$$-2(\cancel{e^x}u' + \cancel{e^x}u)$$
$$+\cancel{e^x}u = 0$$

so $e^x u'' = 0 \Rightarrow u'' = 0$ {normally there's a u' here as well}

$u' = c_1 \quad u = c_1 x + c_2$

so $y = e^x u = e^x(c_1 x + c_2)$

$$= c_1 x e^x + \underbrace{c_2 e^x}_{\text{our given sol}^n}$$

so $y_2 = x e^x$ is our 2nd sol^n

Normally we don't include constant. of integrati

So here $u'' = 0 \Rightarrow u' = 1 \Rightarrow u = x$

Ex2 $\quad y''+y=0 \quad$ 1 sol^n $y=\cos x$

$$y = \cos x\, u$$

$$y' = \cos x\, u' - \sin x\, u$$

$$y'' = \cos x\, u'' - \sin x\, u' - \sin x\, u' - \cos x\, u$$

sub

$$y''+y=0$$

$$\cos x\, u'' - 2\sin x\, u' - \cancel{\cos x\, u} + \cancel{\cos x\, u} = 0$$ {the u terms will alway cancel!!}

let $u'=v$ so $u''=v$

$$\cos x\, v' - 2\sin x\, v = 0$$ ← this is separable

$$\int \frac{dv}{v} = \int \frac{2\sin x}{\cos x}$$ {don't include constant}

$$\ln v = -2\ln \cos x$$

$$v = \cos^{-2} x = \sec^2 x$$

$$u' = v = \sec^2 x \Rightarrow u = \tan x$$

so $y_2 = \cos x \cdot u = \cos x \tan x = \sin x$

GS. $y = c_1 \cos x + c_2 \sin x$

ex 3 $\quad x^2y'' - 3xy' + 4y = 0$

$y_1 = x^2$

let $y = x^2u$, $\quad y' = x^2u' + 2xu$, $\quad y'' = x^2u'' + 2xu'$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 2xu' + 2u$

sub $\quad x^2(x^2u'' + 4xu' + 2u) - 3x(x^2u' + 2xu) + 4x^2u = 0$

$x^4u'' + 4x^3u' + \cancel{2x^2u} - 3x^3u' - \cancel{6x^2u} + \cancel{4x^2u} = 0$

$x^4u'' + x^3u' = 0 \qquad$ cancel x^3 & let $u' = v$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad u'' = v'$

$xv' + v = 0 \qquad$ sep

$\dfrac{dv}{v} = -\dfrac{dx}{x} \Rightarrow \ln v = -\ln x \Rightarrow v = \dfrac{1}{x}$

$u' = \dfrac{1}{x}$ so $\quad u = \ln|x|$ so $\quad y_2 = x^2\ln|x|$

G.S. $\quad y = c_1x^2 + c_2x^2\ln|x|$

ODE

Ex4 $\quad xy'' + (2x+2)y' + 2y = 0 \qquad y_1 = \frac{1}{x}$

$$y = \frac{u}{x} \qquad y' = \frac{xu' - u}{x^2} \qquad y'' = \frac{x^2(xu'' + \cancel{u'} - \cancel{u'}) - 2x(xu' - u)}{x^4}$$

Sub

$$x\,\frac{(x^3u'' - 2x^2u' + 2xu)}{x^4} + (2x+2)\frac{(xu' - u)}{x^2} + \frac{2u}{x}$$

$$u'' - \frac{2u'}{x} + \cancel{\frac{2u}{x^2}} + 2u' - \cancel{\frac{2u}{x}} + \cancel{\frac{2u'}{x}} - \cancel{\frac{2u}{x^2}} + \cancel{\frac{2u}{x}} = 0$$

So $u'' + 2u' = 0 \quad$ let $u' = v$ so $\quad v' + 2v = 0$

$$\frac{dv}{v} = -2dx \qquad \ln v = -2x \qquad (\text{no } c)$$

$$v = e^{-2x} = u' \qquad u = \frac{e^{-2x}}{-2}$$

so $y_2 = \frac{e^{-2x}}{-2x}$ as we will mult. by c
we will absorb $-\frac{1}{2}$ into c

G.S. $\quad y = \frac{c_1}{x} + c_2\frac{e^{-2x}}{x}$