International Journal of Modern Computation, Information and Communication Technology, 2018;1(3):61-68.
ISSN: 2581-5954
http://ijmcict.gjpublications.com

Research Article

Spectral Characterization of Jordan Homomorphisms on Semisimple Banach Algebras

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Abstract

Certain properties of operator algebras have been studied such as boundedness, positivity, surjectivity, linearity, invertibility, numerical range, numerical radius and idempotent property. Of great interest is the study of spectrum of linear mappings. It is therefore necessary to characterize Jordan homomorphisms on semisimple Banach algebras in terms of their spectrum. The objectives of the study are to: Investigate whether Jordan homomorphisms on semisimple Banach algebras are spectral isometries; Investigate whether Jordan homomorphisms are unital surjections on semisimple Banach algebras and to establish the relationship between unital surjections and spectral isometries on semsimple Banach algebras. For us to achieve our objectives we used Kadison's theorem, Gelfand theory and Nagasawa's theorem. The results obtained show that Jordan homomorphism is spectral isometry if it preserves nilpotency also is unital surjection if it preserves Jordan zero products and finally is unital surjective spectral isometry if it preserves commutativity and numerical radius between semisimple Banach algebras. These results are useful in characterizations in quantum mechanics and operator algebras.

Keywords: Jordan homomorphism; Semisimple Banach algebras; Isometry; Morphism.

Introduction

Linear preserver problems have a relatively long history and different kinds [1] and [2]. Some of the most popular linear preserver problems are linear maps preserving problems related to invertibility or spectrum [3]. In [4] they dealt with the problem of characterizing linear maps compressing the numerical range. A counter example was given to show that such a map need not be a Jordan*-homomorphism in general even if the C*-algebras were commutative. Under an auxiliary condition they showed that such a map was a Jordan *-homomorphism. In [5] studied non-linear transformations between the unitary groups of Von Neumann algebras and the twisted subgroups of positive invertible elements in unital C*-algebras with various preserver properties concerning the spectrum, spectral radius and generalized distance measures. They presented several results which showed that those transformations were closely related to the Jordan*-isomorphisms between the underlying full algebras. In [6] the author investigated to what extent a unital spectrally bounded operator from a simple unital C*-algebra of real rank zero onto a unital semisimple Banach algebra was a Jordan epimorphism.

In [7] the author established that all derivations on a semisimple Jordan-Banach algebra were automatically continous. Also it is shown that "almost all" primitive ideals in the algebra were invariant under a given derivation, the general case was reduced to that of primitive Jordan-Banach algebras. In [8] it is proved that every unital, surjective, invertibility preserving map from Von Neumann algebra onto standard operator algebra was a Jordan homomorphism. In [9] it is shown that a continuous derivation on

Received: 05.08.2018; Received after Revision: 17.08.2018; Accepted: 18.08.2018; Published: 04.09.2018

Banach algebra over the real or complex field leaves the primitive ideals of the algebra invariant. Also showed that every (linear) derivation on a semisimple Banach algebra was continuous. Thus every derivation on a semisimple Banach algebra, leaves the primitive ideals of the algebra invariant.

The author in [10] was concerned with certain automatic continuity problems for homomorphisms and derivations on Banach algebras. Cusack showed that if there existed a discontinous homomorphism from Banach algebra or a discontinuous derivation on a semi prime Banach algebra, then there existed a topologically simple radical Banach algebra. Furthermore Cusack showed that there were no Jordan derivations which were not also associative derivations on any semi-prime algebra over a field not of characteristic 2. Moreover it followed that every Jordan derivation on semisimple Banach algebras was a derivation and therefore continuous. In [11] they showed that every surjective n-homomorphism (n-anti-homomorphism) from Banach algebra A into a semisimple Banach algebra B was continuous.

Research Methodology

Definition 2.1 [1, Definition 3.1.2]. The function is **s**urjective (onto) if every element of the codomain is mapped by at least one element of the domain (That is the image and the codomain of the function are equal).

Definition 2.2 [8, Definition 2.8.5]. Let ***A*** be a complex Banach algebra,we say that ***A*** is semisimple if rad (A) = 0.

Theorem 2.3 [7, Theorem 3.3.2]. Nagasawa's theorem which asserts that since T is a bijective spectral isometry, we have that the image under T of the Jacobson radical of ***A*** is exactly the Jacobson radical of ***B***.

Theorem 2.4 [10, Proposition 3.2]. (Kadison's Theorem) Let ***A*** and ***B*** be unital semisimple Banach algebras. Let $T:\boldsymbol{A}\to \boldsymbol{B}$ be a surjective spectral isometry. Then T belongs to the centre ***B*** and its spectrum lies in the unit circle in C.

Theorem 2.5 [5, Theorem 2.1.10]. (Gelfand Theorem) If both ***A*** and ***B*** are commutative unital C*-algebras and T is a unital surjective numerical radius preserving linear map from A to B then r(a)= ||a|| and r(T(a))=||φ(a)||.

Theorem 2.6 [11, Theorem 3.5] When $T:A\to B$ is a surjective linear isometry between two unital C*-algebras A and B, then T_1 is a unitary in B and the mapping x a $(T_1)^{-1}Tx$,x in A is a Jordan *-isomorphism (that is, it preserves additionally self-adjoint elements).

Theorem 2.7 [4, Theorem 3.1]. Let $T:A\to B$ be a spectral isometry between the unital commutative semisimple Banach algebras ***A*** and *B*. We define $\hat{T}:T_A \to T_B$ by $\hat{T}= T_B o T o T_A^{-1}$. Then T is a spectral isometry which is unital or surjective, when T has these properties. Moreover, since spectral radius and norm coincide for continous functions $\hat{T}$ isometry.

Definition 2.8 [9, Definition 2.6]. Automorphism is an isomorphism from a mathematical object to itself. It is in some sense a symmetry of the object and a way of mapping the object to itself while preserving all of its structure.

Definition 2.9 [6, Definition 3.2]. Isometry is a distance preserving transformation between metric spaces, usually assumed to be bijective. A composition of two opposite isometries ia a direct isometry. A reflection in a line is an opposite isometry.

Definition 2.10 [10, Definition 1.4]. If *A* and *B* are algebras then we will call a linear map $T:A\to B$ called a Jordan Homomorphism if T(xy + yx) = T(x)T(y) + T(y) T(x) for every x,y in A

Definition 1.6 [1, Definition 2.2.3]. Morphism refers to a structure preserving map from one mathematical structure to another.

Results and discussions

Proposition 3.1: Let A,B be semisimple Banach algebras. Let ϕ :A $\to$ B be a spectral isometry. Then ϕ is a Jordan homomorphism if it has the property that $\phi p^2= \phi p$ for every projection p in A.

Proof. We have to show that $\phi a^2=\phi a^2$ for every a in A. Let p and q be orthogonal projections in A. Then p+q is a projection,therefore by assumption $\phi p +\phi q=\phi p+\phi q^2=(\phi p+\phi p)(\phi q)+(\phi q)(\phi p)+\phi q)$ it follows that $\phi p)(\phi q)+\phi q)(\phi p)=0$ and hence,$(\phi p)(\phi q)=-(\phi p)(\phi q)(\phi p)=(\phi q)(\phi p)$ since ϕp is idempotent. As a result, ϕp and ϕq are orthogonal idempotents.

Let $a=\sum_{j=1}^{n} \partial jpj$ be a linear combination of mutually orthogonal projections $p_1,\ldots,p_n$ in A. Then $\phi(a^2)=\phi(\sum_{j=1}^{n}\partial^2 jpj)=\sum_{j=1}^{n}\partial^2\phi pj=(\phi a^2)$ for $\phi p_1,\ldots,\phi p_n$ are mutually orthogonal idempotents. By spectral theorem every self-adjoint element a in A_{sa} is the norm-limit of finite linear combinations of mutually orthogonal projections. Hence, the continuity of ϕ entails that $\phi(a^2)=(\phi a)^2$ for every a in A_{sa}. Replacing a by a+b in this identity yields $\phi(ab+ba)=(\phi a)(\phi b)+(\phi b)(\phi a)$ for all a,b in $A_{\{sa\}}$. Suppose $a=a_1+ia_2$ with a_i in A_{sa} is the cartesian decomposition of a in A.
By the above, $\phi(a^2)=\phi(a_1^2-a_2^2+i(a_1a_2+a_2a_1))=(\phi a_1)^2-(\phi a_2)^2+i((\phi a_1)(\phi a_2)+(\phi a_2)(\phi a_1))=(\phi a)^2$. This proves the result.

Theorem 3.2. Let $\phi:A\to B$ be a Jordan homomorphism then ϕ is a spectral isometry if it preserves nilpotency.

Proof. By composing ϕ with the canonical epimorphism $B\to B/\text{rad } B$ since B is semisimple. As a result ϕ is bounded and hence open. Let N>0 be such that, for each $y\in B$, there is x in A with the properties $\phi x=y$ and $\|x\|\leq N\|y\|$. Let m> 0 be such that $r(\phi x) < mr(x)$ for all x in A.
If there exists c> 0 such that $r(a+x)<c\|x\|^{1/n}$ for all x in A with $\|x\| < 1$. Take y in B with $\|y\| < \frac{1}{N}$ and choose x in A such that $\phi x=y$ and $\|x\|< N\|y\|$.
We have $r(\phi a+y)=r(\phi(a+x))<mr(a+x)<mc\|x\|^{1/n}< mcN^{1/n}\|y\|^{1/n}\}$. Thus for some bounded neighbourhood of zero u in A, there is a constant cu> 0 such that $r(a+x)<cu\|x\|^{1/n}$ for all x in u then $(\phi a)^n=0$.

Theorem 3.3. Every spectral isometry is a Jordan homomorphism if it preserves elements with square zero. If e,f are orthogonal idempotents in A,then $(\phi a)(\phi b)+(\phi b)(\phi a)=0$ for all a in eAe,b in fAf which can be written as finite sums of elements with square zero.

Proof. Let a in eAe,b in fAf be written as $a=\sum ai$, $b=\sum bi$ respectively, where ai in eAe and bj in fAf are elements with square zero for all i,j. Then $(ai+bj)^2=0$ for all i,j therefore, by assumption, $(\phi(ai+bj)^2=0$ which yields $(\phi ai)(\phi bj)+(\phi bj)(\phi ai)=0$ for all i,j. Summing over all i,j we find $(\phi a)(\phi b)+(\phi b)(\phi a)=0$.

Corollary 3.4. Let A,B be semisimple Banach algebras and $\phi:A\to B$ be a spectral isometry which is a Jordan homomorphism. If it preserves elements with square zero, then ϕ maps projections in A onto idempotents in B.

Proof. Let p in A be a projection. Suppose at first that both p and 1-p are properly infinite. Then $(\phi p)(1-\phi p)(\phi p)=0$ which is equivalent to $(\phi p)^2=\phi p$. Suppose that p is properly infinite but 1-p is not. Then there is a sub projection f of p such that p : f: p-f, where : denotes equivalence, so that both f and p-f are properly infinite. It follows that 1-f and 1-p+f=1-(p-f) are properly infinite. For example, let z in A be a central projection with z(1-f)< 0. Writing z(1-f)=z(1-p)+z(1-f), we see that z(1-f) is infinite whenever z(p-f)<0 as 1-p and p-f are orthogonal. If z(p-f)=0 then zp=0 since p-f: p. Therefore, z(1-f)=z is infinite in this case too. Hence, 1-f is properly infinite and similarly for 1-(p-f). By step 1 we have $(\phi f)^2=\phi f$ and $(\phi(p-f))^2=\phi(p-f)$ with e=p-f, we also have $\phi(p-f)(\phi f)+(\phi f)\phi(p-f)=0$.
Consequently, $(\phi p)^2=(\phi(p-f)+\phi f)^2=(\phi(p-f)^2+\phi(p-f)(\phi f)+(\phi f)\phi(p-f)+(\phi f)^2=\phi(p-f)+\phi f=\phi p$.
Suppose now that p is not properly infinite but finite. Let z in A be the unique (minimal) projection in A such that zp is properly infinite and (1-z)p is finite, $\phi(zp)$ is idempotent. Since z and 1-z are properly, then we can apply za and (1-z)b and obtain $\phi(za)\phi((1-z)b)+\phi((1-z)b)\phi(za)=0$ a,b in A.
Rearranging we get $\phi(za)\phi(b)+\phi(b)\phi(za)=\phi(za)\phi(zb)+\phi(zb)\phi(za)$, a,b in A……………(1)
Set b=1 since ϕ is unital, it follows that $2\phi(za)=\phi(za)\phi(z)+\phi(z)\phi(za)$ and multiplying this identity on the left by the idempotent $\phi(z)$ as well as on the right and then subtracting the resulting identities, we have $\phi(z)\phi(za)=\phi(za)\phi(z)=\phi(za)$, a in A. Set a=1in (1) Then using the identity just obtained

$\phi(z)\phi(b)\phi(z) = \phi(z)\phi(zb)+\phi(zb)\phi(z) = 2\phi(zb)$ for all b in A

As above, this entails that $\phi(z)\phi(b) = \phi(b)\phi(z) = \phi(zb)$, b in A. In particular,

$\phi(p)\phi(zp) = \phi(p)\phi(z)\phi(p) = \phi(p)\phi(z)^2\phi(p) = \phi(p)\phi(z)\phi(p)\phi(z) = \phi(zp)\phi(zp) = \phi(zp)$

and similarly $\phi(zp)\phi(p)=\phi(zp)$.

From this, we deduce that

$(\phi((1-z)(1-p)))^2 = (1-\phi(z)-(\phi(p)-\phi(zp)))^2$

$= (1-\phi(z))^2+(\phi(p)-\phi(zp))^2-2(1-\phi(z))(\phi(p)-\phi(zp))$

$=1-\phi(z)+\phi(p)^2+\phi(zp)^2-2\phi(p)\phi(zp)$

$=-2(\phi(p)-\phi(z)\phi(p)-\phi(zp)+\phi(z)\phi(zp))$

$= 1-\phi(z)+\phi(p)^2+\phi(zp)-z\phi(p)$

$=(1-\phi(z)-\phi(p)+\phi(zp))+((\phi p)^2-\phi p)$

This gives $(\phi((1-z)(1-p)))^2-\phi((1-z)(1-p)) = (\phi p)^2-\phi p$.

Therefore ϕp is idempotent if and only if ϕq is idempotent, where q is the projection q=(1-z)(1-p). Let z' be a central projection in A. If z'(1-z)=0 then z'q=0 as q<1-z. If z'q<0 is finite, then z'(1-z)p must be infinite as z'q+z'(1-z)p=z'(1-z) which is either infinite or zero and the sum of two orthogonal finite projections is finite since z'(1-z)p is subprojection of the finite projection (1-z)p it follows that z'q is either zero or infinite. By means of this, q is properly infinite whence ϕq is an idempotent by the second part of the proof. Therefore, ϕp is an idempotent.

Finally, suppose that p is finite. Then, 1-p is infinite wherefore $\phi(1-p)=1-\phi p$ is idempotent, which completes the proof.

Lemma 3.5. Let A and B be semisimple Banach algebras. Assume that $\phi: A\rightarrow B$ is a unital surjective map which preserves invertibility. Then T is a Jordan homomorphism.

Proof. Let p_1,p_2 in A be orthogonal Hermitian idempotent since p_1+p_2 is a projection, we have $(\phi(p_1)+\phi(p_2))^2 = \phi(p_1)+\phi(p_2)$ This yields $\phi(p_1)\phi(p_2)+\phi(p_2)\phi(p_1)= 0$. It follows that if H in A is of the form H= $\sum_{j=1}^{n} tjpj$ where t in R and pj are Hermitian idempotents such that pipj=0 if i<j, then $\phi(H^2)=\phi(H)^2$.The set of all Hermitian elements that can be represented as finite real linear combination of mutually orthogonal projections is dense in the set of all Hermitian elements in A. Therefore we have $\phi(H^2)=(\phi(H))^2$ for every Hermitian element H in A. Now replace H by H+K where H and K are both Hermitian, we get (HK + KH)= $\phi(H)\phi(K)+\phi(K)\phi(H)$ since an arbitrary A in A can be written in the form A= H +iK with H,K Hermitian, which imply that $\phi(A^2)=(\phi A)^2$.

Theorem 3.6. Let A and B be semisimple Banach algebras. Let $\phi: A\rightarrow B$ be a unital surjective bounded linear map preserving Jordan zero products. Then ϕ is a Jordan homomorphism J from A onto B such that $\phi(A)=\phi(1)J(A)$ if $\phi(1)$ is an invertible central element of B.

Proof. Let $\phi: A\rightarrow B$ be a bounded linear map such that (TS)$\phi(T)+\phi(T)\phi(S)=0$ for S,T in A with ST+TS=0. Then for any S in A we have $\phi(1)\phi(S)^2 = \phi(S)^2\phi(1)$, $\phi(1)\phi(S)^2+\phi(S)^2\phi(1) = 2\phi(S)^2$. Replacing S by S+T with S,T in A we have

$\phi(1)\phi(S)\phi(T) +\phi(T)\phi(S) = \phi(S)\phi(T)+\phi(T)\phi(S)\phi(1)$

$\phi(1)\phi(ST+TS)+\phi(ST+TS)\phi(1)=2(\phi(S)\phi(T)+\phi(T)\phi(S))$

For each A in A, write A=S+iT with S,T in A. Applying above equations and the linearity of ϕ, we get $\phi(1)\phi(A)^2 = \phi(A)^2\phi(1)$…………(2)

and

$\phi(1)\phi(A^2)+\phi(A^2)\phi(1) = 2\phi(A)^2$ …….(3)

hold for all A in A. Since every element in a semisimple Banach algebra in an algebraic sum of square elements and ϕ is surjective, from (ii) we know that $\phi(1)$ is the center of B. Hence it follows from (2) that $\phi(1)B=B$. In particular, $\phi(1)E=\phi(1)$ for some E in B. So, $\phi(A)^2E = \phi(A^2)\phi(1)E=\phi(A^2)\phi(1)=\phi(A)^2$ for all A in A. Thus BE=B for all B in B. Similarly, EB=B for all B in B. This implies that B is unital with unit E and it follows from $\phi(1)B=B$ that $\phi(1)$ is invertible. Let $J(A)=\phi(1)^{-1}\phi(A)$ for all A in A, then it is easy to verify that J is surjective Jordan homomorphism from A onto B.

Theorem 3.7. Let H and K be Banach algebras and A, B be semisimple Banach algebras on H and K respectively. Let ϕ :A$\rightarrow$B be a unital surjection then ϕ is a Jordan homomorphism if there exists a non-zero scalar c and an invertible bounded linear or conjugate linear operator U:H$\rightarrow$K such that either ϕ(A)=cUAU^{-1},for all A in H or ϕ(A)=cUA^{-1} U^{-1} for all A in H.

Proof. ϕ is injective.

Step 1:ϕ preserves idempotents and rank-one idempotents in both directions. If p in A is an idempotent, then p(1-p)+(1-p)p=0. This implies ϕ(p)(1-ϕ(p))+(1-ϕ(p))ϕ(p)=0, that is ϕ(p)=ϕ(p)^2. Consequently, ϕ(p) is an idempotent. Suppose that p is rank-one while ϕ(p) is not rank one. Then ϕ(p) can be written as a sum of an idempotent and a rank-one idempotent in B. Since ϕ^{-1} satisfies the same property as ϕ. Rank-one idempotent p can also be written as a sum of two non-zero idempotents. This is a contradiction.

Step 2: ϕ preserves rank-one operators in both directions. In particular,ϕ preserves rank-one nilpotent in both directions. Let p be an idempotent of rank-one, then for every non-zero λ in C, we have λ p)(1-p)+(1-p)(λp)=0 which implies that 2ϕ(λp)=ϕ(λ p)ϕ(p)+ϕ(p)ϕ(ϕp). Since ϕ(p) is a rank-one idempotent, we obtain ϕ(λp)ϕ(p)=ϕ(p)ϕ(λp)ϕ(p)=ϕ(p)ϕ(λ p). It follows that ϕ(λp)= ϕ(p) is of rank-one. Especially, there exists fp(λ) in C such that ϕ(λp)= fp(λ)ϕ(p). If A=x$\otimes$f is a nilpotent of rank-one, then there exists f_1 in H' such that f_1(x)=1. Let f_2=f_1-f. Then p_i=x$\otimes f_i$ (i=1,2) are rank-one idempotents and A=p_1-p_2=x$\otimes f_1$-x$\otimes f_2$.

Suppose that ϕ (p_i)=$y_i\otimes g_i$ by step 1, $g_i(y_i)$=1. Notice that p=$\frac{1}{2}$(p_1+p_2) is a rank-one idempotent so ϕ(p)=$\frac{1}{2}$($y_1\otimes g_1$)+($y_2\otimes g_2$) is a rank-one idempotent. Then either y_1,y_2 are linearly dependent or g_1,g_2 are linearly dependent. Assume y_1=y_2=y, then ϕ(A)=y$\otimes g_1$-y$\otimes$ g_2 which is a nilpotent of rank-one.

Step 3: Either

(i) there exists a bijective bounded linear or conjugate linear U:H$\rightarrow$K such that ϕ(A)=UAU^{-1} for every finite rank operator A in H or

(ii) There exists a bijective bounded linear or conjugate linear operator U:H'$\rightarrow$ K such that ϕ(A)=UA'U^{-1} for every finite rank operator A in H. In this case H and K are reflexive.

Since ϕ is additive and preserves rank-one operators, rank-one idempotent and rank-one nilpotent in both directions.

Step 4: for every operator A in A a rank-one idempotent R in H, ϕ(RAR)=ϕ(R)ϕ(A)ϕ(R). By *Step 3*, for every finite rank operator A_0 in H, we have ϕ(RA $_0$R)=ϕ(R)ϕ($A_0\phi$(R)). We have to prove that above equation holds for every A in A. Let R=Z$\otimes$h and p in H with p=x$\otimes$f a rank-one idempotent, where x,Z in H and f,h in H'. Then there exist nilpotents s=x$\otimes$y and T=y$\otimes$f of rank one with y in H, g in H' such that p=sT. Furthermore, Q-Ts=y$\otimes$g is an idempotent of rank-one disjoint with p, and R is a linear combination of p,Q,S and T. For every A in A, let B=(1-p-Q)A(1-p-Q), then we have pB=QB=sB=TB=0 and Bp=BQ=BS=BT=0. Consequently, RB=BR=0. By the property ofϕ, we get ϕ(R)ϕ(B)+ϕ(B)ϕ(R)=0. Since ϕ(R) is an idempotent, a simple computation shows that ϕ(R)ϕ(B)ϕ(R)=0. Using the fact that A-B is of finite rank, we get ϕ(RAR)=ϕ(R(A-B)R)=ϕ(R)ϕ(A-B)ϕ(R)=ϕ(R)ϕ(A)ϕ(R).

Step 5: Either ϕ(A)=UAU^{-1} for every A in A or ϕ(A)=UA'U^{-1} for every A in A.

Suppose that for the operator of finite rank the case (i) of step 3 holds. Let A in A, for any z in H and y in H' and y(z)=1, R=z$\otimes$y in H is an idempotent of rank-one and by step 4, we have t(y(Az)URU^{-1}=t(y(U$^{-1}\phi$(A)Uz))URU^{-1} where t is an identity or the conjugation of C. This yields y(Az)=y(U$^{-1}\phi$(A)Uz) ....(4)

Fix z for a moment. Then (iv) holds for every y in H' with y(z)=1 and so, for every y in H' by linearity. Thus Az=U$^{-1}\phi$(A)Uz is valid for every z in H and the case (1) of the theorem is proved.

Now we assume that case (ii) of step 3 holds for every operator of finite rank. Then for every z in H and y in H' with y(z)=1 by step 4 we get t(y(Az)U(x$\otimes$y)'U^{-1}=t(h((U$^{-1}\phi$(A)U')z)) and therefore y(Az)=y((U$^{-1}\phi$(A)U)'z). Using similar arguments, as above, we obtain A=(U$^{-1}\phi$(A)}U)'. Consequently, the case (2) of the theorem holds true.

Theorem 3.8. Let H and K be Banach algebras and A and B be semisimple Banach algebras on H and K respectively. Let ϕ :A$\rightarrow$B be a unital surjection then ϕ is a Jordan homomorphism if either;

(i) There exists a bijective bounded linear or conjugate linear operator U:H$\rightarrow$ K such that ϕ(A)=UAU^{-1} for all A in A.

(ii) There exists a bijective bounded linear or conjugate linear operator U:H'$\rightarrow$ K such that ϕ(A)=$UA'U^{-1}$ for all A in A. In this case H and K are reflective.

Proof. Let p in B(H) with p^2=p since p(1-p)+(1-p)p=0, we have ϕ(p)ϕ(1-p)+ϕ(1-p)ϕ(p)=0 and consequently ϕ(1)ϕ(p)+ϕ(p)ϕ(1)=2ϕ(p)2. Thus we have $\phi(p)^2\phi(1)+\phi(p)\phi(1)\phi(p)=2\phi(p)^3$ and $\phi(1)\phi(p)^2+\phi(p)\phi(1)\phi(p)=2\phi(p)^3$. These together imply that $\phi(1)\phi(p)^2=\phi(p)^2\phi(1)$.

Similarly, it follows from $\phi(1)^2\phi(p)+\phi(1)\phi(p)\phi(1)=2\phi(1)\phi(p)^2$ and $\phi(p)\phi(1)^2+\phi(1)\phi(p)\phi(1)=2\phi(p)^2\phi(1)$ that is $\phi(p)\phi(1)^2=\phi(1)^2\phi(p)$.

Since every infinite-dimensional Banach algebras has infinite multiplicity then every bounded linear operator on an infinite-dimensional Banach algebra is an algebraic sum of finite many idempotents. Hence we have $\phi(A)\phi^2=\phi(1)^2\phi(A)$holds for every A in H. Therefore, by surjectivity of ϕ(1)^{2} = λ1 for some scalar λ. Let T,S in H with ST=0 for any idempotent p, it follows from Tp(1-p)S+(1-p)STP=0 and that ϕ(TP)ϕ(S)+ϕ(S)ϕ((1-P)S)ϕ(TP)=0.

Thus ϕ(TP)ϕ(S)+ϕ(S)ϕ(TP)=ϕ(TP)ϕ(PS)+ϕ(PS)ϕTP)...(5) holds for every idempotent p. On the other hand T(1-p)pS+pST(1-p)=0 implies that ϕ(T(1-p))ϕ(pS)+ϕ(pS)ϕ(T(1-p))=0 and hence ϕ(T)ϕ(pS)ϕ(T)=ϕ(Tp)ϕ(pS)+ϕ(pS)ϕ(Tp)..(6) for every idempotent p. Combining (5) and (6) we get ϕ(Tp)ϕ(S)+ϕ(S)ϕ(Tp)=ϕ(T)ϕ(pS)+(pS)ϕ(T). For every idempotents p. Hence for every A in H ϕ(TA)θ(S)+ϕ(TA)=ϕ(T)ϕ(AS)+ϕ(AS)ϕ(T)................................(7)

Take T=Q and S=1-Q for some Q in H with Q^2=Q. Then ST=O and from (iii) we get ϕ(QA)ϕ(1-Q)+ϕ(1-Q)ϕ(QA)=ϕ(Q)ϕ(A(1-Q))+ϕ(A(1-Q))ϕ(Q). Thus we see that ϕ(QA)ϕ(1)+ϕ(1)ϕ(QA)-ϕ(Q)ϕ(A)-ϕ(A)ϕ(Q)=ϕ(QA)ϕ(Q)+ϕ(Q)ϕ(QA)-ϕ(Q)ϕ(AQ)-ϕ(AQ)ϕ(Q)

On the other hand, taking T=1-Q and S=Q we obtain from (iii) another equation ϕ(1)ϕ(AQ)+ϕ(AQ)ϕ(1)-ϕ(A)ϕ(Q)-ϕ(Q)ϕ(A)=ϕ(Q)ϕ(AQ)+ϕ(AQ)ϕ(Q)-ϕ(QA)ϕ(Q)-ϕ(Q)ϕ(QA).

Hence ϕ(QA+AQ)ϕ(1)+ϕ(1)ϕ(QA+AQ)=2(ϕ(Q)ϕ(A)+ϕ(A)ϕ(Q)) holds for every idempotent Q. This further implies that ϕ(AB+BA)ϕ(1)+ϕ(1)ϕ(AB+BA)=2(ϕ(A)ϕ(B)+ϕ(B)ϕ(A).....(8) holds for every B in H multiplying (iv) from left and right by ϕ(1) respectively, we see that $\phi(1)^2\phi$(AB+BA)+ϕ(1)ϕ(AB+BA)ϕ(1)=2ϕ(1)(ϕ(A)ϕ(B)+ϕ(B)ϕ(A)) and ϕ(1)ϕ(AB+BA)ϕ(1)+ϕ(AB+BA)$\phi(1)^2$=2(ϕ(A)ϕ(B)+ϕ(B)ϕ(A)ϕ(1)). These two equations, together with the fact that $\phi(1)^2=\lambda 1$, entail that ϕ(1)(ϕ(A)ϕ(B)+ϕ(B)ϕ(A))=(ϕ(A)ϕ(B)+ϕ(B)ϕ(A))ϕ(1)..........(9)

Let A=B in (iv) and (v) then $\phi(1)\phi(A^2)+\phi(A^2)\phi(1)=2\phi(A)^2$.......(10) and $\phi(1)\phi(A)^2=\phi(A)^2\phi(1)$......................(11)

By surjectivity of ϕ, equation (7) implies that ϕ(1) commutes with all idempotent operators and hence there must exist a scalar u such that ϕ(1)=u1 while equation (vi) shows that u<0. Let c=$\frac{1}{u}$ and ψ(.)=cψ(.), then ψ :H$\rightarrow$ K is an additive and surjective preserving Jordan zero products and ψ(1)=1. Moreover for every A in H, $\psi(A^2)=\psi(A)^2$ which implies that ψ is a Jordan homomorphism. Since K is prime, we can see that ψ is either a ring homomorphism or a ring anti-homomorphism. Therefore ψ is a scalar multiple of a surjective ring homomorphism or a surjective ring anti-homomorphism.

We will show that θ is injective. Without loss of generality we assume that θ is a surjective ring homomorphism. We first claim that the null space of θ is closed. For every $0 \neq y$ in K, define $\ker_y(\phi)$={T in H| $\phi(T)_y=0$} which is a left ideal of H and $\ker(\phi)$= I ker $_y(\phi)$. If L is a left ideal such that $\ker_y\phi$ is a proper subset L, then ϕ (L)y is a non-zero invariant linear manifold K. It follows that ϕ(L)y=K. So, there exists T in L such that ϕ(T)y=y for any s in H we have s-sT in ker $_y(\phi) \subset$L. This implies that s in L since sT in L. Therefore, we have L=H and consequently, $\phi_y(\phi)$ is closed and hence $\ker(\phi)$ is closed. Note that the set of ring two-sided ideals, coincides with the set of algebraic two-sided ideals in H. Thus if ϕ is not injective, then the kernel of ϕ is a closed two-sided ideal which contains the ideal consisting of all compact operators. Suppose the dimension of H is N_H, which is an infinite cardinal number $N \leq N_H$, let 1_N={T in H |dim M<N holds for all closed linear subalgebras $M \subseteq$ range (T)}.

Then I_N is a closed two-sided ideal of H and every closed two-sided ideal of H arises in this way. In particular, I_{NH} is the largest one. Therefore, ϕ induces a ring isomorphism from the quotient algebra H/kerϕ onto K. This implies that there is an element A in H such that A+kerϕ is a single element of H/kerϕ . An element T in an semisimple Banach algebra A is single if, for any S,R in A, STR=0 will imply ST=0 or TR=0. For an semisimple Banach algebra A there exists a representation (π,H) of A such that an element T in A is a single element if and only if π(T) is of rank one on H, and consequently, dim TAT=1. Hence (A+kerϕ)H(A+ kerϕ)=AB(H)A + kerϕ is of dimension one modulo kerϕ. Let $N \leq N_H$ be the infinite cardinal number such that kerϕ $=I_N$. Then the range of A contains a close subalgebra of dimension N. By halving the subalgebra into two; each of dimension N, we see that AHA contains two elements linearly independent modulo I_N a contradiction. So, ϕ is injective. Hence we have shown that ϕ is a scalar multiple of a ring isomorphism or a ring anti-isomorphism from H onto K. Thus ϕ is a unital surjective Jordan homomorphism.

Theorem 3.9. Let A and B be semisimple Banach algebras. Let ϕ:A$\to$B be a Jordan homomorphism. Then ϕ is a unital surjective spectral isometry if it preserves commutativity and numerical radius.

Proof. ϕ is invertible. Suppose a in A and let A_1= ⟨a,1⟩ be the closed sub-algebra of A generated by a and 1. Define a linear map ϕ_1:$A_{-1} \to \phi(A_{-1})$ by $\phi_1(x)=\phi$ (x) for all x in A_1. Suppose $\phi(A_1)$ is a subalgebra of B since A_1 is commutative and ϕ preserves commutativity so $\phi(A_1)$ is commutative. Also ϕ_1, preserves numerical radius, therefore ϕ_1 is a Jordan homomorphism, so $\phi(a^2)=\phi(a)^2$. Otherwise let B_1=⟨$\phi(A_1)$⟩ and define a linear map T_1:$B_1 \to T^{-1}B_1$ by $T(y)=T^{-1}(y)$ for all y in B_1, suppose $\phi^{-1}(B_1)$ is a subalgebra of A, since B_1 is commutative, and T^{-1} is numerical radius preserving, therefore T^{-1} is a homomorphism and hence $T(y^2)=T_1(y)^2$, then $T_1(\phi(a)^2)=T_1(\phi(a))^2=a^2$ therefore $\phi(a)^2=\phi(a^2)$. Otherwise, let A_2=⟨$\phi^{-1}(B_1)$⟩ and define a linear map ϕ_2:$A_2 \to \phi(A_2)$ by $\phi_2(x)=\phi(x)$ for all x in A_2. By continuing the process we obtain sequences A_n and B_n commutative subalgebras of A and B respectively such that A_1=⟨a,1⟩, A_n=⟨$\phi^{-1}(B_{n-1})$⟩ B_n= ⟨$\phi(A_n)$⟩ $A_1 \subseteq A_2 \subseteq \ldots \subseteq$A and $B_1 \subseteq B_2 \subseteq \ldots \subseteq$ B. Define A'=$\cup A_n$ and B'=$\cup B_n$ and ϕ' :A'$\to$ B' by $\phi'(x)=\phi(x)$ for every x in A'. A' and B' are commutative and ϕ' is a unital surjective spectral isometry, so ϕ' is Jordan homomorphism and hence $\phi'(a^2)=\phi'(a)^2$, therefore $\phi(a^2)=\phi(a)^2$.

Theorem 3.10. Let ϕ:A$\to$B be a unital surjective spectral isometry between semisimple Banach algebras A and B. Then ϕ is Jordan homomorphism if there is a unitary u in Z(B) and a unital surjective spectral isometry ϕ_1:A$\to$B such that $\phi_a=u\phi_1 a$, a in A.

Proof. Put u=ϕ_1 which is a unitary and set $\phi_1 a=u^{-1}\phi a$, a in A. Since u is central for each a in A, $r(\phi_1 a)<r(u^{-1})r(\phi_a)=r(\phi_a)=r(a)=r(uu^{-1}\phi_a)<r(u)r(u^{-1}\phi_a)= r(\phi_1 a)$ hence ϕ_1 is a unital surjective spectral isometry.

Theorem 3.11. Let A and B be simisimple Banach algebras. Let ϕ:A$\to$B be unital surjective spectral isometry. Then θ is a Jordan homomorphism if ϕ rad(A)=rad(B).

Proof. Take a in rad(A) and y in B such that r(y)=0. Choose x in A with y=ϕx then, r(x)=r(y)=0 it follows that r(ϕa+y)=r(ϕ(a+x))=0 so that ϕa in rad(B). Conversely, take b in rad(B) and let a in A be such that b=ϕa. Let x in A be quasinilpotent. Then r(a+x)=r(ϕ(a+x))= r(b+ϕx)=0. Since ϕx is quasinilpotent. It follows that a in rad(A) therefore b in rad(A). We conclude that ϕrad(A)=rad(B).

Remark 3.12. In particular if both A and B are commutative unital semisimple Banach algebras and ϕ is a unital surjective spectral isometry from A to B, then by Gelfand theorem r(a)=||a|| and r(ϕ(a))=||ϕ(a)|| so r(a)=||a||=v(a) and r(ϕ(a))= ||ϕ (a)||=v(ϕ(a)) for all a in A, therefore r(a)=r(ϕ(a)) and we can use Nagasawa theorem.

Conclusions

Certain properties of operator algebras have been studied such as boundedness, positivity, surjectivity, linearity, invertibility, numerical range, numerical radius and idempotent property. Jordan homomorphisms have been studied by several scholars such as Mathieu, Sorour, Semrl, Braser among others. For instance, Kazempour showed that a linear map on two Banach algebras is a Jordan homomorphism and multiplicative. Furthermore, Martin and Gerhard showed that Jordan homomorphisms between Von Neumann algebras are spectrally bounded. However studies on spectral characterization on semisimple Banach algebras have been done but to a little extent. It is therefore necessary to characterize Jordan homomorphisms on semisimple Banach algebras in terms of their spectrum. This work established that Jordan homomrphism is a unital surjection, spectral isometry and unital surjective spectral isometry on semisimple Banach algebras.

Conflicts of interest

Authors declare no conflict of interest.

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