Math 4315 - PDE's

Solving Non linear 1^{st} order PDEs

$$F(x,y,u,u_x,u_y)=0$$

subject to some boundary condition $u(x,f(x))=g(x)$

the characteristic eq^n's (CE) are:

$$x_s = F_p$$
$$y_s = F_q$$
$$u_s = pF_p + qF_q$$
$$p_s = -F_x - pF_u$$
$$q_s = -F_y - qF_u$$

where F_x, F_y, F_u, F_p & F_q are derivatives of

$$F(x,y,u,p,q)=0$$

where we have let

$$u_x = p,\quad u_y = q$$

The following examples illustrate.

Solve $u_x u_y = 1$ subject to $u(x,0)=x$

so 1st we define F.

$$F = pq - 1$$

Next Calculate derivatives of F

$$F_x = 0, \quad F_y = 0, \quad F_u = 0, \quad F_p = q \quad F_q = p$$

Then construct the CEs

$x_s = q$ from $pq = 1$ the original PDE

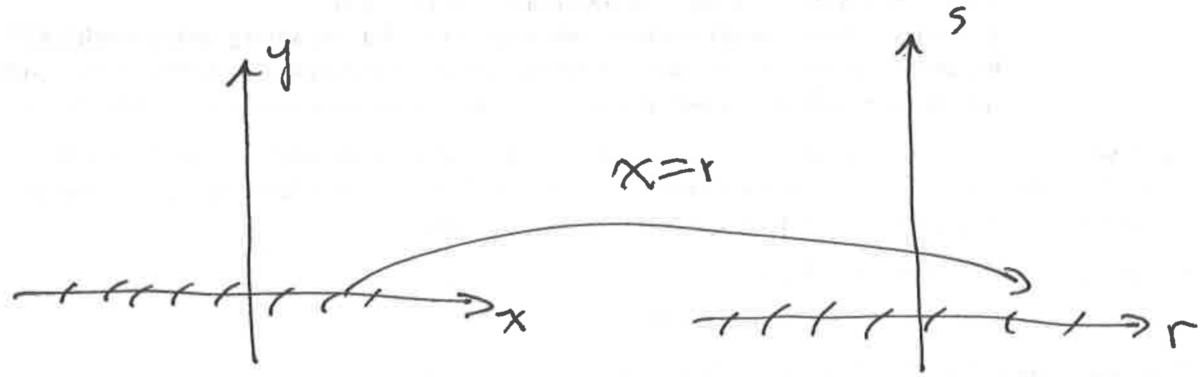
$y_s = p$

$u_s = p \cdot q + q \cdot p = 2pq = 2$

$p_s = 0$

$q_s = 0$

Integrating these will give to 5 arbitrary functions
To find these we need to bring in our BC.



here $y=0$ we pick $s=0$

From $u(x,0)=x$ gives on $s=0$ $x=r, y=0, u=r$

we now need 2 more one for p and one for q if $u(x,0) = x$ then we differeniate wrt x

then $\frac{\partial u}{\partial x}(x,0) = 1$ we ~~will~~ will disuss later what we do on arbitrary bandories $u(x, f(x)) = g(x)$

so on $s=0$, $p=1$

Now we torn to our PDE. $pq = 1$

on the bandary $pq = 1$ still

and since $p = 1 \Rightarrow q = 1$

so now we solve

$x_s = q$ $\quad s=0$
$y_s = p$ $\quad x = r$
$u_s = 2$ $\quad y = 0$
$p_s = 0$ $\quad u = r$
$q_s = 0$ $\quad p = 1$
$\quad q = 1$

Sol^n

(1) $p_s = 0 \Rightarrow p = a(r)$
$s=0$ $p=1 \Rightarrow a = 1$ so

$\boxed{p=1}$

(2) $q_s = 0 \Rightarrow q = b(r)$
$s=0$ $q=1 \Rightarrow b(r) = 1$ so

$\boxed{q=1}$

(3) $u_s = 2 \Rightarrow u = 2s + c(r)$ $\quad s=0$ $u=r \Rightarrow c(r) = r$

$\Rightarrow \boxed{u = 2s + r}$

4) $x_s = q = 1$ $\quad x = s + d(r)$ $\quad s = 0\ x = r \Rightarrow d(r) = r$

$\Rightarrow \boxed{x = s + r}$

5) $y_s = p = 1$ so $y = s + e(r)$ $\quad s = 0\ y = 0 \Rightarrow e(r) = 0$

so $\boxed{y = s}$

Now we have the sol^n parametrically

$$x = s + r, \quad y = s, \quad u = 2s + r$$

and eliminating r & s gives $\quad u = x + y$ – the sol^n

Ex 2 $\quad x u_x - u_y^2 = u \qquad u(x, 0) = -x^2 + x$

so $\quad F = xp - q^2 - u$

$F_x = p, \ F_y = 0, \ F_u = -1, \ F_p = x \ \ F_q = -2q$

CE: $\quad x_s = F_p = x \qquad\qquad p_s = -F_x - pF_u$

$\qquad\qquad\qquad\qquad\qquad\qquad = -p - p(-1) = 0$

$y_s = F_q = -2q$

$u_s = pF_p + qF_q \qquad\qquad q_s = -F_y - qF_u$

$\quad = xp - 2q^2 \qquad\qquad\qquad = 0 - q(-1)$

$\qquad\qquad\qquad\qquad\qquad\qquad = q$

so the CE we need to solve are

	BC.	
$x_s = x$	$s=0$	to find B.C. for p & q
$y_s = -2q$	$x=r$	we use the BC
$u_s = xp - 2q^2$	$y=1$	$u(x,1) = x^2+x$
	$u = r^2+r$	
$p_s = 0$		then $u_x(x,1) = 2x+1$
	$p=?$	
$q_s = q$	$q=?$	so $p = 2r+1$

Now go to PDE. On the boundary

$xp - q^2 - u = 0$ becomes $r(2r+1) - q^2 - (r^2+r) = 0$

$$\Rightarrow \quad 2r^2 + r - q^2 - r^2 - r = 0$$

$$\Rightarrow \quad q^2 = r^2 \Rightarrow \quad q = \pm r \quad \text{two cases}$$

will only consider the 1st & give the sol^n of the second at the end

Now we integrate

(1) $x_s = x \Rightarrow x = a(r)e^s \quad s=0 \; x=r \Rightarrow a=r$

$$\Rightarrow \boxed{x = re^s}$$

(2) $p_s = 0 \cdot p = b(r)$ $\quad s=0 \quad p = 2r+1 \Rightarrow b = 2r+1$

so $\boxed{p = 2r+1}$

(3) $q_s = q$ so $q = c(r)e^{s}$ $\quad s=0 \quad q = r \Rightarrow c = r$

so $\boxed{q = re^{s}}$

(4) $y_s = -2q = -2re^{s}$ so $y = -2re^{s} + d(r)$

$s=0 \quad y=1 \Rightarrow \quad 1 = -2r + d \Rightarrow d = 2r+1$

so $\boxed{y = -2re^{s} + 2r + 1}$

(5) $u_s = xp - 2q^2$ we could bring in PDE but we'll just bring in x, p & q

$= re^{s}(2r+1) - 2r^2e^{2s}$

$u = r(2r+1)e^{s} - r^2e^{2s} + e(r)$

$s=0 \quad u = r^2 + r$ so $\quad 2r^2 + r - r^2 + e = r^2 + r \Rightarrow e = 0$

so $\boxed{u = r(2r+1)e^{s} - r^2e^{2s}}$

So now we have

$$x = re^s, \quad y = -2re^s + 2r + 1$$

$$u = \underset{\substack{\uparrow \\ x}}{(2r+1)re^s} - \underset{\substack{\uparrow \\ x}}{(re^s)^2}$$

Note: From x & y $\quad y + 2x = 2r + 1$

So $\quad u = (y+2x)x - x^2$

$$u = xy + x^2 \leftarrow \text{the sol}^n$$

the second sol^n is $\quad u = x(2-y) + x^2$

Next Class we'll consider example for more general boundary

Ex $\quad u(x, x) = x$

$$u(x, 1-x) = x^2$$

Note: From Calc 3 Chain Rule $\quad u = f(x, y) \quad x = g(r), \ y = h(r)$

$$\frac{du}{dr} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dr}$$