Math 6345 - AODE's

Bifurcation Theory

We now consider ODEs with parameter.

Specifically, we study how critical pts change

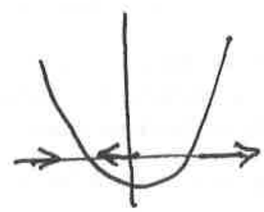
1st order

$$\dot{x} = f(x, \lambda) \quad \lambda \in \mathbb{R}$$

Ex $\quad \dot{x} = x^2 + \lambda$

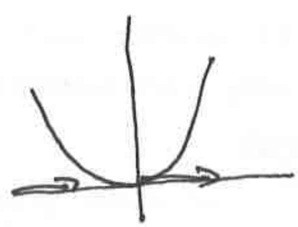
possibilities $\lambda < 0$, $\lambda = 0$, $\lambda > 1$

Ex $\quad \dot{x} = x^2 - 1 \quad$ CP $x = \pm 1$



2 critical pt
1 stable
1 unstable

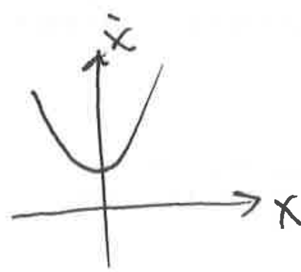
$\dot{x} = x^2 \quad$ CP $\Rightarrow x = 0$



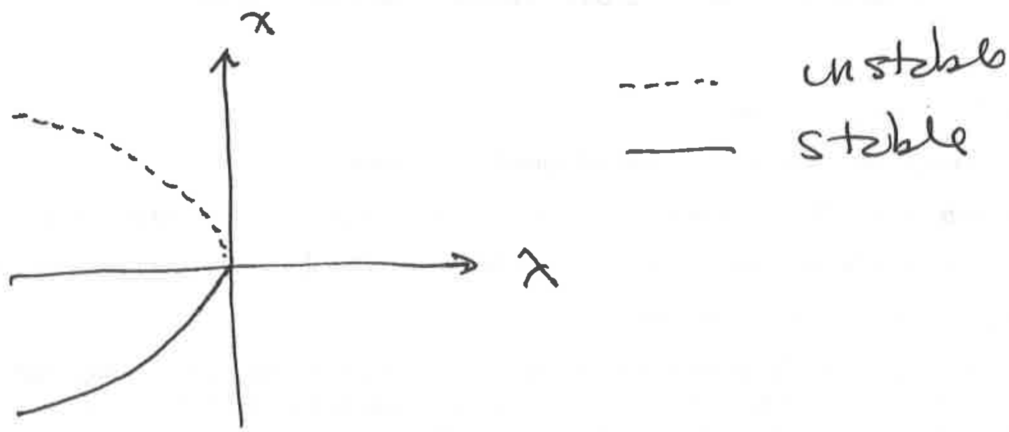
turns to 1 unstable

$\dot{x} = x^2 + 1 \quad$ NO CP

then vanishes



Bifurcation Diagram



This bifurcation is called a <u>saddle-node</u> bif.
the name will be relevant in 2D

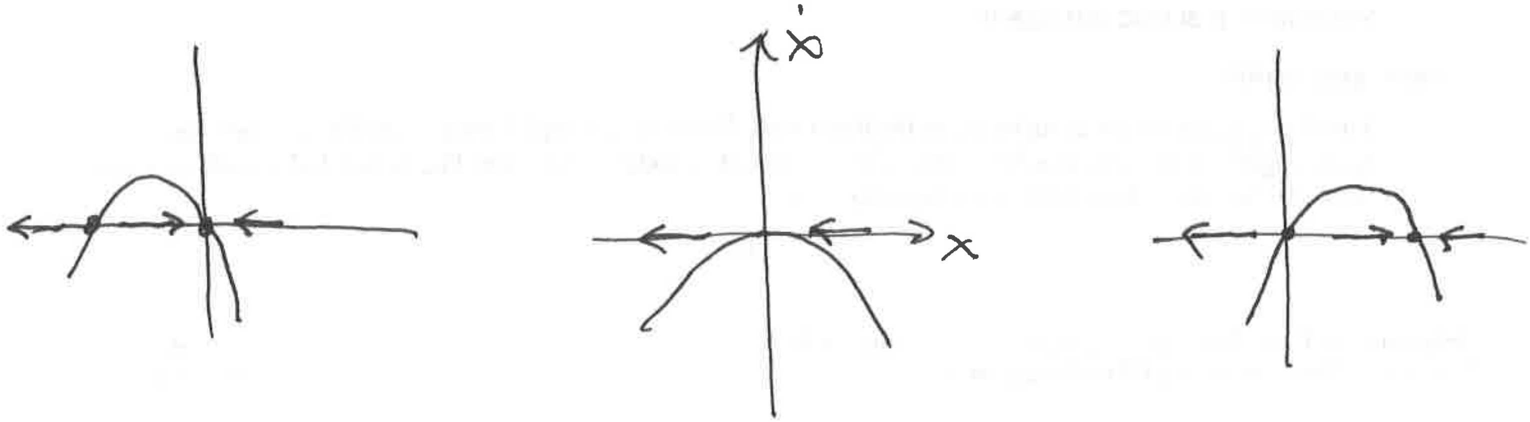
2. <u>Transcritical</u>

stability exchanges

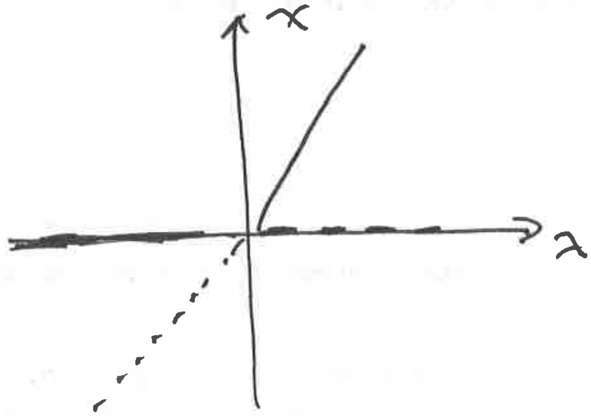
$\dot{x} = \lambda x - x^2$ CP $x=0$, $x=\lambda$

$\lambda < 0$ $\lambda = 0$ $\lambda > 0$

$\dot{x} = -x - x^2$ $\dot{x} = -x^2$ $\dot{x} = x - x^2$



Bifurcation Diagram



3. Pitchfork Bifurcation

$\dot{x} = \lambda x - x^3$ "supercritical"

$\lambda < 0$

$\dot{x} = -x - x^3$

$= -x(1+x^2)$

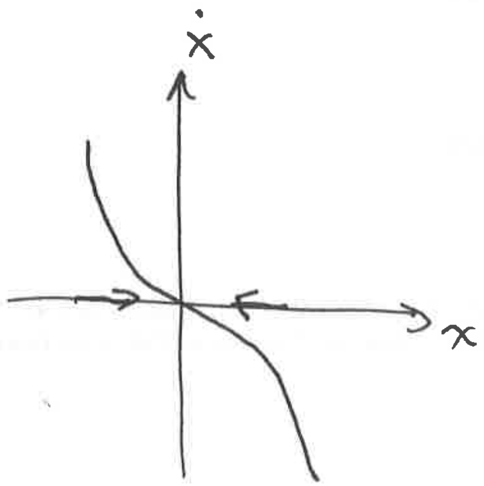
$\lambda = 0$

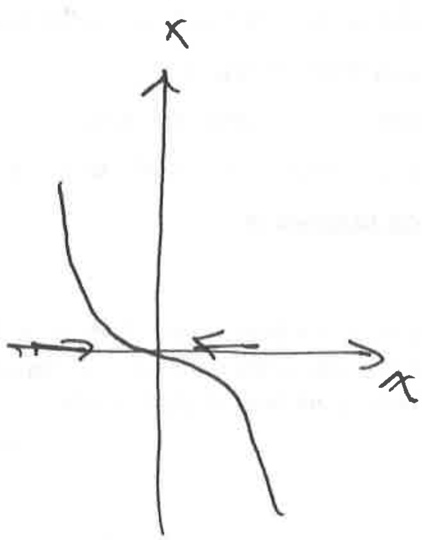
$\dot{x} = -x^3$

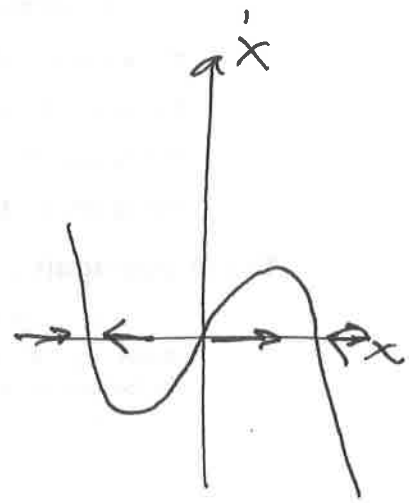
$\lambda > 0$

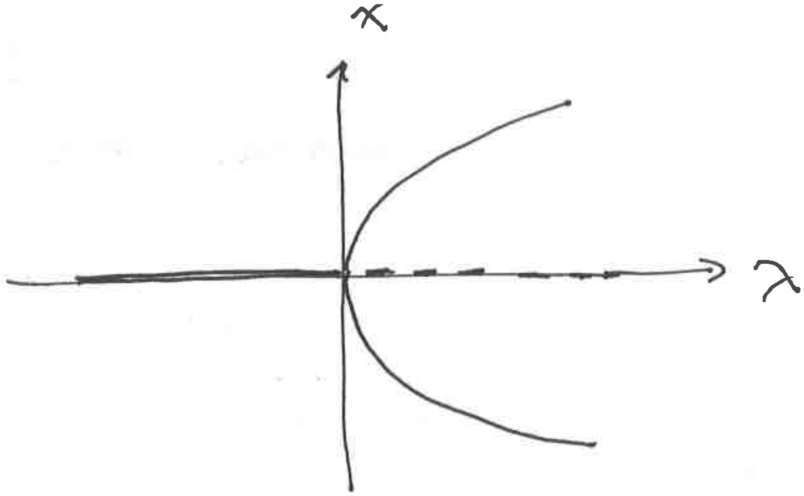
$\dot{x} = x - x^3$

$= x(1-x)(1+x)$





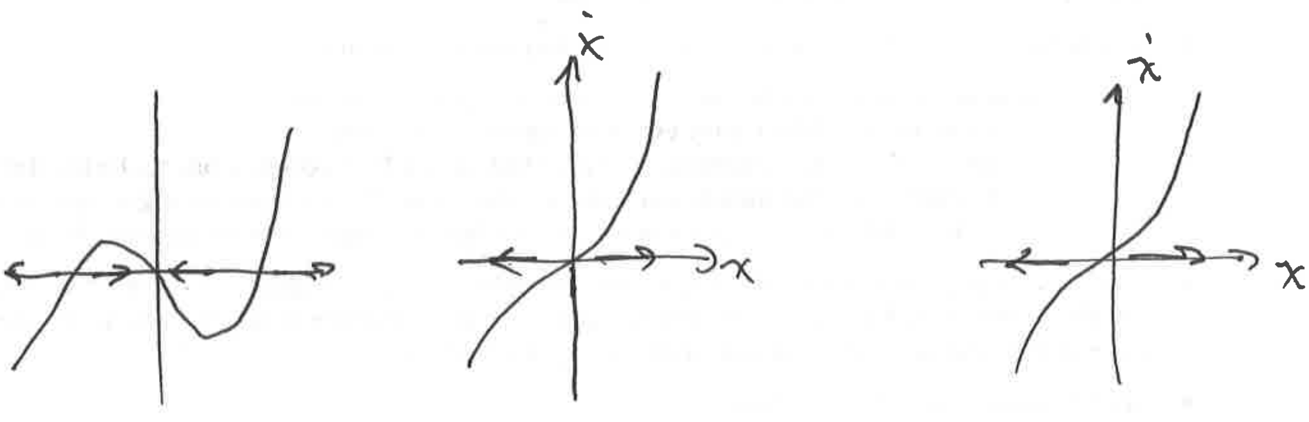


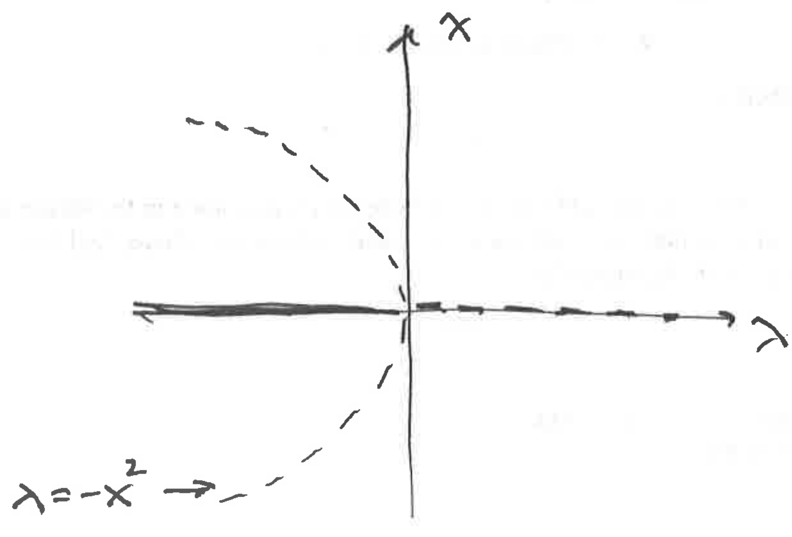


$\dot{x} = \lambda x + x^3$ "subcritical"

$\lambda < 0$ $\lambda = 0$ $\lambda > 0$

$\dot{x} = +x(x^2-1)$ $\dot{x} = x^3$ $\dot{x} = x(1+x^2)$





---- unstable

—— stable

Can things get more complicated?

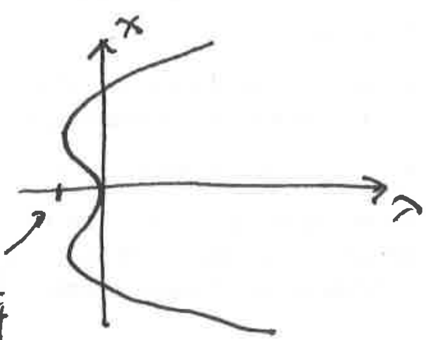
ex $\dot{x} = \lambda x + x^3 - x^5$

$= x(\lambda + x^2 - x^4)$

so CP $x = 0$ and if $\lambda + x^2 - x^4 = 0$

so $x^4 - x^2 - \lambda = 0$

$\therefore \; x^2 = \dfrac{1 \pm \sqrt{1+4\lambda}}{2}$



so if $1 + 4\lambda < 0$ roots 1 real 4 complex

$\lambda = -\frac{1}{4}$ roots 5 real 2 repeated

$0 < 1 + 4\lambda < 1$

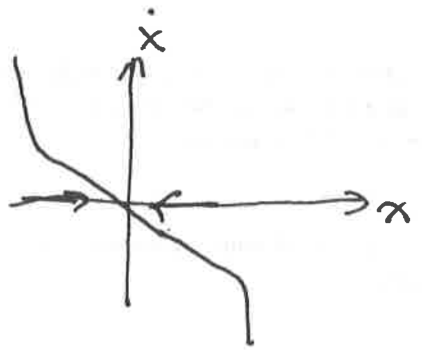
or $-\frac{1}{4} < \lambda < 0$ 5 real distinct roots

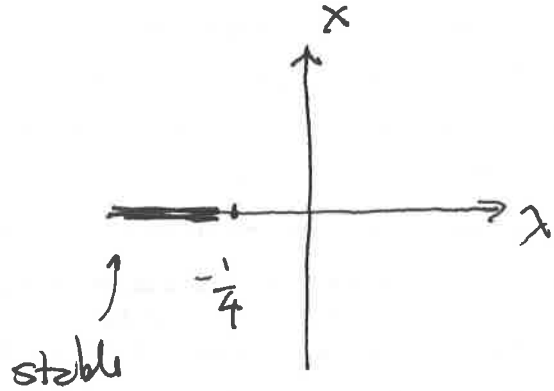
$1 + 4\lambda = 1$ 5 real 1 repeated

$1 + 4\lambda > 1$ 3 real distinct 2 complex we will look at case for each

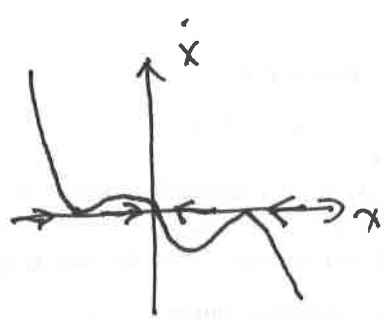
① $\lambda < -\frac{1}{4}$ pick $\lambda = -1$

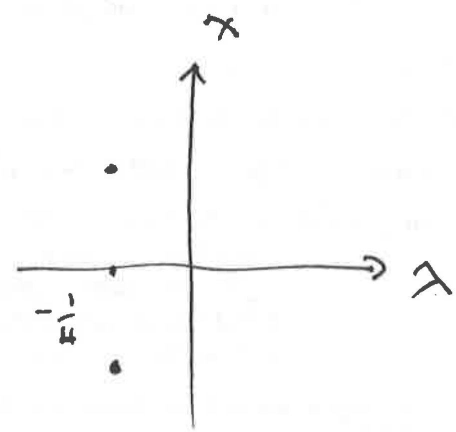
so $\dot{x} = -x + x^3 - x^5$



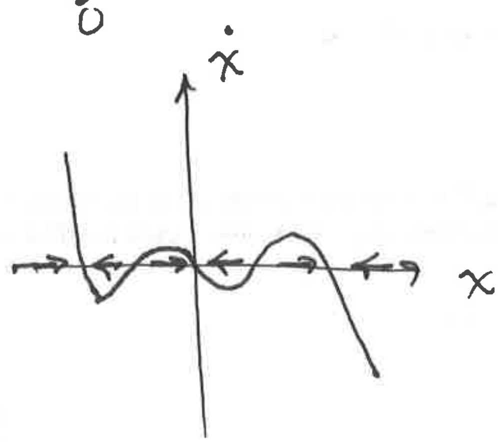


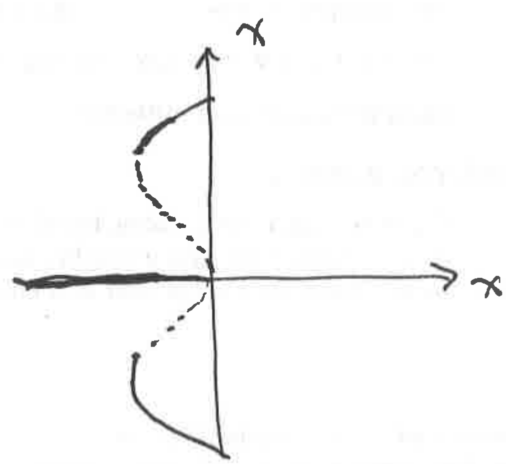
② $\lambda = -\frac{1}{4}$



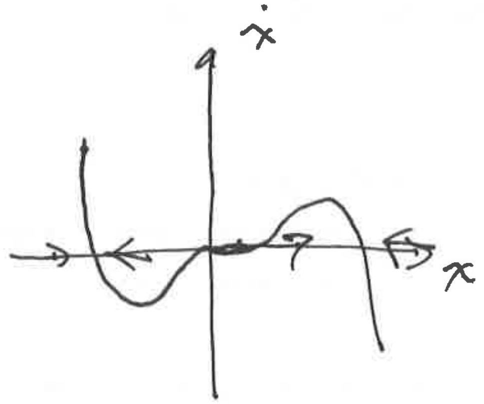


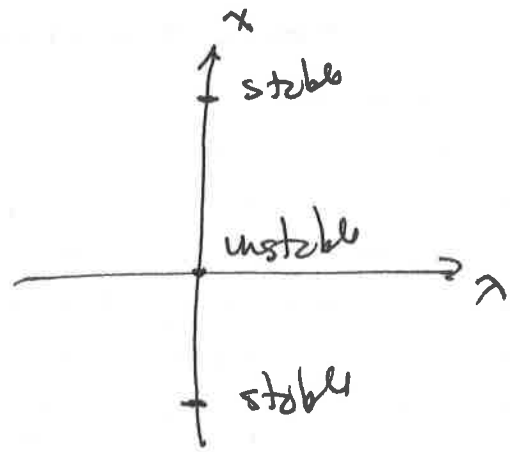
③ $\lambda = -\frac{1}{8}$





④ $\lambda = 0$





⑤ $\lambda = 1$