Math 1497 – Calc 2

We now start a new chapter – Sequences & Series

8.1 (8.2) Sequences

Simply put a sequence is a list of numbers

$$\{1, 2, 3, 4, \ldots\}$$

$$\{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots\}$$

or in general $\{a_1, a_2, a_3, \ldots a_n \ldots\}$

or $\{a_n\}$ for short

a_n – called the generator
n – index

So for example $\{a_n\} = \{\frac{1}{2^n}\}$

as we index $n = 1, 2, 3, \ldots$

we get the terms in the sequence

$$\{\tfrac{1}{2^1}, \tfrac{1}{2^2}, \tfrac{1}{2^3}, \ldots\}$$

Sometimes given the actual sequence we would like to find the generator

ex $\left\{ \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \cdots \right\}$

so we need to see the pattern

$n=1 \quad a_1 = \frac{1}{3}$

$n=2 \quad a_2 = \frac{2}{5}$

$n=3 \quad a_3 = \frac{3}{7}$

$n=4 \quad a_4 = \frac{4}{9}$

numerator is just n

denominator jumps by 2's and start at 3

so $2n+1$ works

and $a_n = \dfrac{n}{2n+1}$ plug in #'s and check for yourself.

Sometimes sequences are given recursively, for example

$$a_{n+1} = \frac{1}{n} a_n, \quad a_1 = 1 \leftarrow \text{a starting value}$$

so $n=1 \quad a_{1+1} = \frac{1}{1} a_1 \Rightarrow a_2 = 1$

$n=2 \quad a_3 = \frac{1}{2} a_2 \Rightarrow a_3 = \frac{1}{2 \cdot 1} = \frac{1}{2}$

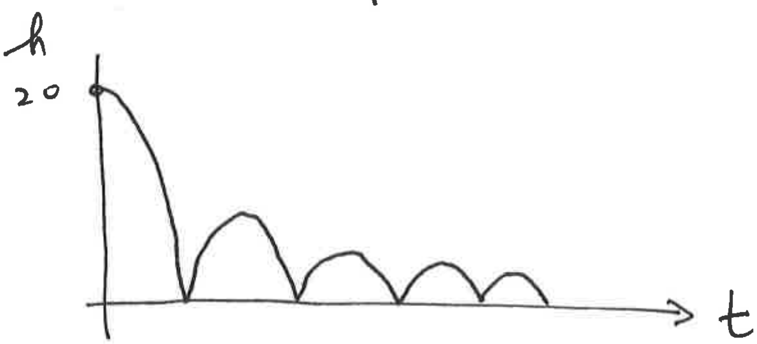
$n=3 \quad a_4 = \frac{1}{3} a_3 = \frac{1}{3} \cdot \frac{1}{2 \cdot 1} = \frac{1}{3 \cdot 2 \cdot 1} = \frac{1}{3!}$

$n=4 \quad a_5 = \frac{1}{4} a_4 = \frac{1}{4 \cdot 3!} = \frac{1}{4!}$ etc.

From this we see the pattern

$$a_n = \frac{1}{n!}, \quad n = 1, 2 \ldots$$

A nice example in the book is the bouncing ball



Suppose we start at $h = 20$ ft and when the ball is released it hits the ground and rebounds and attains a ~~height~~ .8 times it's original.
Suppose this then repeats

$$h_0 = 20$$

$$h_1 = 20 \cdot (0.8) = 16$$

$$h_2 = 16(0.8) = (0.8)^2 20 = 12.80$$

$$h_3 = (0.8)^3 20 = 10.24$$

etc

$$h_n = (0.8)^n 20 \quad \text{for general } n$$

so from graph we see a pattern. If n increases the height decreases and as $n \to \infty$ $h_n \to 0$ so here we see limits again!

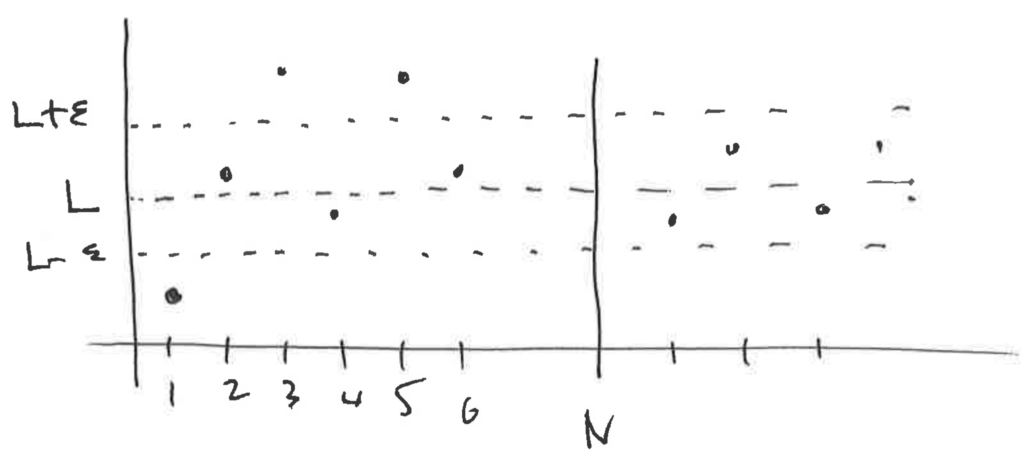
Def^n A sequence $\{a_n\}$ is said to converge to L

if $$\lim_{n\to\infty} a_n = L$$

What does this mean formally. It means there exists a $N > 0$ and $\varepsilon > 0$ such that

$$|a_n - L| < \varepsilon \text{ when } n > N$$

ok. what does this really mean?



$\longrightarrow$ after here every dot is in the band.

so to determine whether a seq. converges (or diverges)

is to determine if

$$\lim_{n\to\infty} a_n \text{ exists}$$

Let's look at some examples

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#10 $\left\{ \dfrac{n^{12}}{3n^{12}+4} \right\}$

$$\lim_{n\to\infty} \frac{n^{12}}{3n^{12}+4} = \lim_{n\to\infty} \frac{1}{3+\frac{4}{n^{12}}}$$

$$\because \lim_{n\to\infty} \frac{1}{n^{12}} = 0 \qquad \lim_{n\to\infty} \frac{1}{3+\frac{4}{n^{12}}} = \frac{1}{3+0} = \frac{1}{3}$$

so the sequence conv.

#12 $\left\{ \dfrac{2e^n+1}{e^n+2} \right\}$ slight variation

$$\lim_{n\to\infty} \frac{2e^n+1}{e^n+2}$$

Can we use L'Hôpital's Rule

Yes - if we replace n with x

n is only defined at the integers

x is a cont^s variable

So $\lim_{x\to\infty} \frac{2e^x+1}{e^x+2} = \text{"}\frac{\infty}{\infty}\text{"}$ LH applies

$\lim_{x\to\infty} \frac{2e^x}{e^x} = 2$ so the seq. conv. to 2

#24 $\left\{ \frac{\ln \frac{1}{n}}{n} \right\}$

$\lim_{x\to\infty} \frac{\ln \frac{1}{x}}{x} = \text{"}\frac{-\infty}{\infty}\text{"}$ so we can use LH

$\lim_{x\to\infty} \frac{\frac{1}{1/x}\cdot\left(\frac{-1}{x^2}\right)}{1} = \lim_{x\to\infty} -\frac{1}{x^2}\cdot x = \lim_{x\to\infty} -\frac{1}{x} = 0$

so the seq. converges to zero!

<u>Some Limit Laws</u>

Assume $\{a_n\}$ $\{b_n\}$ have limits A & B.

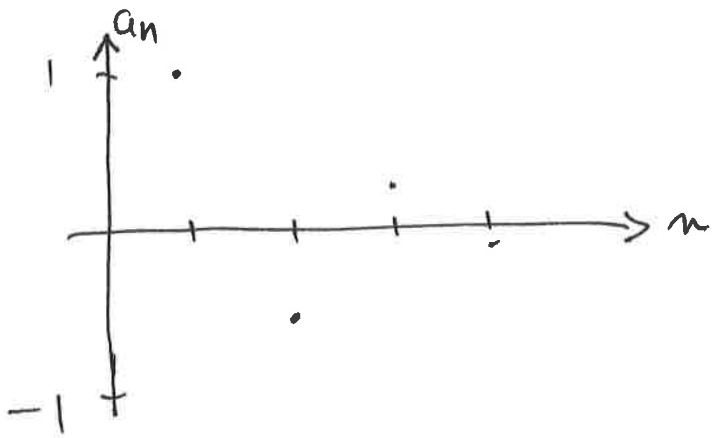
(1) $\lim_{n\to\infty} a_n \pm b_n = A \pm B$

(2) $\lim_{n\to\infty} c\,a_n = cA$ c const

(3) $\lim_{n\to\infty} a_n b_n = AB$, (4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ $(B \neq 0)$

Consider $\left\{ \frac{(-1)^{n+1}}{n} \right\}$

$= \left\{ 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \dots \right\}$



so the sign alternates $+ - + - \dots$

we really can't take the derivative so let's prove another way.

<u>Squeeze Th^m</u> Let $\{a_n\}$ $\{b_n\}$ $\{c_n\}$ be seq

and $a_n \le b_n \le c_n$ for $n > N$ (some N)

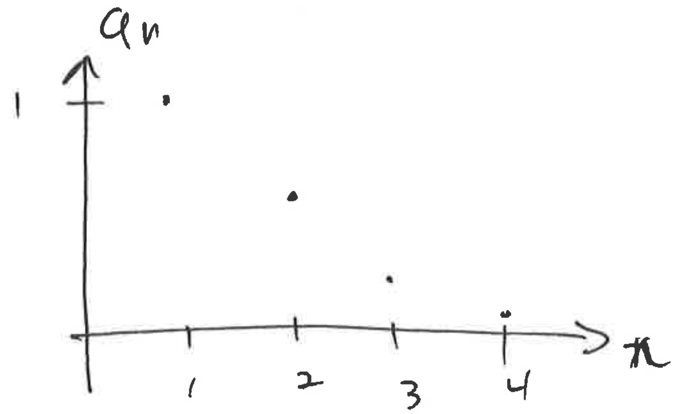
if $\lim_{n\to\infty} a_n = L$ & $\lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$

previous Ex $-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ so by sq^z Th^m $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Consider $\left\{ \frac{1}{n!} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \ldots \right\}$



the terms are decreasing

Let's define this

A seq. $\{a_n\}$ is decreasing if

$$a_{n+1} \leq a_n$$

and increasing if

$$a_{n+1} \geq a_n$$

And strictly decreasing if $a_{n+1} < a_n$

" " increasing if $a_{n+1} > a_n$

Sometimes we can use derivative to determine

this $\quad f'(x) > 0 \quad \{a_n\}$ increasing $\quad a_n = f(n)$

$\qquad f'(x) < 0 \quad \{a_n\}$ decreasing $\quad a_n = f(n)$

In this example we can't so we need to do it directly. It appears the seq. is dec.

so $a_{n+1} \overset{?}{<} a_n$? – we don't know yet!

$$\frac{1}{(n+1)!} \overset{?}{<} \frac{1}{n!} \qquad (n+1)! = (n+1)n!$$

$$\frac{1}{(n+1)n!} \overset{?}{<} \frac{1}{n!} \qquad \text{cancel } n!$$

$$\frac{1}{n+1} \overset{?}{<} \frac{1}{1} \quad \text{or} \quad 1 \overset{?}{<} n+1 \quad \text{or} \quad 0 \overset{?}{<} n$$

Yes

so everything above is correct.

Ex $\left\{\frac{n}{n+1}\right\}$ let $f(x) = \frac{x}{x+1}$ $\quad f' = \frac{1(x+1) - 1 \cdot x}{(x+1)^2}$

$$= \frac{1}{(x+1)^2} > 0$$

so $\}$ is increasing

Boundedness

A seq $\{a_n\}$ is bounded above if

$$a_n \leq M \quad \text{for } n > N$$

and bounded below if

$$m \leq a_n \quad \text{for } n > N$$

so is $\left\{ \frac{n}{n+1} \right\}$ bounded above?

$\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}$

look like $\frac{n}{n+1} \overset{?}{<} 1$ mult. by $n+1$

so $n \overset{?}{<} n+1$ cancel n from each side

$0 \overset{?}{<} 1$ yes

so $\frac{n}{n+1} < 1$ and $\left\{ \frac{n}{n+1} \right\}$ is bounded above by 1.