Linear Algebra

Vectors and Matrices

This is an elementary introduction to vectors and matrices.

A vector is simply an array of numbers written as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{1}$$

where a matrix is written as a collection of arrays, i.e.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \tag{2}$$

In (1), we have a single column of 3 numbers, whereas in (2) we have two rows of 3 numbers in each row. In general we could have a vector of m elements written as

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

are a matrix of m rows and n columns in which we would have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

where a_{mn} is the element in the m^{th} row and n^{th} column.

For those that have seen vectors in either Physics or maybe Calc 3, another way to write them is

$$\vec{u} =< u_1, u_2, u_3 >$$

A natural question is: "Can we add, subtract and multiply both vectors and matrices?"

Vector Addition

To add vectors, we simply add component by component. So, if

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

then $\vec{u} + \vec{v}$ becomes

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 2+5 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Subtraction follows similarly, $\vec{u} - \vec{v}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1-4 \\ 2-5 \\ 3-6 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

Vector Scalar Multiplication

To scale a vector, we simply multiply each component by the scale factor. So,

$$6\vec{u} = 6 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} \qquad -5\vec{v} = -5 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -20 \\ -25 \\ -30 \end{bmatrix}$$

we can also perform both simultaneously, $a\vec{u} + b\vec{v}$ would produce

$$\begin{bmatrix} a+4b \\ 2a+5b \\ 3a+6b \end{bmatrix},$$

Matrix Addition(Subtraction)

To add matrices, again, we simply add(subtract) component by component. So, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 5 & 2 \\ 4 & -3 & 9 \end{bmatrix}$$

then $A + B$ would produce

$$\begin{bmatrix} 1+(-1) & 2+5 & 3+2 \\ 4+4 & 5-3 & 6+9 \end{bmatrix} = \begin{bmatrix} 0 & 7 & 5 \\ 8 & 2 & 15 \end{bmatrix},$$

whereas $A - B$ would produce

$$\begin{bmatrix} 1-(-1) & 2-5 & 3-2 \\ 4-4 & 5-(-3) & 6-9 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 8 & -3 \end{bmatrix}.$$

Matrix Scalar Multiplication

For matrices, we simply multiply each component by the scale factor. So,

$$-2A = -2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \end{bmatrix}$$

Matrix Multiplication

In order to multiply two matrices, it is important that the size of the matrices be of a certain dimension. For example, if

$$A_{mn}\ B_{kl},$$

then we require that $n = k$ and the dimension of the new matrix is $m \times l$. Consider the 2×2 matrix and the 2×3 matrix

$$A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

As A_{22} and B_{23} we can multiply AB but not BA. Thus, the command

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+2\cdot 2 & 1\cdot 3+2\cdot 4 & 1\cdot 5+2\cdot 6 \\ 3\cdot 1+4\cdot 2 & 3\cdot 3+4\cdot 4 & 3\cdot 5+4\cdot 6 \end{bmatrix}$$

giving

$$AB = \begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \end{bmatrix}$$

Determinants

To calculate determinants matrices must be square *i.e.* an $n \times n$ matrix. For 2×2 if A is given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the determinant of A denoted by $detA$ is given by

$$\det A = a\,d - b\,c$$

Inverses

Given an $n \times n$ matrix, if a second matrix B exists such that $AB = I$, the identity matrix. The identity matrix is given by

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

If the matrix B exists, it's called the inverse. Often this is denoted by A^{-1}. For 2×2 matrices, this is given by

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For example, if

$$A = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ -1 & 5 \end{bmatrix}$$

A quick calculation show that

$$\begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 1 & -2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Given a square matrix A, if a vector $\vec{v}$ and scalar λ exists such that

$$A\vec{v} = \lambda\vec{v}$$

then these are called eigenvector and eigenvalues. In order to calculate the eigenvalues, it is necessary to calculate

$$\det|\lambda I - A| = 0 \tag{3}$$

This will give rise to a polynomial for λ. For a given λ, (3) will give rise to the associated eigenvector. Consider the following example. If

$$A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$$

then

$$\det|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -2 \\ -5 & \lambda + 1 \end{vmatrix} = (\lambda + 3)(\lambda - 4) = 0$$

giving rise to the eigenvalues $\lambda = -3, 4$.

If $\lambda = -3$ then

$$\begin{bmatrix} -3 - 2 & -2 \\ -5 & -3 + 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

or

$$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Thus one possible eigenvector is

$$\vec{u} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

If $\lambda = 4$ then

$$\begin{bmatrix} 2 & -2 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Thus one possible eigenvector is

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$