Math 4315 - PDEs

Last class we solved

$$u_x - u_y = 0.$$

Starting with the chain rule

$$u_s = u_x x_s + u_y y_s$$

we choose $x_s = 1,\ y_s = -1$ so $u_s = u_x - u_y = 0$

and solved

$$x_s = 1$$
$$y_s = -1$$
$$u_s = 0$$

This actually works in general.

If $\quad a u_x + b u_y = c, \quad a, b, c$

if $u_s = u_x x_s + u_y y_s$

if
$$\left.\begin{aligned} x_s &= a \\ y_s &= b \\ \text{then} \quad u_s &= c \end{aligned}\right\} \text{ these we can solve}$$

① eliminate s

② eliminate r.

Ex 1 $\quad 2u_x + 3u_y = 1$

If $\quad u_s = u_x x_s + u_y y_s$

if $\quad x_s = 2 \Rightarrow x = 2s + d(r)$, $\quad y_s = 3 \Rightarrow y = 3s + e(r)$, $\quad u_s = 1 \Rightarrow u = s + g(r)$

$$3x - 2y = 3d - 2e = A(r)$$

$$x - 2u = d - 2g = B(r)$$

so $\quad r = A^{-1}(3x - 2y) \Rightarrow x - 2u = B\,A^{-1}(3x - 2y)$

$$\Rightarrow \quad x - 2u = f(3x - 2y)$$

$$u = \frac{x}{2} - \frac{f(3x-2y)}{2} \qquad \text{set } -\frac{f(?)}{2} \to f(?)$$

$$u = \frac{x}{2} + f(3x - 2y)$$

How do we find f? Coming up Friday.

Can we solve 1st order PDE's a little more general than constant coefficient?

For example

$$u_x - u_y = u, \qquad x u_x - y u_y = 2x \; ?$$

These 2 examples are both of the form.

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y) \qquad (*)$$

Again, if we start with

$$u_s = u_x x_s + u_y y_s$$

choosing

$$\left.\begin{aligned} x_s &= a(x,y) \\ y_s &= b(x,y) \\ \text{then} \quad u_s &= c(x,y)u + d(x,y) \end{aligned}\right\} \text{ these are a little more complicated to solve.}$$

these are called "characteristic eq'n's"

<u>Note</u> $x_s = a,\ y_s = b$ so $\dfrac{y_s}{x_s} = \dfrac{b}{a}$ or $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$

<u>ex2</u> $u_x - u_y = u$

$$\left.\begin{aligned} \text{CE} \quad x_s &= 1 \Rightarrow x = s + a(r) \\ y_s &= -1 \Rightarrow y = -s + b(r) \\ u_s &= u \end{aligned}\right\} \begin{aligned} &** \\ &\text{not the same} \\ &a \,\&\, b \text{ as} \\ &\text{in } (*) \uparrow \end{aligned}$$

Now the last one is different!

Recall $\dfrac{dy}{dx} = ky$ so $\dfrac{dy}{y} = k\,dx$ $\quad \ln y = kx + \ln c$

$$y = ce^{kx}.$$

so $u_s = u \Rightarrow \frac{du}{u} = ds$

so $\ln u = s + c(r)$ or $u = \bar{c}(r) e^{s}$

Elliminate s

① $x+y = a+b = A(r)$

$\ln u - x = a(r) - c(r)$

$= B(r)$

$\ln u - x = B(A^{-1}(x+y))$

$= f(x+y)$

$\ln u = x + f(x+y)$

$u = e^{x} \cdot e^{f(x+y)}$

$u = e^{x} g(x+y)$

① $x+y = A(r)$

$\because x = s + a(r)$

$s = x - a(r)$

$u = \bar{c}(r) e^{x - a(r)}$

$= \bar{c}(r) e^{-a(r)} e^{x}$

$= B(r) e^{x}$

$u = e^{x} B(A^{-1}(x+y))$

$u = e^{x} f(x+y)$

if $g = f$

these are the same.

ex 3 $\quad x u_x - y u_y = 2x$

CE $\quad x_s = x$

$y_s = -y$

$u_s = 2x \leftarrow$ we need x here

$x_s = x \Rightarrow x = a(r)e^{s}$

$y_s = y \Rightarrow y = b(r)e^{-s}$

$u_s = 2x = 2a(r)e^{s} \Rightarrow u = 2a(r)e^{s} + c(r)$

So far we have

$x = a(r)e^{s}$

$y = b(r)e^{-s}$

$u = 2a(r)e^{s} + c(r)$

$xy = a(r)e^{s} \cdot b(r)e^{-s} = a(r)b(r) = A(r)$

$u = 2x + c(r) = 2x + c(A^{-1}(xy))$

sol

$u = 2x + f(xy)$