31 May 2012
bj

Klein Geometry

I. Introduction

The Macdowell-Mansouri extension of first order gravity is, according to Derek Wise (arXiv 0611154), based on the Erlangen Program for geometry initiated by Felix Klein in the mid-1800's. It would seem wise for me to understand the basics better. This is an attempt to do so in my own language. It is mathematically unsophisticated, and based on working from specific examples to generalities. I will look at the O(N+1)/O(N) sequence in particular, and specifically on O(5)/O(4). It is big enough to exhibit general patterns, and small enough to be easily manageable.

In Section II we quickly review the O(5)/O(4) example for the obvious choice of spherical coordinates. It exhibits a pattern for the nonvanishing connection coefficients A, which we thereafter adopt in general. I call it the OTAC (one to a customer) pattern, because for a given choice of gauge indices one and only one component (in spacetime) for the connection A turns out to be nonvanishing.

Given the OTAC hypothesis, a more complicated pattern of nontrivial elements of the field strength F emerges, thanks to the nonlinearity present in the relationship of A to F. If one wishes to create maximally symmetric spacetimes, it is only necessary to set $F = 0$. However, we will here choose to be a bit more general. In Sections III - V we will not demand $F = 0$, but only ask that the elements of F themselves exhibit an OTAC pattern very similar to that possessed by the connection A . This constrains the elements of the gauge potential in an interesting way.

However, we do not succeed in finding much of a generalization beyond the $F = 0$ case. It seems that only one component of F is allowed, at least in an easy way, to be nonvanishing, unless the OTAC pattern is abandoned. So in the later sections of this note this attempt is dropped. But it turns out that even the $F = 0$ case yields an interesting pattern. The six degrees of freedom which we end up characterizing the properties of the MM connection A organize themselves into a symplectic structure, i.e. of the trajectory of a "particle" moving in 6-dimensional phase space. Furthermore, as discussed in Section VIII, the MacDowell-Mansouri line element turns out to be trivial and related to the parameters of this symplectic structure. All this deserves much more scrutiny. But that is beyond the scope of this exploratory note.

II. The Maximally Symmetric Connection

Consider the O(5)/O(4) example, where the connection A is a synthesis of a vierbein e and an O(4) connection ω. We write the connection out in component form as follows:

$$A^{AB}_{\mu}(x_i) = -A^{BA}_{\mu}(x_i) = \begin{array}{c}
\left.\begin{array}{c|c|c|c|c|}
AB & & & & \\
\hline
15 & 1 & 0 & 0 & 0 \\
25 & 0 & s_1 & 0 & 0 \\
35 & 0 & 0 & s_1 s_2 & 0 \\
45 & 0 & 0 & & s_1 s_2 s_3 \\
\hline
\end{array}\right\} e \\
\left.\begin{array}{c|c|c|c|c|}
14 & 0 & 0 & 0 & c_1 s_2 s_3 \\
24 & 0 & 0 & 0 & c_2 s_2 \\
34 & 0 & 0 & 0 & c_3 \\
\hline
13 & 0 & 0 & c_1 s_1 & 0 \\
23 & 0 & 0 & c_2 & 0 \\
\hline
12 & 0 & c_1 & 0 & 0 \\
\hline
\mu \rightarrow & x_1 & x_2 & x_3 & x_4
\end{array}\right\} \omega
\end{array}$$

Here

$$s_i = \sin x_i \qquad c_i = \cos x_i$$

We therefore recognize the choice of vierbein we have written down as appropriate for four dimensional spherical coordinates, with the identifications

$$x_1 = \omega \qquad x_2 = \chi \;\; (r = \sin\chi) \qquad x_3 = \theta \qquad x_4 = \varphi$$

$$ds^2 = d\omega^2 + \omega^2\left[d\chi^2 + \sin^2\chi \left[d\theta^2 + \sin^2\theta \, d\varphi^2 \right]\right]$$

Hindsight has also been used to dictate the choice of the connection coefficients. They lead to the result

$$F^{AB}_{\mu\nu} \equiv \partial_\mu A^{AB}_{\nu} - \partial_\nu A^{AB}_{\mu} + \sum_{C=1}^{5}\left(A^{AC}_{\mu} A^{CB}_{\nu} - A^{AC}_{\nu} A^{CB}_{\mu} \right) = 0$$

This is a key property of the MM methodology: spacetimes with maximal symmetry are described in MM language as having flat connections. We will not here bother to demonstrate that the choice of A made above leads to F = 0 ; we address the issue in more general terms in the next section.

Another interesting case is a "Euclidean FRW metric". We write

$$
A_\mu = \begin{array}{c|cccc}
15 & 1 & & & \\
25 & & S_1 & & \\
35 & & & S_1 & \\
45 & & & & S_1 \\
\hline
14 & & & & C_1 \\
24 & & & & 0 \\
34 & & & & 0 \\
\hline
13 & & & C_1 & \\
23 & & & 0 & \\
\hline
12 & & C_1 & & \\
\hline
 & x_1 & x_2 & x_3 & x_4
\end{array}
$$

Again

$$S_1 = \sin x_1 \qquad\qquad C_1 = \cos x_1$$

The line element takes a rather unfamiliar form:

$$ds^2 = dx_1^2 + \sin^2 x_1 \left(dx_2^2 + dx_3^2 + dx_4^2 \right)$$

$$\Rightarrow \qquad d\theta^2 + \sin^2\theta \left(d\varphi_1^2 + d\varphi_2^2 + d\varphi_3^2 \right)$$

But this choice leads to a nonvanishing, but constant, field strength F. Again, we defer detailed examination of this case until later. The OTAC hypothesis is actually violated in this case, because

$$A_\mu^{AB} = 0 \quad \text{if} \quad A \neq 1 \quad \text{and} \quad B \neq 1$$

Such cases are of considerable interest, but do require special attention.

III. Not Quite Maximal Symmetry

In this section we will assume the OTAC property for the connection A, and work out its consequences for the field strength F. Most of this will be done for the specific O(5)/O(4) example, because I personally find the arguments easier to follow. Generalizations will for the most part will rely on common sense. But it should not be difficult for the reader to complete the general proofs of the assertions to be made. We write, with some hindsight regarding notation

$$
A_\mu^{AB} = \begin{array}{c|cccc|}
15 & e_1 & & & \\
25 & & e_2 & & \\
35 & & & e_3 & \\
45 & & & & e_4 \\
\hline
14 & & & & a_{14} \\
24 & & & & a_{24} \\
34 & & & & a_{34} \\
\hline
13 & & & a_{13} & \\
23 & & & a_{23} & \\
\hline
12 & & a_{12} & & \\
\hline
 & x_1 & x_2 & x_3 & x_4
\end{array}
$$

Notice that a nonvanishing component of A requires at least two of the three indices to be equal, and that

$$
A_i^{ik} = 0 \qquad k \neq 5 \qquad i < k
$$

The definition of the field strength, in component form, is

$$
F_{\mu\nu}^{AB} = \begin{array}{c|ccc||cc||c|}
15 & \times & & & \times & & \times \\
25 & & \times & & & \times & \ast \\
35 & & & \times & \ast & \ast & \\
45 & \ast & \ast & \ast & & & \\
\hline
14 & \ast & \checkmark & \checkmark & & & \\
24 & \times & \ast & \checkmark & & & \\
34 & \times & \times & \ast & & & \\
\hline
13 & & & \times & \ast & \checkmark & \\
23 & & & \times & \times & \ast & \\
12 & & \times & & & \times & \ast \\
\hline
 & x_1x_4 & x_2x_4 & x_3x_4 & x_1x_3 & x_2x_3 & x_1x_2
\end{array}
$$

Here we have marked the entries which are identically zero, given the OTAC assumption. Nevertheless, there remain quite a few additional potentially nonvanishing entries. It is at this point that we demand simplifications. In particular, we will require that the only nonvanishing elements of F (for the cases within our range of consideration) are those which are starred. This choice is of course not the most general that can be made, but it is considerably more general than the maximally symmetric option leading to $F = 0$. It also possesses a kind of OTAC structure. However it now the components of the O(N) curvature which are analogous to the vierbein (now a 6-bein!), while the coset components of the curvature are analogous to the O(N) connection ω (but with the roles of spacetime and internal indices interchanged).

So at this point we can concentrate on the constraints imposed by the assumed vanishing of the unstarred components of F. These fall into two categories, as shown. Those marked with an X are of the form

$$\frac{\partial A_k^{ik}}{\partial x_j} = 0$$

They simply restrict the dependence of the connection coefficients on the spacetime parameters. Simple computations, such as exhibited below, lead to the conclusions

$$F_{12}^{15} = \frac{\partial A_1^{15}}{\partial x_2} - \cancel{\frac{\partial A_2^{15}}{\partial x_1}} + \cancel{A_1^{12} A_2^{25}} - \cancel{A_2^{12} A_1^{25}} = 0$$

$$\Rightarrow \frac{\partial e_1}{\partial x_2} = 0$$

Likewise,

$$\frac{\partial e_1}{\partial x_3} = \frac{\partial e_2}{\partial x_3} = 0 \qquad \frac{\partial e_1}{\partial x_4} = \frac{\partial e_2}{\partial x_4} = \frac{\partial e_3}{\partial x_4} = 0$$

Consequently,

$$e_1 = e_1(x_1) \qquad e_2 = e_2(x_1, x_2) \qquad e_3 = e_3(x_1, x_2, x_3)$$

This result obviously generalizes.

$$e_k = e_k(x_1, \cdots, x_k)$$

In a similar way, one finds, for the remaining entries in this category, the constraints

$$a_{ik} = a_{ik}(x_i, \cdots x_k)$$

The remaining constraints, indicated by a check mark, are a little less trivial. For the special case at hand, we simply write them all down:

$$0 = F^{13}_{23} = \frac{\partial A^{13}_2}{\partial x_3} - \frac{\partial A^{13}_3}{\partial x_2} + A^{12}_2 A^{23}_3 - A^{12}_3 A^{23}_2 \quad \Rightarrow \frac{\partial a_{13}}{\partial x_2} = a_{12} a_{23}$$

Likewise,

$$\frac{\partial a_{24}}{\partial x_3} = a_{23} a_{34} \qquad \frac{\partial a_{14}}{\partial x_2} = a_{12} a_{24} \qquad \frac{\partial a_{14}}{\partial x_3} = a_{13} a_{34}$$

Defining

$$p_i \equiv a_{i,(i+1)} (x_i , x_{i+1})$$

we see that

$$\begin{aligned} a_{13} &= c_{13}(x_1, x_3) + \int^{x_2} dx_2' \, a_1(x_1, x_2')\, a_2(x_2', x_3) \equiv c_{13} + \int^2 p_1 p_2 \\ a_{24} &= c_{24}(x_2, x_4) + \int^3 p_2 p_3 \\ a_{14} &= c_{14}(x_1, x_4) + \int^2\!\!\int^3 p_1 p_2 p_3 \end{aligned}$$

There is an obvious generalization:

$$a_{ik} = c_{ik} + \int^{i+1} \cdots \int^{k-1} p_i \cdots p_{k-1}$$

We can now see that the (N - 1) quantities p_i are actually input parameters, while the remaining a_{ik}'s are determined by them (up to the "constants of integration" c_{ik} ,and/or the choices of the lower limits of integration in the sundry integrals).

With these results in hand, we can write down reasonably simple expressions for the components of F which we do wish to retain for consideration. They are as follows:

$$F^{k5}_{ik}(x_1, \cdots x_k) = -\frac{\partial e_k}{\partial x_i} + e_i a_{ik}$$

$$F^{ik}_{ik}(x_1, \cdots x_k) \equiv f_{ik} = \frac{\partial a_{ik}}{\partial x_i} + \sum_{j=1}^{i-1} a_{ji} a_{jk} + e_i e_k$$

While these are reasonably simple, they are nevertheless still nontrivial. However, the dependence of the field strengths F on the variables x_i is strongly constrained.

IV. The Coset Field Strengths

The field strengths containing the 5 index have a simple structure; they are linear in the vierbein parameters. The simplest path forward toward manageable simplicity seems to be to set all these field strengths to zero. The most direct way is by iteration. For example, we have

$$0 = F^{25}_{12} = \frac{\partial e_2}{\partial x_1} - e_1 p_1 \quad \Rightarrow \quad e_2 = \int^{x_1} dx_1' \, e_1(x_1') a_1(x_1', x_2) \equiv \int^1 e_1 a_1$$

$$0 = F^{53}_{23} = \frac{\partial e_3}{\partial x_2} - e_2 p_2 \quad \Rightarrow \quad e_3 = \int^1 e_2 p_2 = \int^1\!\!\int^2 e_1 p_1 p_2$$

Evidently this procedure generalizes, and produces the solution

$$e_k = \int^{x_1} dx_1' \cdots \int^{x_{k-1}} e_1 \, p_1 p_2 \cdots p_{k-1} \equiv \int^1 \cdots \int^{k-1} e_1 \, p_1 p_2 \cdots p_{k-1}$$

This can be checked from the general equation

$$\frac{\partial e_k}{\partial x_i} = \int^1 \cdots \int^{i-1} e_1 \, p_1 \cdots p_{i-1} \int^{i+1} p_i \cdots p_k = e_i a_{ik}$$

We can, if we wish, eliminate all the a_{ik} from the equations by expressing them in terms of derivatives of the e's:

$$a_{ik} = e_i^{-1} \frac{\partial e_k}{\partial e_i}$$

This structure exhibits the content of this step. We seem to be demanding that the connection be Levi-Civita, i.e. expressible in terms of Christoffel symbols. It seems to be this step that is most responsible for connecting the Klein-geometry description to the usual Riemannian - geometry description.

V. The Constant Curvature Hypothesis

We are left with the diagonal curvature terms. These are controlled by nonlinear equations which link the components of F with the parameters e (or equivalently the p's). Before tackling the problem, it is instructive to take the maximally symmetric connection defined in Section II and to see how the condition F = 0 gets satisfied. For our first example, direct substitution gives the following results for the relevant six F's:

$$F^{12}_{12} = \frac{\partial c_1}{\partial x_1} + 0 + (s_1)$$

$$F^{13}_{13} = \frac{\partial}{\partial x_1}(c_1 s_2) + 0 + (s_1 s_2) = F^{12}_{12}\, s_2$$

$$F^{14}_{14} = \frac{\partial}{\partial x_1}(c_1 s_2 s_3) + 0 + (s_1 s_2 s_3) = F^{13}_{13}\, s_3$$

$$F^{23}_{23} = \frac{\partial c_2}{\partial x_2} + (c_1)(c_1 s_2) + (s_1)(s_1 s_2)$$

$$F^{24}_{24} = \frac{\partial}{\partial x_2}(c_2 s_3) + (c_1)(c_1 s_2 s_3) + (s_1)(s_1 s_2 s_3) = F^{23}_{23}\, s_3$$

$$F^{34}_{34} = \frac{\partial c_3}{\partial x_3} + + \left[(c_1 s_2)(c_1 s_2 s_3) + (c_2)(c_2 s_3)\right] + (s_1 s_2)(s_1 s_2 s_3)$$

We have chosen the ordering of these equations with an eye toward simplicity, and as a guide for considering the general case to follow. We see that indeed F = 0.

For the second "Euclidean FRW" example, we find, in passing,

$$F^{12}_{12} = -\frac{\partial A^{12}_2}{\partial x_1} + A^{15}_1 A^{52}_2 = -\frac{\partial c_1}{\partial x_1} - s_1 = F^{13}_{13} = F^{14}_{14} = 0$$

$$F^{23}_{23} = A^{21}_2 A^{13}_3 + A^{25}_2 A^{53}_3 = -c_1^2 - s_1^2 = F^{24}_{24} = F^{34}_{34} = -1$$

Consequently three of the six diagonal curvature components become constant and nonvanishing. This example may comprise a useful model for the general cases we will need in practice. However, we do set this case aside for now.

We now turn to the general case. The first equation reads, in general notation

$$f_{12} \equiv F^{12}_{12} = \frac{\partial p_1}{\partial x_1} + e_1 e_2 = \frac{\partial p_1}{\partial x_1} + e_1 \int^1 e_1 p_1$$

It turns out to be helpful to defer consideration of this case until considering some of the remaining equations. We write

$$f_{13} = \frac{\partial a_{13}}{\partial x_1} + e_1 \int^1 \int^2 e_1 a_1 a_2 = \int^2 \frac{\partial p_1}{\partial x_1} p_2 + e_1 \int^1 \int^2 e_1 p_1 p_2$$

Differentiating this equation yields

$$0 = \frac{\partial f_{13}}{\partial x_2} = \frac{\partial p_1}{\partial x_1} p_2 + \left(e_1 \int^1 e_1 p_1\right) p_2 = f_{12} p_2$$

We see that either $f_{12} = 0$ or $p_2 = 0$. But, given the OTAC hypothesis, none of the p_2 can vanish.

In a similar fashion,

$$0 = \frac{\partial f_{14}}{\partial x_3} = \left[\int^2 \frac{\partial p_1}{\partial x_1} p_2 + e_1 \int^1 \int^2 e_1 p_1 p_2\right] p_3 = f_{13} p_3$$

Finally,

$$\frac{\partial f_{24}}{\partial x_3} = 0 \quad \Rightarrow \quad f_{23} p_3 = 0$$

Considerable simplifications occur by assuming

$$e_1 = 1$$

The more general case appears accessible simply by a change of variables.

Vanishing of f_{12} implies

$$\frac{\partial p_1}{\partial x_1} + \int^1 p_1 = 0$$

Differentiation with respect to x_1 leads to the solution

$$p_1 = c_2(x_2)\cos x_1 \quad \Rightarrow \quad p_1 = \cos x_1 \qquad q_1 = \int^1 p_1 = \sin x_1$$

We set the coefficient c_2 to unity for reasons similar to those for e_1 .

The equation for f_{23} is:

$$f_{23} = \frac{\partial p_2}{\partial x_2} + a_{12}a_{13} + e_2 e_3 = \frac{\partial p_2}{\partial x_2} + p_1\left[c_{13} + \int^2 p_1 a_2\right] + \int^1 e_1 p_1 \iint^{12} e_1 p_1 p_2$$

$$= \frac{\partial p_2}{\partial x_2} + p_1 c_{13} + \int^2 \left[p_1^2 + q_1^2\right] p_2 = \left[\frac{\partial p_2}{\partial x_2} + \int^2 p_2\right] + p_1 c_{13}$$

Differentiation with respect to x_2 leads to

$$p_2 = \cos x_2$$

This in turn implies that the integration constant c_{13} vanishes;

$$\int^2 a_2 = \sin x_2 \qquad c_{13} = 0$$

Next we revisit f_{14}. A similar line of argument ensues:

$$f_{14} = \frac{\partial a_{14}}{\partial x_1} + e_1 e_4 = \frac{\partial c_{14}}{\partial x_1} + \iint^{23} \frac{\partial p_1}{\partial x_1} p_2 p_3 + \iiint^{123} p_1 p_2 p_3 = \frac{\partial c_{14}}{\partial x_1} + \iint^{23}\left[\frac{\partial p_1}{\partial x_1} + \int^1 p_1\right] p_2 p_3$$

Therefore

$$\frac{\partial c_{14}}{\partial x_1} = f_{14}$$

This situation also exists for f_{24}:

$$f_{24} = \frac{\partial a_{24}}{\partial x_2} + a_{12}a_{14} + e_2 e_4 = \int^3 \frac{\partial p_2}{\partial x_2} p_3 + p_1 c_{14} + p_1 \iint^{23} p_1 p_2 p_3 + \int^1 p_1 \iiint^{123} p_1 p_2 p_3$$

$$= p_1 c_{14} + \int^3 \left[\frac{\partial p_2}{\partial x_2} + p_1^2 \int^2 p_2 + \left(\int^1 p_1\right)^2 \int^2 p_2 \right] p_3 = p_1 c_{14} + \int^3 \left[\frac{\partial p_2}{\partial x_2} + \int^2 p_2 \right] p_3$$

$$= p_1 c_{14}$$

The only solution compatible with the previous result is

$$f_{14} = f_{24} = 0 \qquad c_{14} = 0$$

This leaves only f_{34} to consider. Here the situation is actually a little different. The first part of the argument proceeds as before:

$$f_{34} = \frac{\partial a_{34}}{\partial x_3} + a_{13} a_{14} + a_{23} a_{24} + e_3 e_4$$

$$= \frac{\partial p_3}{\partial x_3} + \int^2 p_1 p_2 \iint^{23} p_1 p_2 p_3 + p_2 c_{24} + p_2 \int^3 p_2 p_3 + \iint^{12} p_1 p_2 \iiint^{123} p_1 p_2 p_3$$

$$= p_2 c_{24} + \frac{\partial p_3}{\partial x_3} + \int^3 \left[p_1^2 \left(\int^2 p_2\right)^2 + p_2^2 + \left(\int^1 p_1\right)^2 \left(\int^2 p_2\right)^2 \right] p_3$$

$$= p_2 c_{24} + \left[\frac{\partial p_3}{\partial x_3} + \int^3 p_3 \right]$$

Again

$$p_3 = \cos x_3$$

However, this time we can keep a constant of integration:

$$c_{24} = 0 \qquad e_3 = \int^3 p_3 = \sin x_3 + f_{34}$$

This case already appears in the simplest O(3) / O(2) example. What we learn here is that, within the OTAC hypothesis, nothing more happens when one generalizes to O(N+1) / O(N) . Only one component of F is allowed to become constant and nonvanishing. However, when we generalize, more interesting cases might emerge. We now turn to this more general situation.

VI. Loosening the Constraints

From experience with FRW deSitter metrics, we know that there can be a richer structure for F provided the metric tensor has negative eigenvalues. So we include this generalization in what follows. The relevant line element is

$$ds^2 = -\eta_{55}\,\eta_{AB}\,A^{A5}_{\mu}A^{5B}_{\nu}\,dx^{\mu}dx^{\nu} \Rightarrow ds^2 = \sum_{i=1}^{4} \eta_5 \eta_i\, e_i^2\, dx_i^2$$

We assume

$$\eta_{AB} = \delta_{AB}\,\eta_A \qquad (\eta_A)^2 = +1$$

This choice of metric does not directly affect our definition of A . But the definition of F is affected:

$$F^{AB}_{\mu\nu} = \partial_\mu A^{AB}_{\nu} - \partial_\nu A^{AB}_{\mu} + \sum_C \eta_C \left(A^{AC}_{\mu} A^{CB}_{\nu} - A^{AC}_{\nu} A^{CB}_{\mu} \right)$$

But, in general, the pattern of the subsequent arguments is preserved, with the main change being the presence of $\eta_A\eta_B$ in the equations that gave us sines and cosines. When $\eta_A\eta_B$ = -1, the cosines and sines can be replaced with cosh's and sinh's with very little structural change in the final result. This corresponds, for example, to the k = +1 or k = -1 choices for deSitter geometry. More interesting is what happens if we make the "flat" k = 0 choice. Such cases will be accessible in what follows as well. But most importantly, we recognize that the very essential simplification

$$p_i(x_i, x_{i+1}) \Rightarrow p_i(x_i)$$

will survive the generalization. Consequently, we can ab initio simplify our somewhat cumbersome notation. In particular, we now define A as follows:

$$A^{AB}_{\mu} = \begin{array}{c|cccc}
15 & 1 & & & \\
25 & & q_1 & & \\
35 & & & q_1 q_2 & \\
45 & & & & q_1 q_2 q_3 \\
14 & & & & \eta_1 p_1 q_2 q_3 \\
24 & & & & \eta_2 p_2 q_3 \\
34 & & & & \eta_3 p_3 \\
13 & & & \eta_1 p_1 q_2 & \\
23 & & & \eta_2 p_2 & \\
12 & & \eta_1 p_1 & & \\
\hline
 & 1 & 2 & 3 & 4
\end{array}
\qquad
F^{AB}_{\mu\nu} = \begin{array}{cccccc}
 & & & & & \\
f_{14} & & & & & \\
 & f_{24} & & & & \\
 & & f_{34} & & & \\
 & & & f_{13} & & \\
 & & & & f_{23} & \\
 & & & & & f_{12} \\
\hline
14 & 24 & 34 & 13 & 23 & 12
\end{array}$$

Given this change of notation, we rewrite the equations for the field strengths yet one more time. We also define

$$\dot{p}_i \equiv \frac{\partial p_i}{\partial x_i} \qquad \dot{q}_i \equiv \frac{\partial q_i}{\partial x_i} = p_i$$

It is easily checked that, given these definitions, the only nonvanishing components of F are the f_{ij}, as shown above. And the equations for the f_{ij} 's now get cleaned up to a considerable degree:

$$f_{12} = \eta_1 \dot{p}_1 + \eta_5 q_1 = \eta_1 (\dot{p}_1 + \eta_1 \eta_5 q_1)$$
$$f_{13} = \eta_1 \dot{p}_1 q_2 + \eta_5 q_1 q_2 = \eta_1 (\dot{p}_1 + \eta_1 \eta_5 q_1) q_2 = q_2 f_{12}$$
$$f_{14} = \eta_1 \dot{p}_1 q_2 q_3 + \eta_5 q_1 q_2 q_3 = \eta_1 (\dot{p}_1 + \eta_1 \eta_5 q_1) q_2 q_3 = q_2 q_3 f_{12}$$

$$f_{23} = \eta_2 \dot{p}_2 + \eta_1 p_1^2 q_2 + \eta_5 q_1^2 q_2 = \eta_2 [\dot{p}_2 + \eta_1 \eta_2 (p_1^2 + \eta_1 \eta_5 q_1^2) q_2]$$
$$f_{24} = \eta_2 \dot{p}_2 q_3 + \eta_1 p_1^2 q_2 q_3 + \eta_5 q_1^2 q_2 q_3 = \eta_2 [\dot{p}_2 + \eta_1 \eta_2 (p_1^2 + \eta_1 \eta_5 q_1^2) q_2] q_3 = q_3 f_{23}$$

$$f_{34} = \eta_3 \dot{p}_3 + \eta_1 p_1^2 q_2^2 q_3 + \eta_2 p_2^2 q_3 + \eta_5 q_1^2 q_2^2 q_3$$
$$= \eta_3 [\dot{p}_3 + \eta_2 \eta_3 \{ p_2^2 + \eta_1 \eta_2 (p_1^2 + \eta_1 \eta_5 q_1^2) q_2^2 \} q_3]$$

We see here a strong Hamiltonian flavor emerging; this feature has indeed been the motivation for the p , q notation which we have adopted. In this spirit, define

$$\left\{ \begin{array}{ll} H_1 = p_1^2 + K_1 q_1^2 & K_1 = \eta_5 \eta_1 \\ H_2 = p_2^2 + K_2 q_2^2 & K_2 = \eta_1 \eta_2 H_1 \\ \vdots & K_3 = \eta_2 \eta_3 H_2 \\ & \vdots \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} f_{12} = \eta_1 (\dot{p}_1 + K_1 q_1) \\ f_{23} = \eta_2 (\dot{p}_2 + K_2 q_2) \\ f_{34} = \eta_3 (\dot{p}_3 + K_3 q_3) \end{array} \right.$$

We have chosen an unconventional normalization for these "Hamiltonians" in order that their eigenvalues can be naturally set to either +1, -1, or 0. The "equations of motion" which accompany such "Hamiltonians" are as follows:

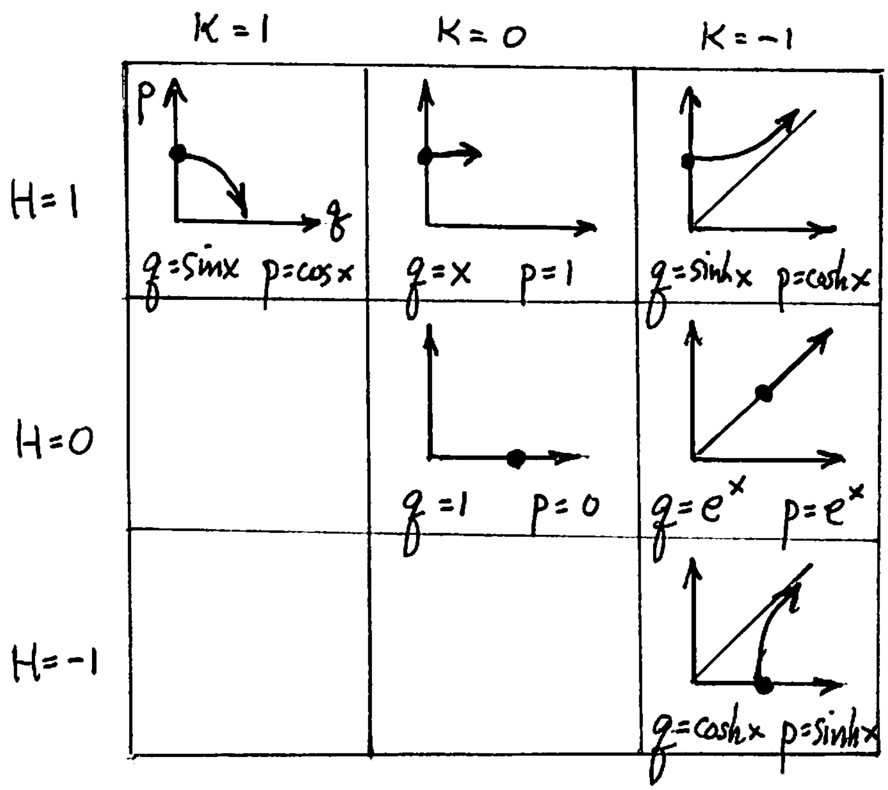
$$\dot{p}_i = -\frac{1}{2}\frac{\partial H_i}{\partial q_i} = -K_i q_i \qquad \dot{q}_i = \frac{1}{2}\frac{\partial H_i}{\partial p_i} = p_i \qquad \dot{H}_i = \frac{dH_i}{dx_i} = 0$$

We see that, when they are satisfied, all the f_{ij} vanish.

Finally, note that the constants of integration present in previous sections have now been assumed to vanish. Doing better is still an interesting option to pursue. But it seems to require going beyond the OTAC hypothesis, which lies outside the scope of this note.

VII. Indefinite Metrics and Vanishing Kappas

Once we stray from the fully Euclidean case, there are a large number of cases to consider. With our Hamiltonian insights, we classify these in terms of the three phase-space plots for the three canonical pairs of q and p which contain all the information present in the cartography. The crucial parameters are the K's and the Hamiltonian eigenvalues, restricted to +1, -1, and 0:



The basic inputs to the scheme are the 5 η's defining the MM internal group structure. Without loss of generality, we can set $\eta_5 = 1$. Furthermore η_4 does not play a direct role in what follows. However, changing the sign of η_4 and leaving everything else unchanged does modify the spacetime metric structure. This issue will return when we see the results of this exercise.

The remaining crucial parameter choices are for H_1 and H_2. Unless at least one of these is set to zero, the considerations we have already made follow with only minor changes. More interesting are cases for which at least one of the H's vanish. Nevertheless, there are many different options to classify.

48

On the next page is a flow chart which illustrates this situation. It hopefully is reasonalbly self-explanatory. There are ~~31~~ distinct paths from the top of the page to the bottom, each of which defines a set of η's (other than η_4), the values of the H's, and therefore the values of the K's. From these ~~[illegible]~~ spacetime line elements can be constructed, thanks to the absence of η_4 in the flow chart. 92

While it is tempting to here provide a catalogue of all these cases, I will not do so. Many of the output spacetime metrics look unphysical. A very simple example is to retreat to the world of generalized O(3) / O(2). Consider our prototypical gauge potential A and field strength F:

$$A^{AB}_{\mu} = \begin{array}{c} 13 \\ 23 \\ 13 \end{array} \begin{pmatrix} 1 & 0 \\ 0 & \sin t \\ 0 & \cos t \end{pmatrix} \qquad F^{AB}_{tx} = \begin{array}{c} 13 \\ 23 \\ 12 \end{array} \begin{pmatrix} 0 \\ -\cos t + \eta_1 \cos t \\ \sin t - \eta_3 \sin t \end{pmatrix}$$
$$\begin{array}{cc} t & x \end{array} \qquad\qquad\qquad tx$$

Choosing $\eta_3 = \eta_1 = 1$, we still may choose $\eta_2 = -1$. This leads to the line element

$$ds^2 = dt^2 - \sin^2 t \, dx^2$$

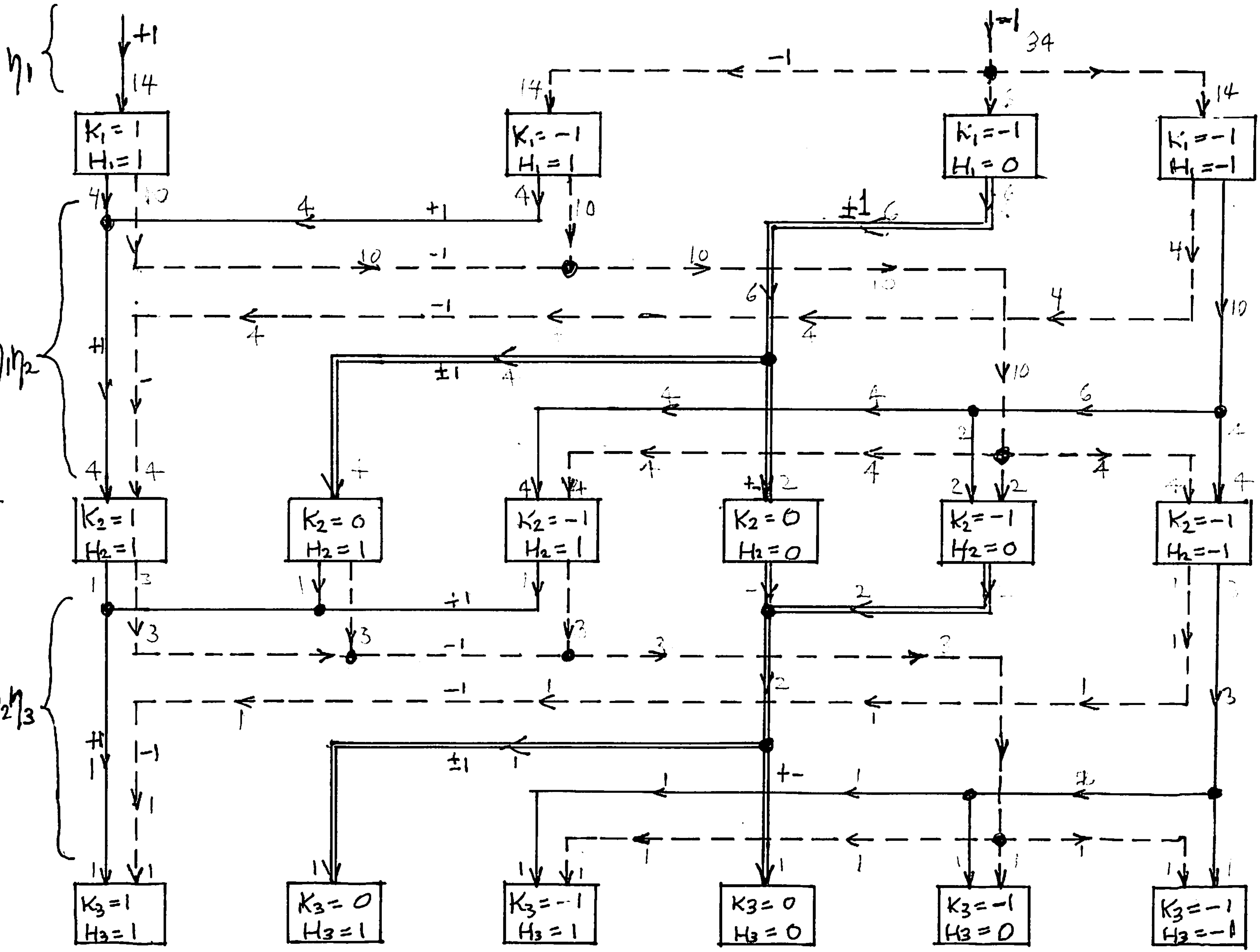
The curvature tensor R which follows from this is nontrivial:

$$R^{12}_{tx} = \sin t$$

However the gauge-invariant curvature remains simple.

$$R^{tx}_{tx} = e_1^{t} e_2^{x} R^{12}_{tx} = 1$$

This kind of consideration needs to be extended to higher dimensions. It is not clear to me, e.g., whether the connection ω which we have constructed is always Levi-Civita. But investigating this further is beyond the scope of this note.


η_1
$\eta_1\eta_2$
$\eta_2\eta_3$
$K_1 = 1$, $H_1 = 1$
$K_1 = -1$, $H_1 = 1$
$K_1 = -1$, $H_1 = 0$
$K_1 = -1$, $H_1 = -1$
$K_2 = 1$, $H_2 = 1$
$K_2 = 0$, $H_2 = 1$
$K_2 = -1$, $H_2 = 1$
$K_2 = 0$, $H_2 = 0$
$K_2 = -1$, $H_2 = 0$
$K_2 = -1$, $H_2 = -1$
$K_3 = 1$, $H_3 = 1$
$K_3 = 0$, $H_3 = 1$
$K_3 = -1$, $H_3 = 1$
$K_3 = 0$, $H_3 = 0$
$K_3 = -1$, $H_3 = 0$
$K_3 = -1$, $H_3 = -1$
+1
-1
34
±1

VIII. The MacDowell-Mansouri Line Element

The parameters K_i which are introduced above have an additional interesting interpretation. Define the MacDowell-Mansouri line element as follows

$$d\sigma^2 = \sum_{\substack{AB=1 \\ (A<B)}}^{10} A_\mu^{AB} A_\nu^{AB} \eta_A \eta_B \, dx^\mu dx^\nu$$

This is easy to evaluate, and turns out to be very simple:

$$d\sigma^2 = \eta_1\eta_5 dx_1^2 + \eta_2\eta_5 dx_2^2 (q_1^2 + \eta_1\eta_5 p_1^2) + \eta_3\eta_5 dx_3^2 (q_1^2 q_2^2 + \eta_1\eta_5 p_1^2 q_2^2 + \eta_2\eta_5 p_2^2)$$
$$+ \eta_4\eta_5 dx_4^2 (q_1^2 q_2^2 q_3^2 + \eta_1\eta_5 p_1^2 q_2^2 q_3^2 + \eta_2\eta_5 p_2^2 q_3^2 + \eta_3\eta_5 p_3^2)$$

The MM "spacetime" is flat, with a signature given by the K's.

$$d\sigma^2 = \eta_1\eta_5 dx_1^2 + \eta_1\eta_2 H_1 dx_2^2 + \eta_2\eta_3 H_2 dx_3^2 + \eta_3\eta_4 H_3 dx_4^2$$
$$= \sum_{i=1}^{4} K_i \, dx_i^2$$

IX. Comments

This study reveals elements of simplicity which deserve further exploration. The symplectic structure in particular should be fleshed out. Two approaches suggest themselves. One is by looking at the covariant-conformal, as well as the Painleve-Gullstrand cartographies, both of which produce elegant descriptions of deSitter space, but do not fall into the OTAC category.

Another approach is to lean on the spinorial, Clifford-algebra description of the MacDowell-Mansouri extension. It clearly has a close linkage to what we have explored in this note. But the details are not at all clear to me at this stage.

In either case, the content of this note does seem to me to be somewhat nontrivial. Further exploration in this direction would seem to be very worthwhile.