Math 2471 – Calc 3

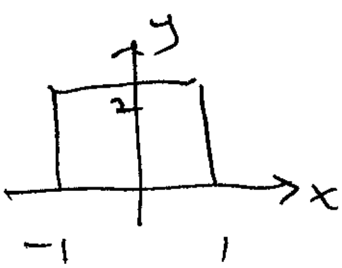
Double Integrals

$$\iint_R f(x,y)\,dA$$

R – region of integration

$dA = dx\,dy$ or $dy\,dx$

Ex. $R = \{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le 2\}$



$$\int_{-1}^{1}\int_{0}^{2}(2x+4y)\,dy\,dx = \int_{-1}^{1} 2xy+2y^2\Big|_0^2\,dx$$

$$= \int_{-1}^{1}(4x+8)\,dx = 2x^2+8x\Big|_{-1}^{1} = (2+8)-(2-8) = 16$$

Could also do $\displaystyle\int_{0}^{2}\int_{-1}^{1}(2x+4y)\,dx\,dy$

In general

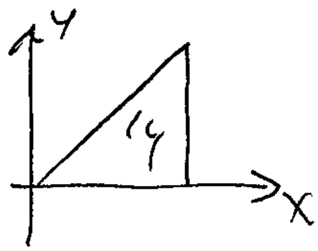
$$\int\int f(x,y)\,dy\,dx$$

curve → curve

pt → pt

Ex 2 $\iint_R 2xy\,dA \qquad R=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le x\}$

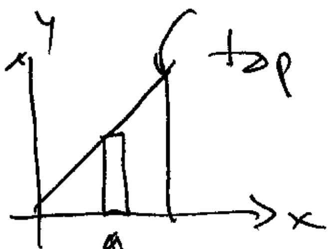
So the region is



$$\int_{x=0}^{1}\int_{y=0}^{x} 2xy\,dy\,dx = \int_0^1 xy^2\Big|_0^x dx = \int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

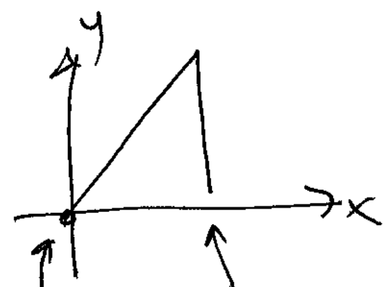
curve $\Rightarrow$ curve

Here



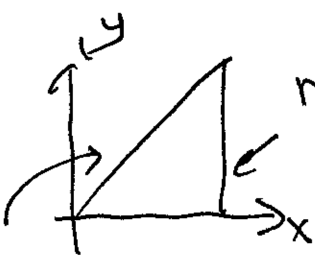
top curve

bottom curve



left pt right pt

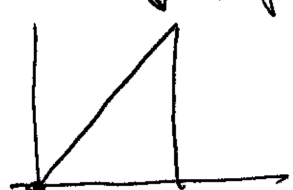
Instead



right curve $x=1$

left curve $x=y$

top pt $y=1$

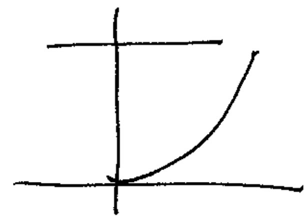


bottom pt $y=0$

so $$\int_{y=0}^{1}\int_{x=y}^{x=1} 2xy\,dx\,dy = \int_0^1 x^2y\Big|_y^1 dy = \int_0^1 y-y^3\,dy = \frac{y^2}{2}-\frac{y^4}{4}\Big|_0^1$$
$$=1/4$$

Ex 3 Given $\int_{x=0}^{2} \int_{y=x^2}^{4} f \, dy \, dx$

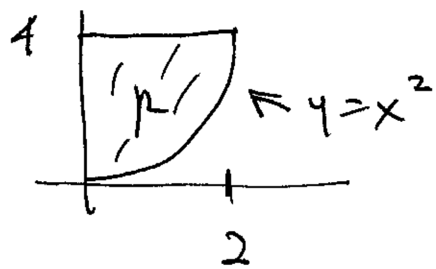
Sketch the region



1st dy curve → curve $y = x^2 \to y = 4$

2nd dx pt → pt $x = 0 \to x = 2$

together



Now switch the order of integration

left curve $x = 0$

right curve $y = x^2$ so $x = \pm\sqrt{y}$ choose +ve (right side of parabola)

bottom pt $y = 0$

top pt $y = 4$

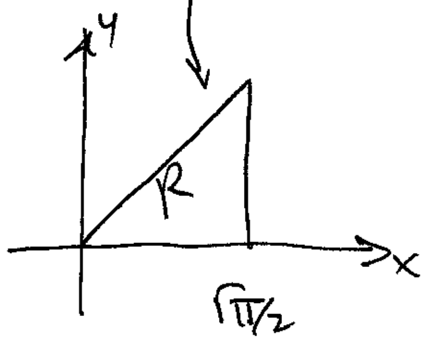
$$\int_0^4 \int_{x=0}^{\sqrt{y}} f(x,y) \, dx \, dy$$

Ex 4 $\displaystyle\int_0^{\sqrt{\pi/2}}\int_y^{\sqrt{\pi/2}} \cos(x^2)\,dx\,dy$

Here, I can't integrate $\cos(x^2)dx$? So we'll switch the order of integration

left curve $x=y$ right curve $x=\sqrt{\pi}$

bottom pt $y=0$ top $y=\sqrt{\pi}$



so $\displaystyle\int_{x=0}^{\sqrt{\pi/2}}\int_{y=0}^{x} \cos(x^2)\,dy\,dx$

pt → pt

bottom → top curve

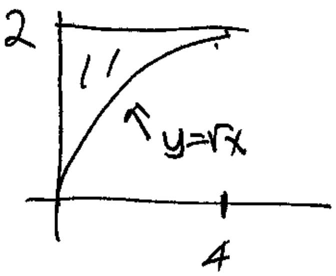
$$\int_0^{\sqrt{\pi/2}} \cos(x^2)y\Big|_0^x dx = \int_0^{\sqrt{\pi/2}} x\cos(x^2)\,dx$$

$$= \frac{\sin(x^2)}{2}\Big|_0^{\sqrt{\pi/2}} = \frac{\sin(\pi/2)}{2} - \frac{\sin 0}{2} = \frac{1}{2} - 0 = \frac{1}{2}$$

Ex 5 Evaluate

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{dy\,dx}{\sqrt{y^3+1}}$$

let $u = y^3+1$



$$\int_0^2 \int_0^{y^2} \frac{dx\,dy}{\sqrt{y^3+1}}$$

left curve $x=0$

right curve $y=\sqrt{x}$

or $x=y^2$

bottom pt $y=0$

top pt $y=2$

$$\int_0^2 \frac{x}{\sqrt{y^3+1}}\Big|_0^{y^2} dy = \int_0^2 \frac{y^2}{\sqrt{y^3+1}}\,dy$$

$$= \frac{2}{3}\sqrt{y^3+1}\Big|_0^2$$

$$= \frac{2}{3}\sqrt{9} - \frac{2}{3}\sqrt{1} = \frac{2}{3}\cdot 3 - \frac{2}{3} = 4/3$$

HW pg