Calculus 3 - Cylindrical and Spherical Coordinates

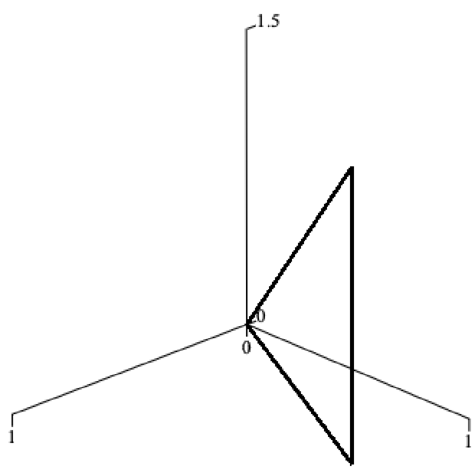
Cylindrical Polar Coordinates

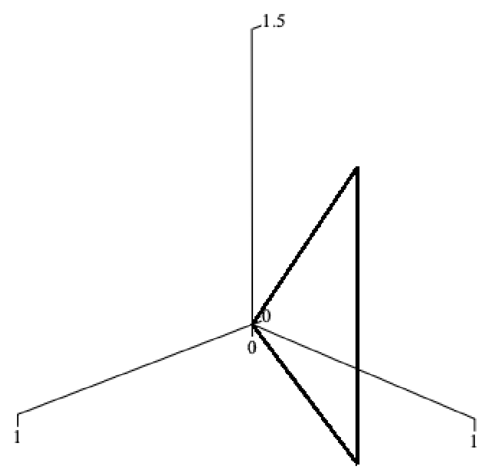
These are similar to the usual polar coordinates that are in 2D but we simply add z to extend to 3D so

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{1}$$

Spherical Polar Coordinates

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi. \tag{2}$$





Triple Integrals

We consider triple integrals

$$V = \iiint_V F(x,y,z)dV \tag{3}$$

in each of these two coordinate systems. We first consider cylindrical and then spherical.

Cylindrical Polar Coordinates

This integral has three main parts:

1. the integrand
2. dV
3. limits

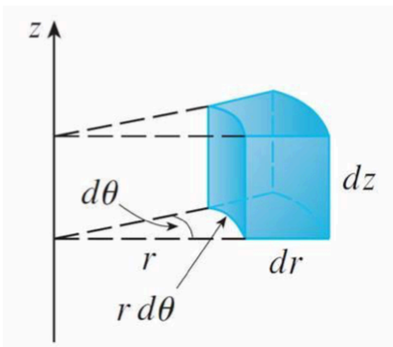
1. The integrand. For this part, we simplify substitute in

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{4}$$

into $F(x, y, z)$ and simplify. So, in general,

$$\iiint_V F(x, y, z)dA = \iiint_V F(r\cos\theta, r\sin\theta, z)dV. \tag{5}$$

2. dV. In Cartesian coordinates, this is $dV = dxdydz$. In cylindrical coords



$$dV = ds\,dr\,dz = rdr\,d\theta\,dz \tag{6}$$

3. Limits of Integration. These ultimately come from the picture of the volume itself. So

$$\int_\alpha^\beta \int_{r_i(\theta)}^{r_o(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r\cos\theta, r\sin\theta, z)rdz\,dr\,d\theta \tag{7}$$

Spherical Polar Coordinates

This integral has three main parts:

$$\iiint\limits_V F(x,y,z)dV \tag{8}$$

1. the integrand
2. dV
3. limits

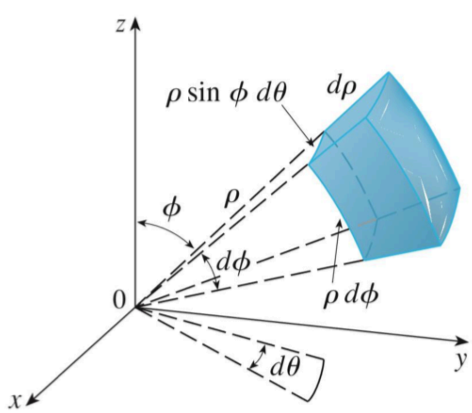
1. The integrand. For this part, we simplify substitute in

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi. \tag{9}$$

into $F(x,y,z)$ and simplify. So, in general,

$$\iiint\limits_V F(x,y,z)dA = \iiint\limits_V F(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)dV. \tag{10}$$

2. dV. In Cartesian coordinates, this is $dV = dxdydz$. In spherical coords



$$dV = ds_1 ds_2\, d\rho = \rho^2 \sin\phi d\rho d\phi d\theta \tag{11}$$

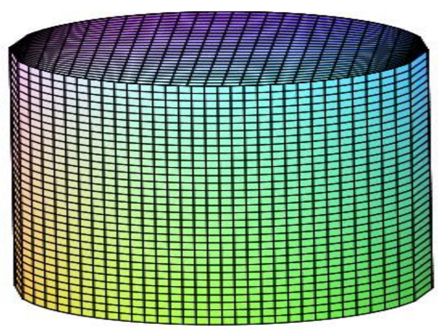
3. Limits of Integration. These ultimately come from the picture of the volume itself. So

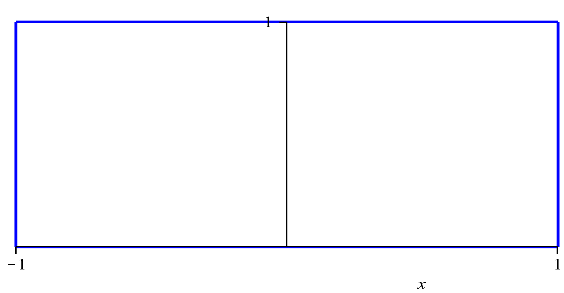
$$\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} F(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi d\rho d\phi d\theta. \tag{12}$$

Example 1. Set up the triple integral for the volume bound by $z = 0, z = 1$, and inside the cylinder $x^2 + y^2 = 1$ and evaluate

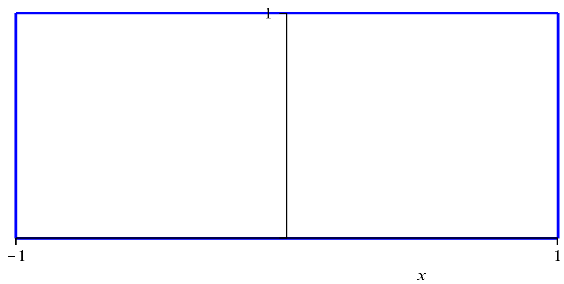
$$\iiint\limits_V dV \tag{13}$$

Soln: We first picture the volume with a side view





Side View



top $z = 1 \Rightarrow \rho\cos\phi = 1$

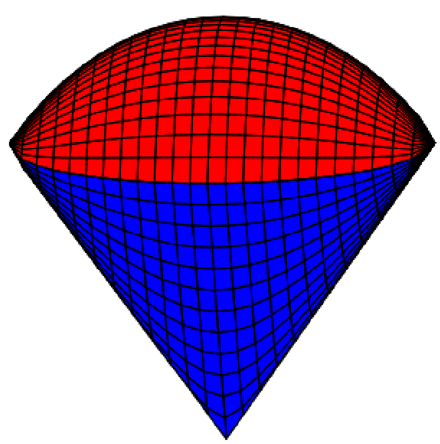
side $x^2 + y^2 = 1 \Rightarrow \rho^2\sin^2\phi = 1$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi d\rho d\phi d\theta + \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\csc\phi} \rho^2\sin\phi d\rho d\phi d\theta$$

Example 1. Set up the triple integral for the volume bound by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$ and evaluate

$$\iiint\limits_V z dV \tag{12}$$

Soln:



We first convert each surface to spherical polar coords. They become $\rho =$

$\sqrt{2}$ and $\phi = \pi/4$. We see that

$$\rho = 0 \to \sqrt{2}, \;\; \phi = 0 \to \pi/4, \;\; \text{and} \;\; \theta = 0 \to 2\pi.$$

$$\begin{aligned}
&\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho \cos\phi \rho^2 \sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{4} \rho^4 \Big|_0^{\sqrt{2}} \sin\phi \cos\phi d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \sin\phi \cos\phi d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{1} \frac{1}{2} \sin^2\phi \Big|_0^{\pi/4} d\theta \\
&= \frac{1}{4} \int_0^{2\pi} d\theta \\
&= \frac{1}{4}\, \theta \Big|_0^{2\pi} = \frac{\pi}{2}.
\end{aligned}$$

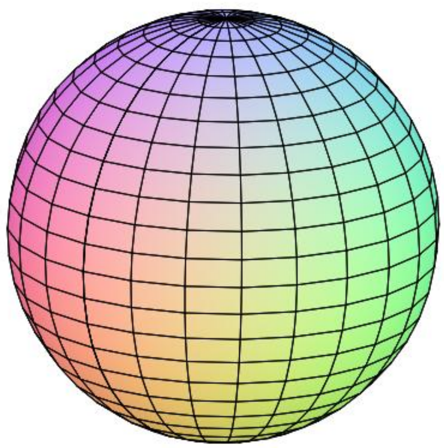
Let us compare with the same setup in cylindrical polar coordinates.

$$\begin{aligned}
&\int_0^{2\pi} \int_0^{1} \int_r^{\sqrt{2-r^2}} z\, r\, dz\, dr d\theta \\
&= \int_0^{2\pi} \int_0^{1} r\, \frac{1}{2} z^2 \Big|_r^{\sqrt{2-r^2}} dr\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^{1} r\, \left(2 - r^2 - r^2\right) dr\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^{1} \left(2r - 2r^3\right) dr\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} r^2 - \frac{1}{2} r^4 \Big|_0^1 d\theta \\
&= \frac{1}{4}\, \theta \Big|_0^{2\pi} = \frac{\pi}{2}.
\end{aligned}$$

Example 2. Set up the triple integral for the volume bound by the sphere $x^2 + y^2 + z^2 = 1$ and evaluate

$$\iiint\limits_V 1dV \tag{12}$$

Soln:



We first convert each surface to spherical polar coords. They become $\rho = 1$. We see that $\rho = 0 \to 1, \;\; \phi = 0 \to \pi, \;\;$ and $\;\; \theta = 0 \to 2\pi$.

$$\begin{aligned}
&\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{3}\rho^3 \Big|_0^1 \sin\phi d\phi d\theta \\
&= \frac{1}{3} \int_0^{2\pi} - \cos\phi \Big|_0^{\pi} d\phi d\theta \\
&= \frac{1}{3} \times 2 \int_0^{2\pi} d\theta \\
&= \frac{1}{3} \times 2 \times 2\pi = \frac{4}{3}\pi
\end{aligned}$$