ONTARIO MATH CIRCLES
ANNUAL ARML TEAM SELECTION TEST
2019

Instructions

1. Do not open this competition package until you are told to do so.
2. This competition contains 40 problems to be answered in 120 minutes. This is open to all high school and middle school students.
3. Print your name clearly on your answer sheet. Phone number and e-mail addresses are requested to contact top finishers about being on the Ontario Math Circles ARML team. If you do not wish to be consider for this team, you may leave those blank. Please print these, especially e-mail address, legibly.
4. Print clearly and legibly on the answer package. You must leave all answers in exact form. For example, π would be correct while 3.1415 would not. You must simplify your answers as much as possible. For example, $\frac{6}{4}$ must be simplified to $\frac{3}{2}$ and all denominators must be rationalized. Perfect squares must be removed from radicals. For example, $\sqrt{99}$ must be written as $3\sqrt{11}$. Trig functions of standard arguments must be evaluated. Frequently, several equivalent expressions will be considered correct. For example, $\frac{3}{2}$, $1\frac{1}{2}$, and 1.5 will all be considered correct.
5. Each correct answer is worth 1 point. A blank and an incorrect answer are both worth 0 points. There are no fractional points.
6. Only pencils, erasers, and pens are allowed. **Electronic devices and calculators are not allowed.** Everything else must be approved by the proctor.
7. You may keep your question sheet and scratch work.
8. The full results of this competition will be posted on the Toronto Math Circles' website. It will only display your name, school, grade, and score.
9. The prize for top scorers will be a numerical amount to be deducted from one of their future ARML trip fees. This amount is not transferable nor redeemable for cash.
10. ***Do not discuss the problems or solutions from this contest for the next 7 days.***
11. GOOD LUCK!

1. Evaluate the sum $0+1+2-3-4-5+6+7+8-\cdots-2019$.

2. What is the smallest prime divisor of $2017^{2017}+2019^{2019}$?

3. Determine the number of ways to assign 3 red and 3 blue to the integers $1,2,3,4,5,6$ such that no two consecutive integers have the same colour.

4. Let l be the line that passes through the origin and $(1,2)$. Find the shortest distance from the point $(0,10)$ to l.

5. Define the operation $*$ with $a*b=a^2-2^b$. Solve the following inequality for all real solutions

$$t*t<t*1\leq t$$

6. Find the unit's digit of 3^{3^3}.

7. How many positive integers less than 2018 is divisible by 2 or 3?

8. The system of equations

$$\begin{cases} x^2+y^2=100 \\ x^2+y^2+180=20(x+y) \end{cases}$$

intersect at two distinct points A and B. Determine the equation of the line that passes through A and B.

9. Let $f(x)=ax^2+bx+c$ with $a\neq 0$. If $f(f(0))=f(0)$, find the smallest possible value of the discriminant of $f(x)$.

10. Consider the list of 2019 numbers $111, 1011, 10011, 100011, 1000011, \ldots$. How many of these numbers are divisible by 33?

11. Determine the number of ways to assign red, blue, and green to the integers $1,2,3,4,5,6$ such that no two consecutive integers have the same colour.

12. Find all ordered pairs of real numbers (x,y) which satisfy

$$2\log x=\log x^2+2\log y$$

13. Consider the following system of inequalities

$$\begin{cases} -5<y\leq\lfloor x\rfloor \\ 0\leq x<10 \end{cases}$$

Find the area of the region.

14. Find the minimum value of $x6^{\frac{1}{x}}+\frac{1}{x}6^x$ for $x>0$.

15. Find all positive integers n such that 7^n+147 is a perfect square.

16. In $\triangle ABC$, $\angle B=90^\circ$, $\angle A=60^\circ$, and $AB=1$. Let $\angle B$ be trisected to create three smaller triangles. If r_1,r_2,r_3 are the inradii of these three triangles, find $r_1+r_2+r_3$.

17. Compute $\sum_{k=1}^{\infty}\frac{k\pi}{\pi^k}$.

18. How many ways are there to choose 3 numbers from $1,2,3,\ldots,10$ such that no 2 of the 3 numbers are consecutive?

19. Find the sum of all positive integers n for which $f(n)=n^4-360n^2+400$ is a prime number.

20. If $x>y>0$ and $2\log_{10}(x-y)=\log_{10}x+\log_{10}y$, what is the value of $\frac{x}{y}$.

21. Find the tens digit of the sum

$$1!+2!+3!+\cdots+2019!$$

22. In $\triangle ABC$, D is a point on BC such that $\angle BAD = 30^\circ$ and $\angle DAC = 15^\circ$. If $AB = 3\sqrt{2}$ and $AC = 6$, find the length of AD.

23. Find positive integers a, b if for every $x, y \in [a, b]$ then $\frac{1}{x} + \frac{1}{y} \in [a, b]$.

24. Let $a_n = n^2 + 2n + 50$, $n = 1, 2, \ldots$. Let d_n be the largest positive integer that is a divisor of both a_n and a_{n+1}. Find the maximum possible value of d_n for $n = 1, 2, \ldots$.

25. Tom writes down the integers from 1 to 100, inclusive, on a chalkboard and then erases every number that contains a prime digit. Compute the sum of the digits that remain on the chalkboard.

26. Let the tangent line passing through a point A outside the circle with center O touches the circle at B and C. Let BD be the diameter of the circle. Let the lines CD and AB meet at E. If the lines AD and OE meet at F, find $|AF|/|FD|$.

27. Let $A = \{z : z^{\binom{5}{2}} = 1\}$ and $B = \{w : w^{\binom{10}{2}} = 1\}$. Determine the number of distinct elements in $\{zw : z \in A, w \in B\}$.

28. Determine the number of ways to assign each of the integers 1 to 8 to $a_1, a_2, \ldots, a_8$ such that $|a_n - n| \leq 1$ for $n = 1, 2, \ldots, 8$.

29. Find the number of positive integer pairs (m, n) with $m, n \leq 50$ such that $m + n + 1$ is prime and divides $2\left(m^2 + n^2\right) - 1$.

30. Let AB and CD be two segments length 1. If they intersect at O such that $\angle AOC = 60^\circ$, find the minimum of $AC + BD$.

31. How many 6-digit numbers whose leftmost digit is 1 have exactly two pairs of identical digits (and no digit occurs three or more times)?

32. Given that G is the centroid of $\triangle ABC$, $GA = 2\sqrt{3}$, $GB = 2\sqrt{2}$, $GC = 2$. Find the area of $\triangle ABC$.

33. Find the number of positive integers n such that $n^2 \leq 2019$ is the product of all positive proper divisors of n.

34. Let $f(x)$ be a rational coefficient polynomial with $(1+i)\frac{\sqrt{6}}{4} + (1-i)\frac{\sqrt{2}}{4}$ as a root. Find the minimum possible positive degree of $f(x)$.

35. Find all real numbers $x \in \left[0, \frac{\pi}{2}\right]$, such that $(2 - \sin 2x)\sin(x + \frac{\pi}{4}) = 1$.

36. An integer $n \geq 2$ is called friendly if there exists a family $A_1, A_2, \ldots, A_n$ of subsets of the set $\{1, 2, \ldots, n\}$ such that:
(1) $i \notin A_i$ for every $i = 1, 2, \ldots, n$;
(2) $i \in A_j$ if and only if $j \notin A_i$, for every distinct $i, j \in \{1, 2, \ldots, n\}$;
(3) $A_i \cap A_j$ is non-empty, for every $i, j \in \{1, 2, \ldots, n\}$.
Determine the smallest friendly number.

37. On the Cartesian plane the curve (C) has equation $x^2 = y^3$. A line d varies on the plane such that d always cut (C) at three distinct points with x-coordinates x_1, x_2, x_3. Determine the maximum possible value of $\sqrt[3]{\frac{x_1x_2}{x_3^2}} + \sqrt[3]{\frac{x_2x_3}{x_1^2}} + \sqrt[3]{\frac{x_3x_1}{x_2^2}}$.

38. Let $f(x, y) = \frac{x+y}{(x^2+1)(y^2+1)}$. Find the maximum value of $f(x, y)$ for all $x, y \in \mathbb{R}$.

39. For any $x_i \geq 0$, $i = 1, 2, \ldots, 2018$, if let $x_{2019} = x_1$, find the minimum value of

$$\sum_{k=1}^{2018} \sqrt{\frac{1}{(x_k+1)^2} + \frac{x_{k+1}^2}{(x_{k+1}+1)^2}}$$

40. Find the maximum number of edges of a 4 dimensional cube that are cut by a hyperplane.