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Gravitational Waves in the First Order Formalism

I. Introduction

This note is self-pedagogical, and is meant simply to familiarize myself on how gravitational waves emerge from the Einstein-Cartan action. Our motivation is to eventually introduce the Holst term and a fermion condensate, and see how the description of gravitational waves is modified with these symmetry-violating terms included.

II. Relevant Degrees of Freedom

We write the vierbein in the following form:

$$\epsilon_\mu^A = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1+\epsilon_x^1) & \epsilon_y^1 & 0 \\ 0 & \epsilon_x^2 & (1+\epsilon_y^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This should suffice to capture the transverse-traceless piece of the metric which describes a gravitational wave propagating in the z direction. (All degrees of freedom are assumed to only depend upon z and t.)

The only assumption regarding the connection ω_μ^{AB} is that we may choose temporal gauge:

$$\omega_t^{AB} = 0$$

Our first job is to compute the curvature tensor R :

$$R_{\mu\nu}^{AB} = \partial_\mu \omega_\nu^{AB} - \partial_\nu \omega_\mu^{AB} + \omega_\mu^{AC} \eta_C \omega_\nu^{CB} - \omega_\nu^{AC} \eta_C \omega_\mu^{CB}$$

The Minkowski metric in the above equation is of course

$$\eta_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta_A \delta_{AB}$$

From the above formulae the computation of the sundry components of R is straightforward, and the result is, in general notation (The dot indicates a time derivative, and the prime indicates a derivative with respect to z.)

$$R_{tx}^{AB} = (\dot{\omega}_x^{AB}) \qquad R_{ty}^{AB} = (\dot{\omega}_y^{AB}) \qquad R_{tz}^{AB} = (\dot{\omega}_z^{AB})$$

$$R_{xy}^{AB} = \qquad -\omega_x^{0A}\omega_y^{0B} + \omega_y^{0A}\omega_x^{0B} + \sum_{i=1}^{3}\left(\omega_x^{iA}\omega_y^{iB} - \omega_y^{iA}\omega_x^{iB}\right)$$

$$R_{zx}^{AB} = (\omega_x^{AB}) - \omega_z^{0A}\omega_x^{0B} + \omega_x^{0A}\omega_z^{0B} + \sum_{i=1}^{3}\left(\omega_z^{iA}\omega_x^{iB} - \omega_x^{iA}\omega_z^{iB}\right)$$

$$R_{zy}^{AB} = (\omega_y^{AB})' - \omega_z^{0A}\omega_y^{0B} + \omega_y^{0A}\omega_z^{0B} + \sum_{i=1}^{3}\left(\omega_z^{iA}\omega_y^{iB} - \omega_y^{iA}\omega_z^{iB}\right)$$

We write the Einstein-Cartan Lagrangian density (with no cosmological term!) as follows:

$$\mathcal{L} = \mathcal{L}_{(tx)} + \mathcal{L}_{(ty)} + \mathcal{L}_{(tz)} + \mathcal{L}_{(zx)} + \mathcal{L}_{(zy)} + \mathcal{L}_{(xy)}$$
$$\equiv \left[R_{tx}^{AB}e_y^C e_z^D + R_{ty}^{AC}e_x^B e_z^D + R_{tz}^{AD}e_x^B e_y^C + R_{zx}^{DB}e_y^C e_t^A + R_{zy}^{DC}e_x^B e_t^A + R_{xy}^{BC}e_z^D e_t^A\right]\epsilon_{ABCD}$$

The first and second terms are

$$\mathcal{L}(tx) = R_{tx}^{01}e_y^2 e_z^3 - R_{tx}^{02}e_y^1 e_z^3 = (\dot{\omega}_x^{01}) + (\dot{\omega}_x^{01})e_y^2 - (\dot{\omega}_x^{02})e_y^1$$
$$\mathcal{L}(ty) = R_{ty}e_x e_z - R_{ty}e_x e_z = (\dot{\omega}_y^{02}) + (\dot{\omega}_y^{02})e_x^1 - (\dot{\omega}_y^{01})e_x^2$$

The linear terms of course can be neglected. In some of the subsequent terms we will also neglect terms of third order in the perturbation. This does occur in the next term. We also do not display terms containing e_x^3 or e_y^3 which clearly vanish.

$$\mathcal{L}_{(tz)} = R_{tz}^{03}\left(e_x^1 e_y^2 - e_x^2 e_y^1\right) + \cdots$$
$$= (\omega_z^{03})\left[1 + \epsilon_x^1 + \epsilon_y^2\right] + 3^{\text{rd}} \text{ order}$$

In the next two terms, $e_t \Rightarrow \overset{\circ}{e}_t = 1$, providing some simplification.

$$\mathcal{L}_{(zx)} = R^{31}_{zx} e^{2}_{y} - R^{32}_{zx} e^{1}_{y}$$

$$= (\omega^{31}_{x})' + (\omega^{31}_{x})' \epsilon^{2}_{y} - (\omega^{32}_{x})' \epsilon^{1}_{y} - \omega^{03}_{z}\omega^{01}_{x} + \omega^{03}_{x}\omega^{01}_{z} + \omega^{23}_{z}\omega^{21}_{x} - \omega^{23}_{x}\omega^{21}_{z} + 3^{\underline{rd}}\text{ order}$$

$$\mathcal{L}_{(zy)} = R^{32}_{zy} e^{1}_{x} - R^{31}_{zy} e^{2}_{x}$$

$$= (\omega^{32}_{y})' + (\omega^{32}_{y})' \epsilon^{1}_{x} - (\omega^{31}_{y})' \epsilon^{2}_{x} - \omega^{03}_{z}\omega^{02}_{y} + \omega^{03}_{y}\omega^{02}_{z} + \omega^{13}_{z}\omega^{12}_{y} - \omega^{13}_{y}\omega^{12}_{z} + 3^{\underline{rd}}\text{ order}$$

In the next and final term, there is the additional simplification coming from $e_z \Rightarrow e^3_z = 1$.

$$\mathcal{L}_{(xy)} = R^{12}_{xy} e^{3}_{z} e^{0}_{t} = R^{12}_{xy}$$

$$= -\omega^{01}_{x}\omega^{02}_{y} + \omega^{01}_{y}\omega^{02}_{x} - \omega^{31}_{x}\omega^{32}_{y} + \omega^{31}_{y}\omega^{32}_{x}$$

We may now assemble the full Lagrangian density:

$$\begin{aligned}
\mathcal{L} = {} & (\dot{\omega}^{01}_{x})\epsilon^{2}_{y} - (\dot{\omega}^{02}_{x})\epsilon^{1}_{y} + (\dot{\omega}^{02}_{y})\epsilon^{1}_{x} - (\dot{\omega}^{01}_{y})\epsilon^{2}_{x} \\
& + (\dot{\omega}^{03}_{z})\epsilon^{1}_{x} + (\dot{\omega}^{03}_{z})\epsilon^{2}_{y} \\
& + (\omega^{31}_{x})'\epsilon^{2}_{y} - (\omega^{32}_{x})'\epsilon^{1}_{y} + (\omega^{32}_{y})'\epsilon^{1}_{x} - (\omega^{31}_{y})'\epsilon^{2}_{x} \\
& - \omega^{03}_{z}\omega^{01}_{x} + \omega^{03}_{x}\omega^{01}_{z} + \omega^{23}_{z}\omega^{21}_{x} - \omega^{23}_{x}\omega^{21}_{z} \\
& - \omega^{03}_{z}\omega^{02}_{y} + \omega^{03}_{y}\omega^{02}_{z} + \omega^{13}_{z}\omega^{12}_{y} - \omega^{13}_{y}\omega^{12}_{z} \\
& - \omega^{01}_{x}\omega^{02}_{y} + \omega^{01}_{y}\omega^{02}_{x} + \omega^{31}_{x}\omega^{32}_{y} - \omega^{31}_{y}\omega^{32}_{x}
\end{aligned}$$

There are 22 parameters, leading to the same number of equations of motion. Before displaying them, we note that schematically (ignoring index hell) they are of the form

$$\mathcal{L} \sim \mathring{k}\epsilon + c'\epsilon + kk + cc$$

Here we have labeled the elements of the connection having a gauge index of zero as k 's (extrinsic curvatures) and the remainder as c's (contortion and/or curvature). The variational principle will yield equations of the form

$$\delta k: \quad \mathring{\epsilon} = k \Rightarrow \ddot{\epsilon} = \mathring{k}$$
$$\delta c: \quad -\mathring{\epsilon} = c \Rightarrow -\mathring{\epsilon}' = c'$$
$$\delta \epsilon: \quad \mathring{k} = -c'$$
$$\Rightarrow \ddot{\epsilon} = \epsilon''$$

The strategy for simplifying the equations of motion will be to eliminate the k's by taking a time derivative of their defining equations, and to eliminate the c's by taking a space derivative of their equations. As shown above, the result should be of the form of wave equations for the vierbein parameters ϵ .

The relevant equations of motion are displayed below:

$\delta\omega$:

	x	y	z
01	$(\mathring{\epsilon}^2_y) = -\omega^{03}_z - \omega^{02}_y$	$-(\mathring{\epsilon}^2_x) = \omega^{02}_x$	$0 = \omega^{03}_x$
02	$-(\mathring{\epsilon}^1_y) = \omega^{01}_y$	$(\mathring{\epsilon}^1_x) = -\omega^{03}_z - \omega^{01}_x$	$0 = \omega^{03}_y$
03	$0 = \omega^{01}_z$	$0 = \omega^{02}_z$	$(\mathring{\epsilon}^1_x) + (\mathring{\epsilon}^2_y) = -\omega^{01}_x - \omega^{02}_y$
31	$0 = -(\epsilon^2_y)' + \omega^{32}_y$	$0 = (\epsilon^2_x)' + \omega^{12}_z - \omega^{32}_x$	$0 = -\omega^{12}_y$
32	$0 = (\epsilon^2_y)' - \omega^{12}_z - \omega^{31}_y$	$0 = -(\epsilon^1_x)' + \omega^{31}_x$	$0 = \omega^{12}_x$
12	$0 = \omega^{32}_z$	$0 = -\omega^{31}_z$	$0 = -\omega^{32}_x + \omega^{31}_y$

$\delta\epsilon$:

	x	y
1	$0 = (\mathring{\omega}^{02}_y) + (\mathring{\omega}^{03}_z) + (\omega^{32}_y)'$	$0 = -(\mathring{\omega}^{02}_x) - (\omega^{32}_x)'$
2	$0 = -(\mathring{\omega}^{01}_y) - (\omega^{31}_y)'$	$0 = (\mathring{\omega}^{01}_x) + (\mathring{\omega}^{03}_z) + (\omega^{31}_x)'$

These 22 equations organize themselves into groups. There are 8 equations which directly set variables to zero.

$$\omega_x^{03} = \omega_x^{12} = \omega_y^{03} = \omega_y^{12} = \omega_z^{01} = \omega_z^{02} = \omega_z^{31} = \omega_z^{32} = 0$$

The 14 remaining equations separate into two groups of 7, corresponding to the "$+$" and " X " graviton polarization states.

There are 7 which relate the " X " degrees of freedom to each other.:

$$(\dot{\epsilon}_x^2) = -\omega_x^{02} \qquad (\epsilon_x^2)' = \omega_x^{32} - \omega_z^{32} \qquad (\dot{\omega}_x^{02}) = -(\omega_x^{32})'$$

$$(\dot{\epsilon}_y^1) = -\omega_y^{01} \qquad (\epsilon_y^1)' = \omega_y^{31} + \omega_z^{12} \qquad (\dot{\omega}_y^{01}) = -(\omega_y^{31})'$$

$$\omega_x^{32} = \omega_y^{31}$$

The remaining 7 equations relate the "$+$" degrees of freedom to each other.

$$(\dot{\epsilon}_x^1) = -\omega_x^{01} - \omega_z^{03} \qquad (\epsilon_x^1)' = \omega_x^{31} \qquad (\dot{\omega}_z^{03}) = -(\dot{\omega}_x^{01}) - (\omega_x^{31})'$$

$$(\dot{\epsilon}_y^2) = -\omega_y^{02} - \omega_z^{03} \qquad (\epsilon_y^2)' = \omega_y^{32} \qquad (\dot{\omega}_z^{03}) = -(\dot{\omega}_y^{02}) - (\omega_y^{32})'$$

$$\epsilon_x^1 + \epsilon_y^2 = -\omega_x^{01} - \omega_y^{02}$$

Examination of these equations requires that $\omega_z^{30} = 0$. It also appears that ω_z^{12} will not satisfy a wave equation. Therefore we set it to zero. So we find that all components of ω_z can be chosen to vanish.

$$\omega_z^{30} = 0 \qquad \omega_z^{12} = 0 \qquad \Rightarrow \qquad \omega_z^{AB} = 0$$

As a consequence, we can express everything in terms of the two graviton polarization degrees of freedom as follows:

$$\epsilon^{\times} \equiv \epsilon_x^2 = \epsilon_y^1 \qquad \epsilon^{+} \equiv \epsilon_x^1 = -\epsilon_y^2$$

$$k^{\times} \equiv \omega_x^{02} = \omega_y^{01} \qquad k^{+} \equiv \omega_x^{01} = -\omega_y^{02}$$

$$c^{\times} \equiv \omega_x^{32} = \omega_y^{31} \qquad c^{+} \equiv \omega_x^{31} = -\omega_y^{32}$$

The 14 equations are thereby reduced to two, essentially identical, sets of three equations:

$$\dot{\epsilon}^{\times} = -(k^{\times})' \qquad (\epsilon^{\times})' = c^{\times} \qquad \dot{k}^{\times} = -(c^{\times})'$$

$$\dot{\epsilon}^{+} = -(k^{+})' \qquad (\epsilon^{+})' = c^{+} \qquad \dot{k}^{+} = -(c^{+})'$$

These can be reduced to the desired wave-equation form:

$$\ddot{\epsilon}^{\times} = (\epsilon^{\times})'' \qquad \ddot{\epsilon}^{+} = (\epsilon^{+})''$$

III. Inclusion of a Cosmological Background

We now redo the previous exercise for FRW Cosmology, with special attention to the deSitter-space special case. However, this time we can use the experience gained in the previous exercise by reducing the number of variables and the number of nonvanishing entries in the connection ω ab initio. We limit our attention to the case of the "$+$" polarization, and write for the vierbein and the connection

$$
e^{A}_{\mu} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & (a+\epsilon) & 0 & 0 \\ 0 & 0 & (a-\epsilon) & 0 \\ 0 & 0 & 0 & a \end{pmatrix}_{+} \qquad \begin{aligned} a &= a(t) \\ \epsilon &= \epsilon(t,z) \end{aligned}
$$

$$
\omega^{AB}_{\mu} = \begin{array}{c} 01 \\ 02 \\ 03 \\ 31 \\ 32 \\ 12 \end{array} \begin{pmatrix} 0 & (K+k) & 0 & 0 \\ 0 & 0 & (K-k) & 0 \\ 0 & 0 & 0 & K \\ 0 & c & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{+} \qquad \begin{aligned} K &= K(t) \\ k &= k(t,z) \\ c &= c(t,z) \end{aligned}
$$

This time the Riemann tensor takes the form

$$
R^{AB}_{\mu\nu} = \begin{array}{c|cccccc} & & & & & & \\ 01 & (\dot{K}+\dot{k}) & 0 & 0 & (k'-Kc) & 0 & 0 \\ 02 & 0 & (\dot{K}-\dot{k}) & 0 & 0 & (-k'+Kc) & 0 \\ 03 & 0 & 0 & \dot{K} & 0 & 0 & 0 \\ 31 & \dot{c} & 0 & 0 & K^2 & 0 & 0 \\ 32 & 0 & -\dot{c} & 0 & 0 & (-c'-K^2+Kk) & 0 \\ 12 & 0 & 0 & 0 & 0 & 0 & (-K^2+k^2-c^2) \\ \hline & tx & ty & tz & zx & zy & xy \end{array}_{+}
$$

The six terms for the Lagrangian are now

$$\mathcal{L}_{(+)}\quad R^{01}_{tx}e^{2}_{y}e^{3}_{z}+R^{02}_{ty}e^{1}_{x}e^{3}_{z}+R^{03}_{tz}e^{1}_{x}e^{2}_{y}+R^{31}_{zx}e^{2}_{y}e^{0}_{t}+R^{32}_{zy}e^{1}_{x}e^{0}_{t}+R^{12}_{xy}e^{3}_{z}e^{0}_{t}$$

$$=(\mathring{K}+\mathring{k})\,a(a-\epsilon)+(\mathring{K}-\mathring{k})\,a(a+\epsilon)+\mathring{K}(a^{2}-\epsilon^{2})$$

$$+N\left[(c'-K^{2}-Kk)(a-\epsilon)+(-c'-K^{2}+Kk)(a+\epsilon)+(-K^{2}+k^{2}-c^{2})a\right]$$

The sum can be classified as follows

$$\mathcal{L}_{(+)}=3\mathring{K}a^{2}-3K^{2}aN$$
$$-\epsilon^{2}\mathring{k}+2Kk\epsilon N$$
$$-2\mathring{k}\epsilon-2c'\epsilon+(k^{2}-c^{2})aN$$

Even in the absence of the background extrinsic curvature terms, and upon setting N(t) = a(t) = 1, K = 0, there are several notable features present in this result. First of all, the Lagrangian does reduce to the previous case, without approximation. Therefore, the solution found in the previous section is actually an exact solution of the field equations. However, the solution does not satisfy the vacuum Einstein equations. This can be seen by computing the metric version of the Riemann tensor, and from it the Ricci/Einstein tensor:

"Riemann"

$$R^{\alpha\beta}_{\mu\nu}=e^{\alpha}_{A}e^{\beta}_{B}R^{AB}_{\mu\nu}=\begin{array}{c}tx\\ ty\\ tz\\ zx\\ zy\\ xy\end{array}\begin{bmatrix}\frac{\mathring{k}}{(1+\epsilon)} & 0 & 0 & \frac{-k'}{(1+\epsilon)} & 0 & 0\\ 0 & \frac{-\mathring{k}}{(1-\epsilon)} & 0 & 0 & \frac{-k'}{(1-\epsilon)} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \frac{\mathring{c}}{(1+\epsilon)} & 0 & 0 & \frac{c'}{(1+\epsilon)} & 0 & 0\\ 0 & \frac{-\mathring{c}}{(1-\epsilon)} & 0 & 0 & \frac{c'}{(1-\epsilon)} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(k^{2}-c^{2})}{(1-\epsilon^{2})}\end{bmatrix}$$

"Ricci"

$$R^{\alpha\nu}_{\mu\nu}=R^{\alpha}_{\mu}=\begin{array}{c}t\\ x\\ y\\ z\end{array}\begin{pmatrix}\frac{-2\mathring{k}\epsilon}{(1-\epsilon^{2})} & 0 & 0 & \frac{-2\mathring{c}\epsilon}{(1-\epsilon^{2})}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \frac{-2k'\epsilon}{(1-\epsilon^{2})} & 0 & 0 & \frac{-2c'\epsilon}{(1-\epsilon^{2})}\end{pmatrix}$$

The exact solutions have the form

$$\begin{aligned} \epsilon &= h(z-t) \\ k &= -\dot{\epsilon} = h'(z-t) \\ c &= \epsilon' = h'(z-t) \end{aligned}$$

Therefore the Ricci tensor is traceless and equal to the Einstein tensor:

$$R_\mu^{\ \alpha} = R_\mu^{\ \alpha} - \frac{1}{2} g_\mu^{\ \alpha} R_\nu^{\ \nu} \equiv G_\mu^{\ \alpha} = \frac{2hh''}{(1-h^2)} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$$

This solution has all the properties we might ask of the energy-momentum tensor describing the gravitational wave:

$$T_z^{\ 0} = -T_0^{\ z} = T_{0z} = T_{z0}$$

Indeed, we would anticipate that a computation done entirely within the metric formalism, but done to higher orders in the amplitude h, would end up with an Einstein tensor of this form. But it does seem remarkable that this occurs automatically, without having to put in anything by hand.

There is a general argument which states that, under circumstances such as these, one should have recovered the vacuum Einstein equations from the Einstein-Cartan formalism, given that no explicit source terms were put in. This argument would also lead to the conclusion that the torsion component of the connection ω vanishes, as a consequence of the variational principle. Given that we have violated these conclusions, it is essential that the origin of the discrepancy be identified.

It is not hard to locate a "problem". In the general argument, all 24 components of the connection are to be subjected to the variational principle. On the other hand, we set $\omega_t = 0$ ab initio, as well as setting $\omega_z = 0$ later on. This led us to a violation of the vacuum equations; the Einstein tensor does not vanish. While one might suspect that the solution we have found contains torsion, this seems not the case. To check this, one can simply take the metric $g_{\mu\nu}$ implied by our choice of vierbein, and compute the Christoffel symbols and Riemann/Ricci tensors using the textbook algorithms. At least for the case of the "+" mode of the gravitational wave, the result is simply the form of the Einstein tensor written above. Since it can be identified with the energy-momentum expected to be carried by gravitational waves, there seems no problem in presuming that this procedure does lead to a physically sensible description of gravitational waves, albeit obtained by the well-known (but all too rare in practice) "guess the answer" method. The more general case of a mixture of "+" and " X " modes has more index hell and was not checked (Maple or Mathematica is the appropriate tool.). However, it is very likely that the energy-momentum tensor, when multiplied by $\sqrt{-\text{ii}\, g_{ii}}$, will simply be the sum of the two contributions.The consequences of this description are discussed in somewhat more general terms in the next section.

IV. Einstein-Cartan Formalism in Temporal Gauge

Before trying to synthesize our first-order description of gravitational waves with FRW cosmology, we shall first explore the formal structure of the first order formalism. This will lead to some notational simplicities, as well as some general results. We do make the following general, simplifying assumptions:

The vierbein e has the following form:

$$e^A_\mu = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} N(t) & 0 & 0 & 0 \\ 0 & e^1_x & e^1_y & e^1_z \\ 0 & e^2_x & e^2_y & e^2_z \\ 0 & e^3_x & e^3_y & e^3_z \end{pmatrix}$$

The connection ω has the following form

$$\omega^{AB}_\mu = \begin{array}{c} 01 \\ 02 \\ 03 \\ 23 \\ 31 \\ 12 \end{array} \begin{pmatrix} 0 & k^1_x & k^1_y & k^1_z \\ 0 & k^2_x & k^2_y & k^2_z \\ 0 & k^3_x & k^3_y & k^3_z \\ 0 & c^1_x & c^1_y & c^1_z \\ 0 & c^2_x & c^2_y & c^2_z \\ 0 & c^3_x & c^3_y & c^3_z \end{pmatrix}$$

It follows that the Riemann tensor has the following form

$$R^{0A}_{ti} = \dot{k}^A_i$$

$$R^{BC}_{ti} = \dot{c}^A_i$$

$$R^{0A}_{ij} = \partial_i k^A_j - \partial_j k^A_i + k^B_i c^C_j - k^C_i c^B_j$$

$$R^{BC}_{ij} = \partial_i c^A_j - \partial_j c^A_i - k^B_i k^C_j + k^C_i k^B_j - c^B_i c^C_j + c^C_i c^B_j$$

In these equations, the indices A, B, and C are assumed to take only the values 1, 2, or 3. Furthermore we shall demand A $\neq$ B $\neq$ C $\neq$ A, and that they are in cyclic order, i.e. either 123, 231, or 312. Space indices i, j, and k are subjected to the same conditions.

The Einstein-Cartan (Palatini) Lagrangian density has many terms, as well as 27 variational parameters. But it can be written in a quite compact form

$$\mathcal{L} = \frac{3M_{pl}^2}{8\pi}\left[|ee\dot{k}| - N\left(|ecc| - |ekk| + H^2|eee| - |(\partial e)c|\right)\right]$$

We have included here the correct normalization for a change, as well as the cosmological term. Our notation in this equation is as follows:

$$|abc| = \frac{1}{6}\sum_{\substack{ABC=1\\ ijk=1}}^{3} \epsilon_{ABC}\epsilon^{ijk} a_i^A b_j^B c_k^C$$

$$|(\partial a)b| = \frac{1}{3}\sum_{ijk=1}^{3} \epsilon^{ijk}(\partial_i a_j^A) b_k^A \equiv |b(\partial a)|$$

Note that these constructions satisfy the following symmetry properties:

$$|abc| = |bac| = |bca|$$

The first-order Lagrangian has only one term which contains time derivatives. This suggests going to a "Hamiltonian" language by defining

$$\mathcal{L} = \frac{3M_{pl}^2}{8\pi}|ee\dot{k}| + U(e,k,c)$$

$$U = \frac{3M_{pl}^2}{8\pi} N(t)\left[|ekk| - |ecc| - H^2|eee| + |(\partial e)c|\right.$$

It is easily seen that the Lagrange equations take the schematic form

$$\delta N: \quad U(e,k,c) = 0$$

$$\delta k: \quad \frac{3M_{pl}^2}{8\pi}\frac{\partial}{\partial \dot{k}}|ee\dot{k}| = \frac{\partial U}{\partial k}$$

$$\delta e: \quad \frac{3M_{pl}^2}{8\pi}\frac{\partial}{\partial e}|ee\dot{k}| + \frac{\partial U}{\partial e} = 0$$

$$\delta c: \quad \frac{\partial U}{\partial c} = 0$$

The variation with respect to lapse N(t) requires that the "Hamiltonian" U (we use this notation, because the variable H is already taken.) vanishes. The role of the 9 extrinsic curvatures k is that of "momenta". The 9 "coordinate" variables conjugate to the k's are quadratic forms built from the vierbein and are essentially areas. The remaining degrees of freedom called c (for space curvature and/or contorsion) will create equations of constraint.

To warm up, we review FRW cosmology in this formalism. Symmetry implies

$$k_i^A \Rightarrow \delta_i^A K(t) \qquad c_i^A \Rightarrow \delta_i^A C(t) \qquad e_i^A \Rightarrow \delta_i^A a(t)$$

Therefore the Lagrangian simplifies to the form

$$\mathcal{L} = \frac{3M_{pl}^2}{8\pi}\left[a^2\dot{K} + NaK^2 - NaC^2 - NH^2a^3\right]$$

The variational principle leads to four equations of motion/constraint:

$$\begin{aligned}
\delta N:&\quad a\left[K^2 - C^2 - H^2a^2\right] = 0\\
\delta k:&\quad 2a\dot{a} = 2NaK\\
\delta c:&\quad -2NaC = 0\\
\delta a:&\quad 2a\dot{K} + N\left(K^2 - C^2 - 3H^2a^2\right) = 0
\end{aligned}$$

The deSitter space solution then follows immediately.

$$N = 1 \qquad C = 0 \qquad K = \dot{a} = Ha \qquad a = e^{Ht}$$

Note that the role of N(t) is significant---it provides the FRW equation directly. In general, for the FRW description, it suffices to set N to unity once the variation has been performed. However, we will later find it convenient to go to conformal coordinates by setting N(t) = a(t). When this is eventually done, we will redefine the time variable as η , consistent with conventional notation.

For the gravitational-wave example, we neglected the cosmological term

$$H \Rightarrow 0$$

We also did not consider N and e_z^3 as dynamical variables; we shall not allow them to participate here either. Upon performing the remaining variations, we now find the following expressions:

$$\begin{aligned}
\delta k:&\quad -2|e\,\dot{e}\,\delta k| + 2N|e\,k\,\delta k| = 0\\
\delta c:&\quad 2|e\,c\,\delta c| - |(\partial e)\,\delta c| = 0\\
\delta e:&\quad 2|e\,\dot{k}\,\delta e| - N|c\,c\,\delta e| + N|k\,k\,\delta e| + N|(\partial c)\,\delta e| = 0
\end{aligned}$$

Upon examination of the equations involving δk, we immediately see that they are solved simply:

$$k_i^A = \mathring{e}_i^A$$

If there is no degeneracy in the 9 relevant linear equations, this is a unique solution. Otherwise it is one of many, presumably related by gauge transformations.

Examination of the equations involving c are linear constraint equations. But they must be handled with care. In component form, they take the form

$$|e\, c\, \delta c| = \frac{1}{2}|(\partial e)\,\delta c|$$

$$\Rightarrow \frac{1}{6}\epsilon^{ijk}\epsilon_{ABC}\, c_i^A\, e_j^B = \frac{1}{6}\epsilon^{ijk}(\partial_i e_j^C)$$

However, the matrix c is not diagonal. The nonvanishing elements are

$$c_x^2 \neq 0 \qquad\qquad c_y^1 \neq 0$$

The variational principle yields two and only two nontrivial constraint equations. After index hell, the results are as follows:

$$\delta c: \qquad \begin{aligned} e_z^3\, c_y^1 &= \partial_z e_y^2 \\ e_z^3\, c_x^2 &= -\partial_z e_x^1 \end{aligned}$$

The final application of the variational principle involves the transverse components of the vierbein:

$$\delta e: \qquad \begin{aligned} 2[e_z^3 k_y^2 - \partial_z c_y^1]\,\delta e_x^1 &= 0 \\ 2[e_z^3 k_x^1 - \partial_z c_x^2]\,\delta e_y^2 &= 0 \end{aligned}$$

With these results, one readily reconstructs the wave equation and again finds that it is an exact solution of the equations of motion.

$$k_x^1 = -k_y^2 = h'(z-t)$$

$$c_x^2 = c_y^1 = h'(z-t)$$

$$e_x^1 = 1 + h(z-t)$$

$$e_y^2 = 1 + h(z-t)$$

$$e_z^3 = 1$$

We now turn to the issue at hand, namely to include the FRW cosmological solution as a background metric for the gravitational waves. Our notation will be

$$W_i^{0A} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\!\begin{bmatrix} (K+k_+) & k_\times & 0 \\ k_\times & (K-k_+) & 0 \\ 0 & 0 & K \end{bmatrix} \qquad \omega_i^{BC} = \begin{array}{c} 23 \\ 31 \\ 12 \end{array}\!\begin{bmatrix} c_+ & c_\times & 0 \\ c_\times & -c_+ & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_i^A = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\!\begin{bmatrix} (a+\epsilon_+) & \epsilon_\times & 0 \\ \epsilon_\times & (a-\epsilon_+) & 0 \\ 0 & 0 & a \end{bmatrix}$$

$$N = a(\eta) \qquad K = H a(\eta) \text{ (for deSitter space only)}$$

Note that we here commit to the conformal language. The main reason is that the wave equation for massless bosons in deSitter space has a simple solution when expressed in the conformal coordinates, and we hope that this simplification will be manifest in this first-order formalism.

We have also introduced additional streamlining of our notation. The 3 x 3 matrix degrees of freedom (e, k, and c) will have the above structure, each of which contains at most three independent degrees of freedom:

$$b_i^A = \begin{bmatrix} (B+b_+) & b_\times & 0 \\ b_\times & (B-b_+) & 0 \\ 0 & 0 & B \end{bmatrix}$$

It is then straightforward to simplify the structure of our "double forms". The result is

$$\begin{aligned} |abc| = {} & ABC \\ & -\tfrac{1}{3}\left[a_+ b_+ C + b_+ c_+ A + c_+ a_+ B \right] \\ & -\tfrac{1}{3}\left[a_\times b_\times C + b_\times c_\times A + c_\times a_\times B \right] \end{aligned}$$

The additional form which involves the torsion/curvature also simplifies as follows:

$$|(\partial a)\, b| = -\tfrac{2}{3}\left(a'_\times b_+ - a'_+ b_\times \right)$$

Our FRW/gravitational wave Lagrangian can now be written in streamlined form as follows; index-hell has been vanquished!

$$\mathcal{L} = \frac{3M_{pl}^2}{8\pi}\left(\begin{array}{l}(a^2\mathring{K} + a^2K^2 - a^4H^2) \\ -\frac{1}{3}(\epsilon_+^2+\epsilon_\times^2)\mathring{K} - \frac{2}{3}(\epsilon_+\mathring{k}_+ + \epsilon_\times\mathring{k}_\times)\,a \\ +\frac{a^2}{3}(c_+^2 + c_\times^2 - k_+^2 - k_\times^2) \\ -\frac{2}{3}(\epsilon_+k_+ + \epsilon_\times k_\times)Ka \\ +(\epsilon_+^2+\epsilon_\times^2)H^2a^2 \\ +\frac{2}{3}(\epsilon_\times' c_+ - \epsilon_+' c_\times)a\end{array}\right)$$

We emphasize that the lapse N has here been equated to the scale factor $a(\eta)$. Therefore the FRW equations are modified. They now read

$$\delta K:\qquad 2a[-\mathring{a} + aK] = 0$$

$$\delta a:\qquad 2a[\mathring{K} + K^2 - 2a^2H^2] = 0$$

The solution for deSitter space is well-known:

$$a = \frac{1}{H\eta}\qquad K = \frac{\mathring{a}}{a} = -\frac{1}{\eta}$$

The gravitational-wave degrees of freedom again only appear in the Lagrangian to quadratic order. Therefore their equations of motion, now with the FRW degrees of freedom representing a background metric, will again be linear:

$$\delta k_+:\quad -\frac{2}{3}(\mathring{\epsilon}_+ a + \epsilon_+\mathring{a}) = -\frac{2}{3}(a^2k_+ + \epsilon_+Ka) = -\frac{2}{3}(a^2k_+ + \epsilon_+\mathring{a})$$

Therefore $\frac{2}{3}(a^2c_\times - \epsilon_+'a) = 0$

$$\mathring{\epsilon}_+ = k_+a\qquad \epsilon_+' = c_\times a$$

The third equation can again be simplified down to wave-equation form

$$\delta\epsilon_+:\quad 0 = -\frac{2}{3}[\epsilon\mathring{K} - \mathring{k}a - kKa + 3\epsilon H^2a^2 + c'a]$$

$$= \frac{2}{3}\left[-2\epsilon_+a^2H^2 + \epsilon_+\left(\frac{\mathring{a}}{a}\right)^2 - \mathring{\mathring{\epsilon}}_+ + \frac{\mathring{\epsilon}_+\mathring{a}}{a} - \frac{\mathring{\epsilon}_+\mathring{a}}{a} + 3\epsilon_+H^2a^2 + \epsilon_+''\right]$$

Therefore

$$\mathring{\mathring{\epsilon}}_+ - \epsilon_+'' = \epsilon_+\left[\left(\frac{\mathring{a}}{a}\right)^2 - 2H^2a^2 + 3H^2a^2\right] = \epsilon_+\left[\left(\frac{\mathring{a}}{a}\right)^2 + H^2a^2\right]$$

For deSitter space, we find the simple equation

$$\ddot{\epsilon}_+ - \epsilon_+'' = \frac{2\epsilon_+}{\eta^2}$$

It is known as the Mukhanov-Sasaki equation (cf Baumann TASI lectures and/or his website notes on inflation). It admits a closed solution for a plane wave moving in the z direction with comoving momentum p :

$$\epsilon_+ = e^{-ip(\eta - z)}\left[1 - \frac{i}{p\eta}\right]$$

$$c_x = e^{-ip(\eta - z)}\left[ip + \frac{1}{\eta}\right]\cdot H\eta = He^{-ip(\eta - z)}\left[1 + ip\eta\right]$$

$$k_+ = e^{-ip(\eta - z)}\left[-ip - \frac{1}{\eta} + \frac{i}{p\eta^2}\right]\cdot H\eta = -He^{-ip(\eta-z)}\left[1 + ip\eta - \frac{i}{p\eta}\right]$$

The parameters ϵ_+ and k_+ can be considered, from the Hamiltonian point of view, a canonical pair. I suspect the parameters c_x play a special role as well in inflation theory---in the Baumann description they appear to be related to the gauge-invariant perturbation he denotes as $\mathcal{R}$. If this is right, the formalism we are using should be very well-suited for describing inflation.

If one wants to generalize the above description beyond deSitter space, allowing a more general FRW expansion, source terms need to be included. For our purposes, all that needs to be done is to generalize the cosmological term to a general function of $|e\ e\ e|$ .We write for the modified Lagrangian

$$\mathcal{L} = \frac{3M_{pl}^2}{8\pi}\left[|ee\dot{k}| - N|ecc| + N|ekk| + N|(\partial e)c|\right] - N|eee|\,\rho(|eee|)$$

$$N|eee|\,\rho(|eee|) \cong Na^3\rho(a^3) - Na(\epsilon_+^2 + \epsilon_x^2)\left[\rho(a^3) + a^3\rho'(a^3)\right]$$

We have expanded the function ρ out to linear order in order to accommodate the modifications to the gravitational-wave equation. Note that in the deSitter limit

$$\rho(a^3) \rightarrow \frac{3M_{pl}^2H^2}{8\pi} = \rho_{DE} \qquad \rho' = 0$$

At the FRW level, the equations of motion are as follows:

$$\delta N: \quad aK^2 - \frac{8\pi}{3M_{pl}^2} a^3 \rho = 0$$

$$\delta K: \quad 2a\dot{a} = 2NaK$$

$$\delta a: \quad 2a\dot{K} + NK^2 - \frac{8\pi N}{3M_{pl}^2}\left[3a^2\rho(a^3) + 3a^5\rho'(a^3)\right] = 0$$

When we set N = 1 , the FRW equation immediately follows from the first two of the equations.

$$N = 1 \qquad K = \dot{a} \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3M_{pl}^2}\rho$$

We see that the final, "pressure" equation, as expected, follows automatically from the first two:

$$\delta a: \quad 2a\ddot{a} + \dot{a}^2 - \frac{8\pi a^2}{M_{pl}^2}(\rho + a^3\rho') = 0$$

$$0 = \frac{d}{dt}\left(a\dot{a}^2 - \frac{8\pi}{3M_{pl}^2} a^3\rho\right) = \dot{a} + 2a\dot{a}\ddot{a} - \frac{8\pi}{3M_{pl}^2}(3a^2\rho + 3a^5\rho')\dot{a} = 0$$

The equations for the gravitational waves are again better written down in the conformal-coordinate description. Before writing them down, we reduce the FRW equations for this case. The first two equations now read

$$\delta N: \quad aK^2 - \frac{8\pi}{3M_{pl}^2} a^3\rho = 0$$

$$\delta K: \quad 2a\dot{a} = 2a^2K$$

Therefore

$$N = a(\eta) \qquad K = \frac{\dot{a}}{a} \qquad K^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3M_{pl}^2} a^2 \rho$$

Again the third, "pressure" equation follows automatically from the first two:

$$\delta a: \quad a(2\dot{K} + K^2) - \frac{8\pi a^3}{M_{pl}^2}\left[\rho + a^3\rho'\right] = 0$$

$$\frac{8\pi a^2\dot{a}}{M_{pl}^2}(\rho + a^3\rho') = \frac{d}{d\eta}\left[\frac{8\pi}{3M_{pl}^2} a^3\rho\right] = \frac{d}{d\eta}(aK^2) = \frac{d}{d\eta}\left(\frac{\dot{a}^2}{a}\right) = \frac{2\dot{a}\ddot{a}}{a} - \frac{\dot{a}^3}{a^2} = \dot{a}\left[2\dot{K} + K^2\right]$$

We now write down the gravitational-wave equations. The first two agree with the previous case, because the source term ρ is not involved.

$$\delta k_+: \quad \dot{\epsilon}_+ = k_+ a$$

$$\delta c_\times: \quad \dot{\epsilon}_+ = c_\times a$$

Reduction of the third "pressure" equation leads to the final form of the wave equation:

$$\delta\epsilon_+ : \quad 0 = -\frac{2}{3}\left[\epsilon_+ \mathring{K} + \mathring{k}_+ a + k_+ K a - c'_x a\right] + \frac{8\pi a^2}{3M_{pl}^2} \cdot 2\epsilon_+\left(\rho + a^3 \rho'\right)$$

$$= -\frac{2}{3}\left[\epsilon_+ \mathring{K} + \mathring{\mathring{\epsilon}}_+ - \mathring{k}_+ \mathring{a} + k_+ \mathring{a} - \epsilon''_+ - \epsilon_+\left(2\mathring{K} + K^2\right)\right]$$

Therefore

$$\mathring{\mathring{\epsilon}}_+ - \epsilon''_+ = \epsilon_+\left(-\mathring{K} + 2\mathring{K} + K^2\right) = \epsilon_+\left(\frac{\mathring{\mathring{a}}}{a}\right)$$

This equation agrees with the generalized Mukhanov-Sasaki equation as quoted by Baumann.

To go further would lead us into a description of inflation theory which is based on the first-order formalism. This topic is an extremely attractive one for me to pursue. But it is beyond the scope of this note. Instead, we now turn to the question of adding discrete symmetry violation to this description.

V. Inclusion of the Holst Term

The Holst term in the Lagrangian, which violates discrete symmetries, has a structure similar to the Einstein-Cartan (Palatini) term. The combined Lagrangian is

$$\mathcal{L} \sim \epsilon^{\mu\nu\lambda\sigma} R_{\mu\nu}^{AB} e_\lambda^C e_\sigma^D \left[\epsilon_{ABCD} - \frac{2}{\gamma}\eta_{AC}\eta_{BD}\right]$$

The coupling parameter γ is known as the Barbero-Immirzi parameter.

We again assume temporal gauge for the connection and the "space-dominance" assumption for the vierbein.

$$\omega_t^{AB} = 0 \qquad e_t^A = 0 \qquad e_i^0 = 0$$

$$(A,B = 1,2,3) \qquad (A = 1,2,3) \qquad i = (x,y,z)$$

We will also include "$\vec{j}\cdot\vec{\omega}$" source terms which are linear in the connection variables k and c.

It is then straightforward to expand the Lagrangian in terms of double forms. After considerable algebra and care with numerical factors, the result is as follows:

$$\frac{8\pi}{3M_{pl}^2}\mathcal{L} = \left[|ee\mathring{k}| - N|ecc| + N|ekk| + N|(\partial e)c| - 2N\hat{H}_K|eek|\right]$$

$$+ \frac{1}{\gamma}\left[|ee\mathring{c}| + 2N|ekc| - N|(\partial e)k| - 2N\gamma\hat{H}_c|eec|\right]$$

$$- \frac{8\pi}{3M_{pl}^2} N|eee|\,\rho(|eee|)$$

The only dynamical variable is a "generalized extrinsic curvature" $\tilde{k}$, especially familiar to the loop quantum gravity community:

$$\tilde{k} = k + \frac{c}{\gamma}$$

We therefore recast the Lagrangian in terms of this variable, and find

$$\frac{8\pi}{3M_{pl}^2}\mathcal{L} = |ee\mathring{\tilde{k}}| - N\left(\frac{1+\gamma^2}{\gamma^2}\right)|ecc| + N|e\tilde{k}\tilde{k}| - \frac{N}{\gamma}|(\partial e)\tilde{k}| + N\left(\frac{1+\gamma^2}{\gamma^2}\right)|(\partial e)c|$$
$$-2N\hat{H}_k|ee\tilde{k}| + \frac{2N}{\gamma}(\hat{H}_k - \gamma\hat{H}_c)|eec| - \frac{8\pi}{3M_{pl}^2}N|eee|\,P(|eee|)$$

Again we only consider the "FRW + transverse-traceless" case. This allows a relatively simple expansion of the sundry double forms.

$$|ee\mathring{\tilde{k}}| = a^2\mathring{\tilde{K}} - \frac{1}{3}(\epsilon_+^2 + \epsilon_\times^2)\mathring{\tilde{K}} - \frac{2}{3}(\epsilon_+\mathring{\tilde{k}}_+ + \epsilon_\times\mathring{\tilde{k}}_\times)a$$

$$-N\left(\frac{1+\gamma^2}{\gamma^2}\right)|ecc| = -N\left(\frac{1+\gamma^2}{\gamma^2}\right)\left[aC^2 - \frac{1}{3}(c_+^2 + c_\times^2)a - \frac{2}{3}(\epsilon_+c_+ + \epsilon_\times c_\times)C\right]$$

$$N|e\tilde{k}\tilde{k}| = N\left[a\tilde{K}^2 - \frac{1}{3}(\tilde{k}_+^2 + \tilde{k}_\times^2)a - \frac{2}{3}(\epsilon_+\tilde{k}_+ + \epsilon_\times\tilde{k}_\times)\tilde{K}\right]$$

$$-\frac{N}{\gamma}|(\partial e)\tilde{k}| = \frac{2}{3}\frac{N}{\gamma}(\epsilon'_\times\tilde{k}_+ - \epsilon'_+\tilde{k}_\times)$$

$$N\left(\frac{1+\gamma^2}{\gamma^2}\right)|(\partial e)c| = -\frac{2}{3}N\left(\frac{1+\gamma^2}{\gamma^2}\right)(\epsilon'_\times c_+ - \epsilon'_+ c_\times)$$

$$-2N\hat{H}_k|ee\tilde{k}| = -2NHu\left[a^2\tilde{K} - \frac{1}{3}(\epsilon_+^2 + \epsilon_\times^2)\tilde{K} - \frac{2}{3}(\epsilon_+\tilde{k}_+ + \epsilon_\times\tilde{k}_\times)a\right]$$

$$\frac{2N}{\gamma}(\hat{H}_k - \gamma\hat{H}_c)|eec| = \frac{2N}{\gamma}Hv\left[a^2C - \frac{1}{3}(\epsilon_+^2 + \epsilon_\times^2)C - \frac{2}{3}(\epsilon_+c_+ + \epsilon_\times c_\times)a\right]$$

$$-\frac{8\pi}{3M_{pl}^2}N|eee|P(|eee|) = -\frac{8\pi N}{3M_{pl}^2}\left\{a^3P(a^3) - a(\epsilon_+^2 + \epsilon_\times^2)\left[P(a^3) + a^3P'(a^3)\right]\right\}$$

We have in the above expressions introduced the notation

$$Hu = \hat{H}_k \qquad Hv = (\hat{H}_k - \gamma\hat{H}_c)$$

There are now 10 parameters to be varied. We begin with the FRW limit:

$$\delta\tilde{K}: \quad 0 = -2a\mathring{a} + 2Na\tilde{K} - 2N(Hu)a^2$$

$$\delta C: \quad 0 = -2N\left(\frac{1+\gamma^2}{\gamma^2}\right)aC + \frac{2N}{\gamma}(Hv)a^2$$

$$\delta N: \quad 0 = -\left(\frac{1+\gamma^2}{\gamma^2}\right)aC^2 + a\tilde{K}^2 - 2(Hu)a\tilde{K} + \frac{2}{\gamma}(Hv)a^2C - \frac{8\pi}{3M_{pl}^2}a^3\rho$$

After some algebra, these equations simplify to simple expressions:

$$\tilde{K} = \mathring{a} + a(Hu)$$

$$C = \frac{\gamma a}{(1+\gamma^2)}(Hv)$$

$$\left(\frac{\mathring{a}}{a}\right)^2 = \frac{8\pi}{3M_{pl}^2}\left[\rho + (Hu)^2 - \frac{1}{(1+\gamma^2)}(Hv)^2\right]$$

The first two of these express the "generalized extrinsic curvature" K and the contorsion C in terms of the vierbein parameters and the source parameters $\hat{H}$ The third equation is the FRW equation, and shows that the only effect of the presence of the sources is to renormalize the input Hubble parameter.

There is a fourth variation to take:

$$\delta a: \quad 0 = 2a\mathring{\tilde{K}} - N\left(\frac{1+\gamma^2}{\gamma^2}\right)C^2 + N\tilde{K}^2 - 4N(Hu)a\tilde{K} + \frac{4N}{\gamma}(Hv)aC - \frac{8\pi N}{3M_{pl}^2}\cdot 3a^2(\rho + a^3\rho')$$

As usual, this follows from the FRW equation and provides no new constraint on the dynamics.

In preparation for the 6 variations that lead to the gravitational wave equations, we redo the above exercise in the conformal language, where N is set equal to a(η).

$$\tilde{K} = \left(\frac{\mathring{a}}{a}\right) + a(Hu)$$

$$C = \frac{\gamma a}{(1+\gamma^2)}(Hv)$$

$$\left(\frac{\mathring{a}}{a}\right)^2 = a^2\left[\frac{8\pi}{3M_{pl}^2}\rho + (Hu)^2 - \frac{(Hv)^2}{(1+\gamma^2)}\right]$$

The first four of the equations are relatively simple, and as usual allow us to solve explicitly for the connection variables k and c .

$$\delta\tilde{k}_+ : \quad 0 = \frac{2}{3}\left(\mathring{\epsilon}_+ a + \epsilon_+ \mathring{a}\right) - \frac{2}{3}a^2\tilde{k}_+ - \frac{2}{3}\epsilon_+ a\tilde{K} + \frac{2a}{3\gamma}\epsilon'_\times + \frac{4}{3}(Hu)\epsilon_+ a^2$$

$$\delta\tilde{k}_\times : \quad 0 = \frac{2}{3}\left(\mathring{\epsilon}_\times a + \epsilon_\times \mathring{a}\right) - \frac{2}{3}a^2\tilde{k}_\times - \frac{2}{3}\epsilon_\times a\tilde{K} - \frac{2a}{3\gamma}\epsilon'_+ + \frac{4}{3}(Hu)\epsilon_\times a^2$$

$$\delta c_+ : \quad 0 = \frac{2}{3}a\left(\frac{1+\gamma^2}{\gamma^2}\right)\left[a c_+ + \epsilon_+ C\right] - \frac{2}{3}a\left(\frac{1+\gamma^2}{\gamma^2}\right)\epsilon'_\times - \frac{4a}{3\gamma}(Hv)\epsilon_+$$

$$\delta c_\times ; \quad 0 = \frac{2}{3}a\left(\frac{1+\gamma^2}{\gamma^2}\right)\left[a c_\times + \epsilon_\times C\right] + \frac{2}{3}a\left(\frac{1+\gamma^2}{\gamma^2}\right)\epsilon'_+ - \frac{4a}{3\gamma}(Hv)\epsilon_\times$$

After some algebra, these can be written as follows:

$$a\tilde{k}_+ = \mathring{\epsilon}_+ + \frac{\epsilon'_\times}{\gamma} + a(Hu)\epsilon_+$$

$$a\tilde{k}_\times = \mathring{\epsilon}_\times - \frac{\epsilon'_+}{\gamma} + a(Hu)\epsilon_\times$$

$$a c_+ = \epsilon'_\times + \frac{a\gamma}{(1+\gamma^2)}(Hv)\epsilon_+$$

$$a c_\times = -\epsilon'_+ + \frac{a\gamma}{(1+\gamma^2)}(Hv)\epsilon_\times$$

Finally, the last two equations for the vierbein components will determine the structure of the wave equation:

$$\begin{aligned}\delta\epsilon_+ : \quad 0 = & -\frac{2}{3}\epsilon_+\mathring{\tilde{K}} - \frac{2}{3}\mathring{\tilde{k}}_+ a + \frac{2}{3}a\left(\frac{1+\gamma^2}{\gamma^2}\right)c_+ C - \frac{2}{3}a\tilde{k}_+\tilde{K} + \frac{2a}{3\gamma}\tilde{k}'_\times \\ & - \frac{2}{3}\left(\frac{1+\gamma^2}{\gamma^2}\right)a c'_\times + \frac{4a}{3}(Hu)\left[\epsilon_+\tilde{K} + \tilde{k}_+ a\right] - \frac{4a}{3\gamma}(Hv)\left[\epsilon_+ C + c_+ a\right] \\ & + \frac{16\pi}{3M_{pl}^2}a^2\epsilon_+\left(\rho + a^3\rho'\right)\end{aligned}$$

$$\begin{aligned}\delta\epsilon_\times : \quad 0 = & -\frac{2}{3}\epsilon_\times\mathring{\tilde{K}} - \frac{2}{3}\mathring{\tilde{k}}_\times a + \frac{2}{3}a\left(\frac{1+\gamma^2}{\gamma^2}\right)c_\times C - \frac{2}{3}a\tilde{k}_\times\tilde{K} - \frac{2a}{3\gamma}\tilde{k}'_+ + \frac{2}{3}\left(\frac{1+\gamma^2}{\gamma^2}\right)a c'_+ \\ & + \frac{4a}{3}(Hu)\left[\epsilon_\times\tilde{K} + \tilde{k}_\times a\right] - \frac{4a}{3\gamma}(Hv)\left[\epsilon_\times C + c_\times a\right] + \frac{16\pi}{M_{pl}^2}a^2\epsilon_\times\left(\rho + a^3\rho'\right)\end{aligned}$$

Since these equations are linear, we can go to complex wave-equation notation. We assume a plane-wave structure with momentum p in the positive z direction. Therefore we make the replacements

$$\epsilon_+' = ip\epsilon_+ \qquad \tilde{k}_+' = ip\tilde{k}_+$$

$$\epsilon_\times' = ip\epsilon_\times \qquad \tilde{k}_\times' = ip\tilde{k}_\times$$

This allows us to solve for the c 's in terms of the ϵ 's.

$$c_+ = \frac{ip}{a}\epsilon_\times + \left(\frac{\gamma}{1+\gamma^2}\right)(Hv)\,\epsilon_+$$

$$c_\times = -\frac{ip}{a}\epsilon_+ + \left(\frac{\gamma}{1+\gamma^2}\right)(Hv)\,\epsilon_\times$$

Thanks to considerable hindsight, it turns out that choice of a circular-polarization basis in all cases leads to a significant amount of simplification. This is implemented by the following assumptions:

$$\epsilon_\times = \lambda\epsilon_+$$

$$\tilde{k}_\times = \lambda\tilde{k}_+ \qquad \lambda^2 = -1$$

$$c_\times = \lambda c_+$$

It is easily seen that all the previous equations for the gravitational-wave amplitudes are consistent with these assumptions. We are left with only two independent equations:

$$\mathring{\tilde{k}}_+ = \left[-\tilde{K} + 2a(Hu) + \frac{ip\lambda}{\gamma}\right]\tilde{k}_+ + \left[X - \frac{p^2}{a}\left(\frac{1}{\gamma^2}+1\right) - \frac{2ip\lambda}{\gamma}\right]\epsilon_+$$

$$\mathring{\epsilon}_+ = a\tilde{k}_+ - \left[a(Hu) + \frac{ip\lambda}{\gamma}\right]\epsilon_+$$

The entry marked $\mathbb{X}$ is a bit lengthy:

$$\mathbb{X} = \frac{1}{a}\left[-\mathring{\tilde{K}} + 2a(Hu)\tilde{K} - \frac{3\mathring{a}^2}{(1+\gamma^2)}(Hv)^2 + \frac{8\pi}{M_{pl}^2}a^2\left(\rho + \mathring{a}^3\rho'\right)\right]$$

If we use the FRW "pressure" equation, this simplifies down to

$$a\mathbb{X} = \mathring{\tilde{K}} - 2a\tilde{K}(Hu) + \tilde{K}^2 = \left(\frac{\ddot{a}}{a}\right) + \mathring{a}(Hu) - \mathring{a}^2(Hu)^2$$

In practice, it turns out that we will be interested in two special cases. Before reheating, the spacetime can be taken to be deSitter, but with nonvanishing source terms. In this case, we know $a(\eta)$ explicitly.

$$a = \frac{-1}{H\eta} \qquad \mathring{a} = \frac{1}{H\eta^2} \qquad \ddot{a} = \frac{-2}{H\eta^3}$$

Therefore we also know $\mathbb{X}$ and K explicitly in terms of the time variable η and the source parameters.

$$\mathbb{X} = -\frac{H}{\eta}(1+u)(2-u) \qquad \tilde{K} = \frac{-(1+u)}{\eta}$$

The equations of motion then take the form (for the deSitter case only!)

$$\mathring{\tilde{k}}_+ = \left[\frac{(1-u)}{\eta} + \frac{ip\lambda}{\gamma}\right]\tilde{k}_+ + \left[\frac{-H}{\eta}(1+u)(2-u) + p^2 H\eta\left(\frac{1}{\gamma^2}+1\right) - \frac{2ip\lambda}{\gamma}Hv\right]\epsilon_+$$

$$\mathring{\epsilon}_+ = -\frac{\tilde{k}_+}{H\eta} + \frac{u}{\eta}\epsilon_+ - \frac{ip\lambda}{\gamma}\epsilon_+$$

We will assume that the source terms are only present before reheating, and that only the present-day dark energy remains thereafter. Therefore the post-reheating description only depends on the (nontrivial) behavior of the FRW scale factor $a(\eta)$. Nevertheless, the equations of motion again simplify considerably:

$$\mathring{\tilde{k}}_+ = -\left(\frac{\mathring{a}}{a}\right)\tilde{k}_+ + \left[\left(\frac{\ddot{a}}{a^2}\right) - \left(\frac{p^2}{a}\right)\left(\frac{1}{\gamma^2}+1\right)\right]\epsilon_+ + \frac{ip\lambda}{\gamma}\tilde{k}_+$$

$$\mathring{\epsilon}_+ = a\tilde{k}_+ - \frac{ip\lambda}{\gamma}\epsilon_+$$

We begin with the source-free FRW case:

$$\ddot{\epsilon}_+ = \dot{a}\tilde{k}_+ + a\dot{\tilde{k}}_+ - \frac{ip\lambda}{\gamma}\dot{\epsilon}_+$$

$$= \dot{a}\tilde{k}_+ + a\left\{-\left(\frac{\dot a}{a}\right)\tilde k_+ + \left[\left(\frac{\ddot a}{a^2}\right) - \frac{p^2}{a}\left(\frac{1}{\gamma^2}+1\right)\right]\epsilon_+ + \frac{ip\lambda}{\gamma}\tilde k_+\right\} - \frac{ip\lambda}{\gamma}\dot\epsilon_+$$

$$= \left(\frac{\ddot a}{a}\right)\epsilon_+ - p^2\left(\frac{1}{\gamma^2}+1\right)\epsilon_+ + \frac{ip\lambda}{\gamma}\left(a\tilde k_+ - \dot\epsilon_+\right)$$

The solution agrees with what we found previously:

$$\ddot\epsilon_+ + p^2\epsilon_+ = \left(\frac{\ddot a}{a}\right)\epsilon_+ - \frac{p^2}{\gamma^2}\epsilon_+ + \left(\frac{ip\lambda}{\gamma}\right)\left(\frac{ip\lambda}{\gamma}\right)\epsilon_+ = \left(\frac{\ddot a}{a}\right)\epsilon$$

In the presence of the sources, the corresponding deSitter equation of motion is as follows:

$$\ddot\epsilon_+ = \frac{1}{H\eta}\tilde k_+ - \frac{1}{H\eta}\dot{\tilde k}_+ - \frac{u}{\eta^2}\epsilon_+ + \frac{u}{\eta}\dot\epsilon_+ - \frac{ip\lambda}{\gamma}\dot\epsilon_+$$

$$= \frac{1}{H\eta}\tilde k_+ - \frac{1}{H\eta}\left\{\left[\frac{(1-u)}{\eta} + \frac{ip\lambda}{\gamma}\right]\tilde k_+ + \left[\frac{-H}{\eta}(1+u)(2-u) + p^2 H\eta\left(\frac{1}{\gamma^2}+1\right) - \frac{2ip\lambda}{\gamma}(Hv)\right]\epsilon_+\right\}$$

$$- \frac{u}{\eta}\epsilon_+ + \left(\frac{u}{\eta} - \frac{ip\lambda}{\gamma}\right)\dot\epsilon_+$$

$$= \left(\frac{u}{\eta} - \frac{ip\lambda}{\gamma}\right)\left(\dot\epsilon_+ + \frac{\tilde k_+}{H\eta}\right) + \epsilon_+\left[\frac{1}{\eta^2}(2+u-u^2) - p^2\left(\frac{1}{\gamma^2}+1\right) + \frac{2ip\lambda v}{\eta\gamma} - \frac{u}{\eta^2}\right]$$

$$= \left(\frac{u}{\eta} - \frac{ip\lambda}{\gamma}\right)^2\epsilon_+ + \epsilon_+\left[\frac{1}{\eta^2}(2+u-u^2) - p^2\left(\frac{1}{\gamma^2}+1\right) + \frac{2ip\lambda v}{\eta\gamma} - \frac{u}{\eta^2}\right]$$

$$= \epsilon_+\left[\frac{2ip\lambda}{\eta\gamma}(v-u) - p^2 + \frac{1}{\eta^2}(2+u-u^2) + \frac{1}{\eta^2}(u^2-u)\right]$$

The final wave equation is gloriously simple:

$$\ddot{\epsilon}_+ + p^2 \epsilon_+ = \left[\frac{2}{\eta^2} + \frac{2ip\lambda}{\eta\gamma}(v-u)\right]\epsilon_+$$
$$= 2\left[\frac{1}{\eta^2} - \frac{ip\lambda}{\eta\gamma}\left(\frac{\gamma \hat{H}_c}{H}\right)\right]\epsilon_+$$
$$= 2\left[\frac{1}{\eta^2} - \frac{ip\lambda}{\eta}\left(\frac{\hat{H}_c}{H}\right)\right]\epsilon_+$$

Remarkably, the only correction to the "normal", symmetric behavior does not depend upon the value of the Barbero-Immirzi parameter γ. It also requires the existence of a source for torsion (more precisely, for contorsion).

The previous case admitted an exact solution for the wave equation. We now search for its generalization.

The equation of motion is actually that of a harmonic oscillator with time-dependent frequency which passes through zero near horizon crossing and thereafter becomes imaginary. To clean up notation, we write

$$\tau = p\eta$$
$$f = -i\lambda\left(\frac{\hat{H}_c}{H}\right) = f^*$$

Note that, under a space reflection, which interchanges left- and right-handed chiralities, the parameter f changes sign.

The equation of motion is, in this streamlined notation,

$$\frac{d^2\epsilon_+}{d\tau^2} = -\left[1 - \frac{2f}{\tau} - \frac{2}{\tau^2}\right]\epsilon_+ = -\omega^2(\tau)\,\epsilon_+$$

For $f = 0$, the two independent solutions are the real and imaginary parts of the expression

$$\epsilon_+ = e^{-i\tau}\left(1 - \frac{i}{\tau}\right)$$

It will be useful to consider the phase space description of this system. The canonical momentum is evidently just $\dot{\epsilon}_+$. This allows a quick visualization of the time evolution of a classical ensemble and/or a quantum Wigner function.

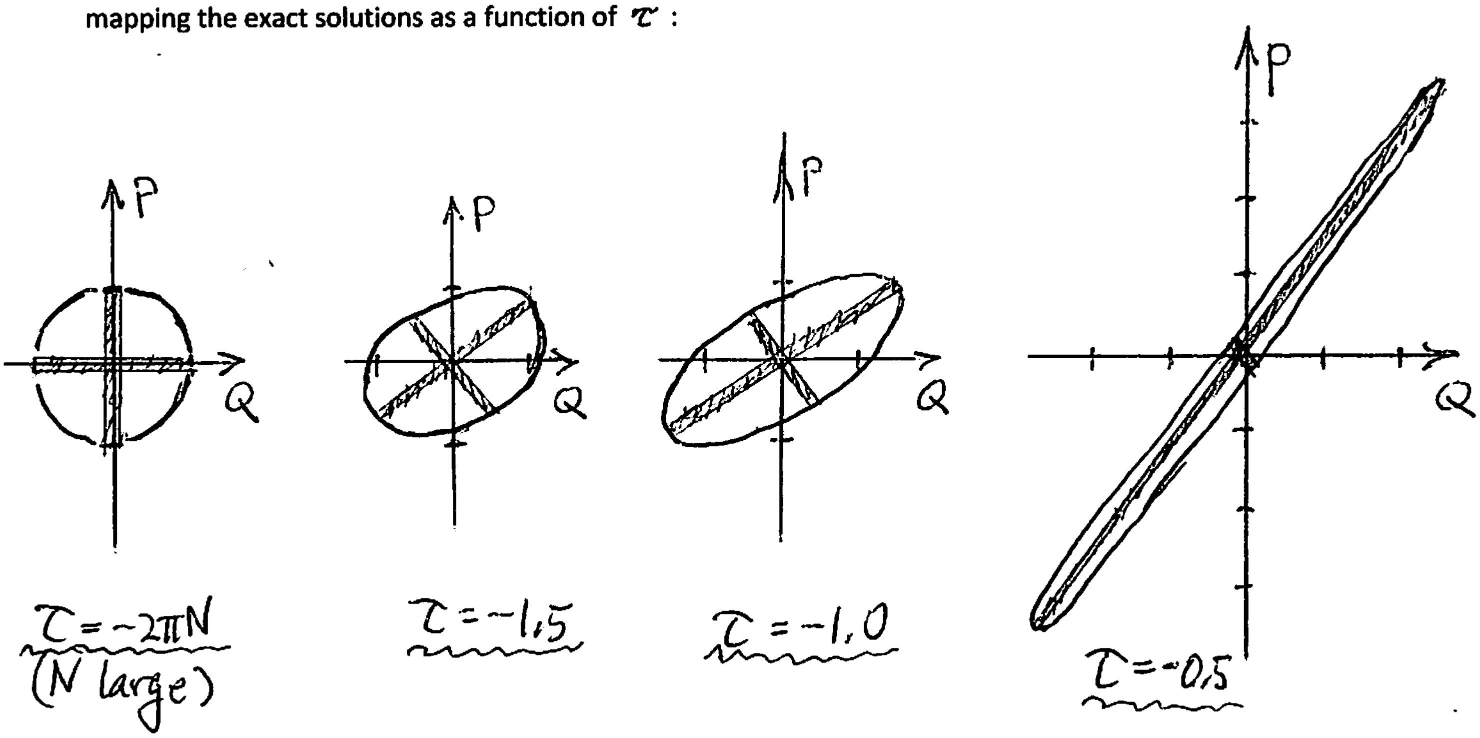
This exact solution can be written

$$\epsilon_+ = Q_1 + iQ_2 \qquad Q_1 = \left(\cos\tau - \frac{\sin\tau}{\tau}\right) \qquad P_1 = \sin\tau\left(-1+\frac{1}{\tau^2}\right) - \frac{\cos\tau}{\tau}$$

$$\mathring{\epsilon}_+ = {}_1 + iP_2 \qquad Q_2 = -\left(\sin\tau + \frac{\cos\tau}{\tau}\right) \qquad P_2 = \cos\tau\left(- + \frac{1}{\tau^2}\right) - \frac{\sin\tau}{\tau}$$

The pairs (Q_1, P_1) and (Q_2, P_2) are solutions of the Hamiltonian equations of motion. They will serve as a basic skeleton for the description of a more general phase-space architecture.

When τ is large (and negative!), we have a simple harmonic oscillator. Its natural phase-space structure is circular, both at the classical level, and at the quantum level, provided a Bunch-Davies vacuum structure is assumed. As the frequency parameter ω^2 decreases to zero and then changes sign, the phase space is squeezed and rapidly turns into a filamentary structure. This can be seen by mapping the exact solutions as a function of τ :



The shaded areas indicate an ensemble of degrees of freedom which are in proportion to (Q_1, P_1) or to (Q_2, P_2) and with relative amplitudes in the interval $(-1, 1)$. Note that the pair (Q_1, P_1) describes the thickness of the filament, while the pair (Q_2, P_2) describes its length. The area of the filament is described by the quantity $(Q_1 P_2 - Q_2 P_1) = -1$, independent of τ. This quantity, the Wronskian, is an expression of the Liouville theorem, which is applicable to this system.

Not surprisingly, the crossover from round to filamentary occurs when the frequency parameter vanishes, namely when $\tau = -1.4$. When we include the contorsion parameter f , this crossover point will occur at a different value of τ. This means that the squeezing process is either delayed or advanced relative to the standard description. This will in turn lead to a "length of the filament" at small τ which is f–dependent. But it is these lengths which control the polarization properties of the gravitational beam after it "crosses the horizon".

The polarization properties of the graviton beam are essentially the same as for photons, because they have only two helicities. In each case they are most conveniently encoded in terms of the Stokes matrix. We write, for photons and gravitons respectively

Photons

$$\epsilon_i = \begin{pmatrix} \epsilon_{\rightarrow} \\ \epsilon_{\uparrow} \end{pmatrix}$$

Gravitons

$$\epsilon_i = \begin{pmatrix} \epsilon_{+} \\ \epsilon_{\times} \end{pmatrix}$$

Stokes Matrix

$$P_{ij} = \epsilon_i \epsilon_j^* = P_{ji}^* = \frac{\epsilon}{2}\left(I + \vec{\sigma}\cdot\vec{S}\right)_{ij}$$

$$\vec{\sigma}\cdot\vec{S} = \begin{pmatrix} Q & U - iV \\ U + iV & -Q \end{pmatrix}$$

For a pure state, the determinant of the Stokes matrix vanishes. For pure linear polarized photon beams,

$$P = \frac{1}{2}\begin{pmatrix} 1 + \cos\phi & \sin\phi \\ \sin\phi & 1 - \cos\phi \end{pmatrix}$$

For pure, circularly polarized photon beams

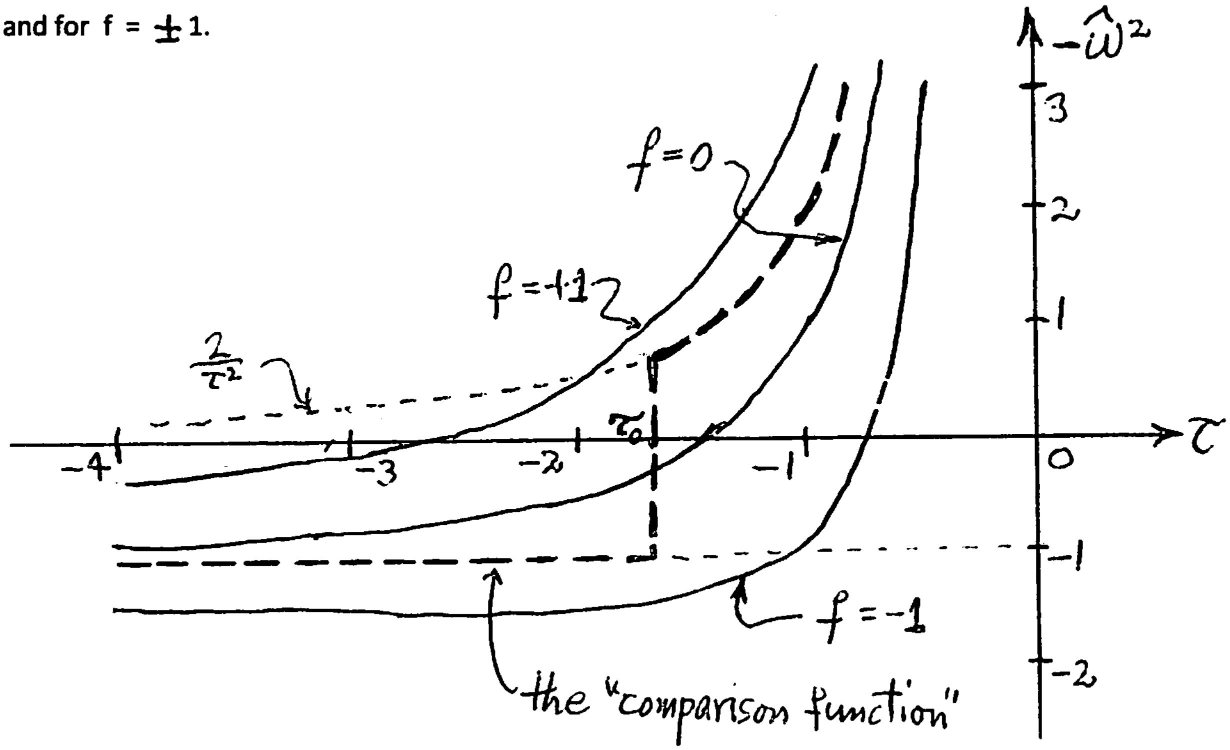
$$P = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}$$

The case of interest here is for circularly polarized gravitational beams, and the Stokes parameter of interest is V .

This parameter can be expressed directly in terms of the "length of the filaments", as discussed above:

$$V = \frac{|\ell(f)|^2 - |\ell(-f)|^2}{|\ell(f)|^2 + |\ell(-f)|^2}$$

When f is nonvanishing, the solutions of the wave equation are more complicated. They are best found numerically, and I leave that task to others. But it is not so hard to bound the "filament length" parameter via simple, analytic means. In the figure below we plot the "frequency function" $-\hat{\omega}^2$ for f = 0 and for f = $\pm$1.



Also plotted (the heavy dashed line) is a "comparison function"

$$-\hat{\omega}^2 = \begin{cases} -1 & |\tau| > |\tau_0| \\ \frac{2}{\tau^2} & |\tau| < |\tau_0| \end{cases}$$

Given this comparison function, there is a corresponding easy solution:

$$\hat{\epsilon} = \begin{cases} e^{-i\tau} & |\tau| \geq |\tau_0| \\ a\tau^2 + \frac{b}{\tau} & |\tau| \leq |\tau_0| \end{cases}$$

After matching the solutions at $\tau = \tau_0$ by equating the values of $\hat{\epsilon}$ and $\dot{\hat{\epsilon}}$ across the discontinuity there, we obtain

$$a = \frac{1}{3\tau_0^2}(1 - i\tau_0)\, e^{-i\tau_0}$$

$$b = \frac{\tau_0}{3}(2 + i\tau_0)\, e^{-i\tau_0}$$

The quantity of interest is the square of the "filament length" :

$$\ell^2 = \left| \lim_{\tau \to 0} \tau \epsilon(\tau) \right|^2$$

For the comparison-function solution

$$\ell^2(\tau_0) = |b|^2 = \frac{\tau_0^2}{9}(4 + \tau_0^2)$$

As long as the comparison function $-\hat{\omega}^2$ is less than $-\omega^2(f)$ for f = 1, and for all τ, the value of ℓ^2 for the f = 1 solution will be larger than $|b|^2$. The most stringent limit will occur for $\tau_0 = -2$, implying

$$\ell^2(f)\Big|_{f=1} > \frac{32}{9} \cong 3.5$$

In the same way, as long as the comparison function $-\hat{\omega}^2$ is larger than $-\omega^2(f)$ for f = −1, and for all τ, the value of ℓ^2 for f = −1 will be less than $|b|^2$. The most stringent limit occurs for $\tau_0 = -1$, implying

$$\ell^2(f)\Big|_{f=-1} < \frac{5}{9}$$

Evidently, for $|f| = 1$, we have very large polarization:

$$V(|f|)\Big|_{f=1} \equiv \frac{[\ell^2(1) - \ell^2(-1)]}{[\ell^2(1) + \ell^2(-1)]} \approx \left[1 - \frac{2\ell^2(-1)}{\ell^2(+1)}\right] > \frac{22}{32} \approx 0.7$$

When $|f| \ll 1$, the same kind of analysis does not succeed, This emphasizes the need for careful numerical work. But even in the absence of this work, we can conclude that for $|f| \gtrsim 1$, large values of the Stokes parameter V are indicated.

In terms of the source parameters, the implications are interesting. Recall that

$$\hat{H}_k \equiv Hu \qquad\qquad \hat{H}_c \equiv Hv = \pm fH$$

$$1 = u^2 - [u - \gamma v]^2$$

If the "simple" scenario is adopted, we have additional constraints:

$$u = 1 = \gamma v$$

It follows that there is a correlation between the value of $|f|$ and that of the Barbero-Immirzi parameter γ:

$$|V| = |f| = \left|\frac{1}{\gamma}\right|$$

Therefore, large circular-polarization effects, namely large values of the Stokes parameter V , will in the "simple" scenario occur if and only if the Barbero-Immirzi parameter $|\gamma|$ is not large compared to unity.

$$1 \geq |V| \geq 0{,}5 \iff |\gamma| \lesssim 1$$

In the next section we will study how this result gets expressed in terms of the properties of fermion-condensate sources.

VI. Dirac-Fermion Source Terms

In the first order formalism, the Dirac gamma matrices live only within the internal O(3,1) gauge group and have no spacetime dependence. The Dirac Lagrangian density takes the form

$$\mathcal{L}_{Dirac} = \frac{i}{2}(1 - i\alpha)\bar{\Psi}\gamma^A e_A^\mu\left(\partial_\mu + \omega_\mu^{BC}\frac{\gamma_B\gamma_C}{2}\right)\Psi + h.c.$$

We have included a phase factor $(1-i\alpha)$ out in front. In the Minkowky-space limit it does not affect anything. But in curved spacetime, as pointed out by Freidel et al (arXiv 0507253), it does have an effect. (We have normalized this factor so that the Dirac action reduces to standard form in the Minkowski-space limit.)

We will assume that these fermions form a condensate, such that

$$\rho_A \equiv \langle \bar{\Psi}\gamma_5\gamma_0\Psi \rangle \neq 0 \qquad \rho_V \equiv \langle \bar{\Psi}\gamma_0\Psi \rangle \neq 0$$

The Dirac Lagrangian density reduces to

$$\mathcal{L}_{Dirac} \Rightarrow -3Na^2\left[\alpha\rho_V K + \rho_A C\right]$$

This form allows us to identify the condensate densities with the source terms used in the previous sections:

$$\frac{3M_{pl}^2}{8\pi}(-2N\hat{H}_K)a^2K = -3Na^2\alpha\rho_V K$$

$$\frac{3M_{pl}^2}{8\pi}(-2N\hat{H}_C)a^2C = -3Na^2\rho_A C$$

Consequently

$$\hat{H}_k = \frac{4\pi\alpha}{M_{pl}^2}\rho_V \qquad \hat{H}_c = \frac{4\pi}{M_{pl}^2}\rho_A$$

The two conditions for our "simple" scenario, where all of the inflationary dark energy is contributed by the condensate, is

$$\hat{H}_K = \gamma \hat{H}_c \quad \Rightarrow \quad \alpha\rho_V = \gamma\rho_A$$

$$\hat{H}_K = H \quad \Rightarrow \quad \alpha\rho_V = \frac{HM_{pl}^2}{4\pi}$$

If we demand a fully chiral condensate

$$|\rho_V| = |\rho_A|$$

it follows that

$$|\alpha| = |\gamma|$$

We also found, for the simple scenario,

$$|f| = \left|\frac{\hat{H}_c}{H}\right| = \left|\frac{1}{\gamma}\right|$$

This allows us to solve for the condensate density:

$$|\rho_V| = |\rho_A| = \frac{HM_{pl}^2}{4\pi|\gamma|}$$

We conclude that a fully chiral, Lorentz-violating, fermionic condensate is sufficient to generate a large value for the gravitational-wave Stokes parameter V emergent from the inflationary era, provided only that the magnitude of the Barbero-Immirzi parameter γ is not large compared to unity.

VII. Summary

1) The goals of this note have been met, and in a surprisingly user-friendly way.
2) The choice of temporal gauge is very appropriate for the problem at hand.
3) The gravitational-wave solution in Minkowski space is exact, and is physically reasonable.
4) The use of double forms in the temporal-gauge first-order formalism provides considerable—but not complete—computational relief.
5) Inclusion of the Holst term and of sources is a smooth generalization, leading to remarkably simple results.
6) The source of torsion, needed to provide a large Stokes parameter V, is provided quite naturally by a Lorentz-violating, pure chiral fermion condensate.
7) These results need to be recast in a shorter, more intelligible form. That, along with a discussion of inflation theory, will be the next step.