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All Years

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First Published in 1996 (Revised in 2012) by:

Mathematics Publishing Company,
'The Cottage'
45, Blackstaff Road,
Clough,
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BT30 8SR.

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What other people think of our material

Below is a selection of written comments received from people who have used our material.

Key Stage 2 (9-11 year olds)

Extracts from a head teacher's letter:

'… very well received by parents, teachers and pupils …'
'… self contained…'
'… highly structured …'
'… all children including the less well able are helped …'
'…to develop concepts through a series of clearly defined steps …'
'… increased confidence for pupils …'
'… parents find user friendly as worked examples are given …'
'… language and notation are simple and clearly defined …'

From a 10 year old pupil (boy):

'… the material describes the working out in a way that is easy. The worked examples are laid out very clearly …'

GCSE (15-16 year olds)

From a 15 year old pupil (boy):

'… simple and easy way to learn maths …'
'… careful explanations of each topic…'
'… also questions to make sure you know and understand what you have learned, and each question has a worked answer to check everything you have covered …'
'… so you are never left without any help …'

From a 15 year old pupil (girl):

'…easy to understand …'
'…clear and concise …'
'…thoroughly recommended …'

GCSE Additional (15-16 year olds)

From a 16 year old pupil (girl):

'… GCSE Mechanics was very helpful …'

'*… clearly explained and easy to understand …*'
'*… well laid out …*'
'*… well structured …*'
'*… I would not hesitate to use these again …*'

From a 16 year old pupil (boy):

'*… self-explanatory and easy …*'
'*… laid down basis of skill required …*'
'*… helped me consolidate …*'
'*… succinct and effective …*'
'*… boosted my confidence …*'
'*… contributed significantly towards helping me to prepare for exams …*'

GCE Advanced (18 year olds)

From a 19 year old university student (man):

'*… may I put on record my appreciation …*'
'*… your material… gave me help and reinforcement …*'
'*… increasing my confidence to pursue my maths …*'
'*… I am now enjoying life at university …*'

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<u>Course No. 2 (Age 9 to 11)</u> **Key Stage 2 Mathematics**	**Comprising:** **- 18 sections** **- 18 practice papers (with full worked answers)**
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<u>SAMPLE 1</u>: KEY STAGE 2 - SECTION 2 – Fractions

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Just as you and your friend are about to divide up the Choco Bar, 2 more friends drop in!
Oh! Dear! What do you do? Again, you would want to share the bar with your 3 friends, wouldn't you?

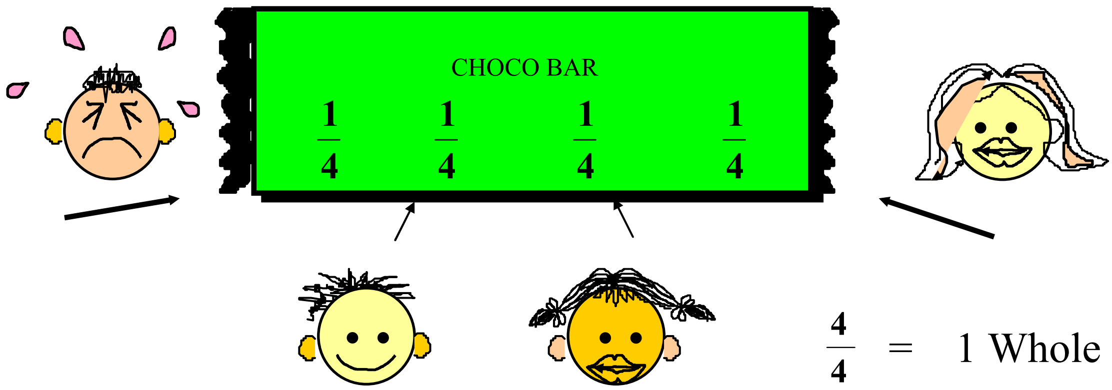
Now that the bar must be broken up into 4 equal parts, you get just **one quarter** of a bar each.

A **quarter** is written as $\frac{1}{4}$ and since **4 quarters make one whole**, we have:

$$4 \times \frac{1}{4} \quad \text{or} \quad \frac{4}{4} = 1 \text{ whole.}$$

Notice that **2 quarters** is the same as **1 half**, so:

$$\frac{2}{4} = \frac{1}{2}$$



$$\frac{4}{4} = 1 \text{ Whole}$$

$$\frac{2}{4} = \frac{1}{2}$$

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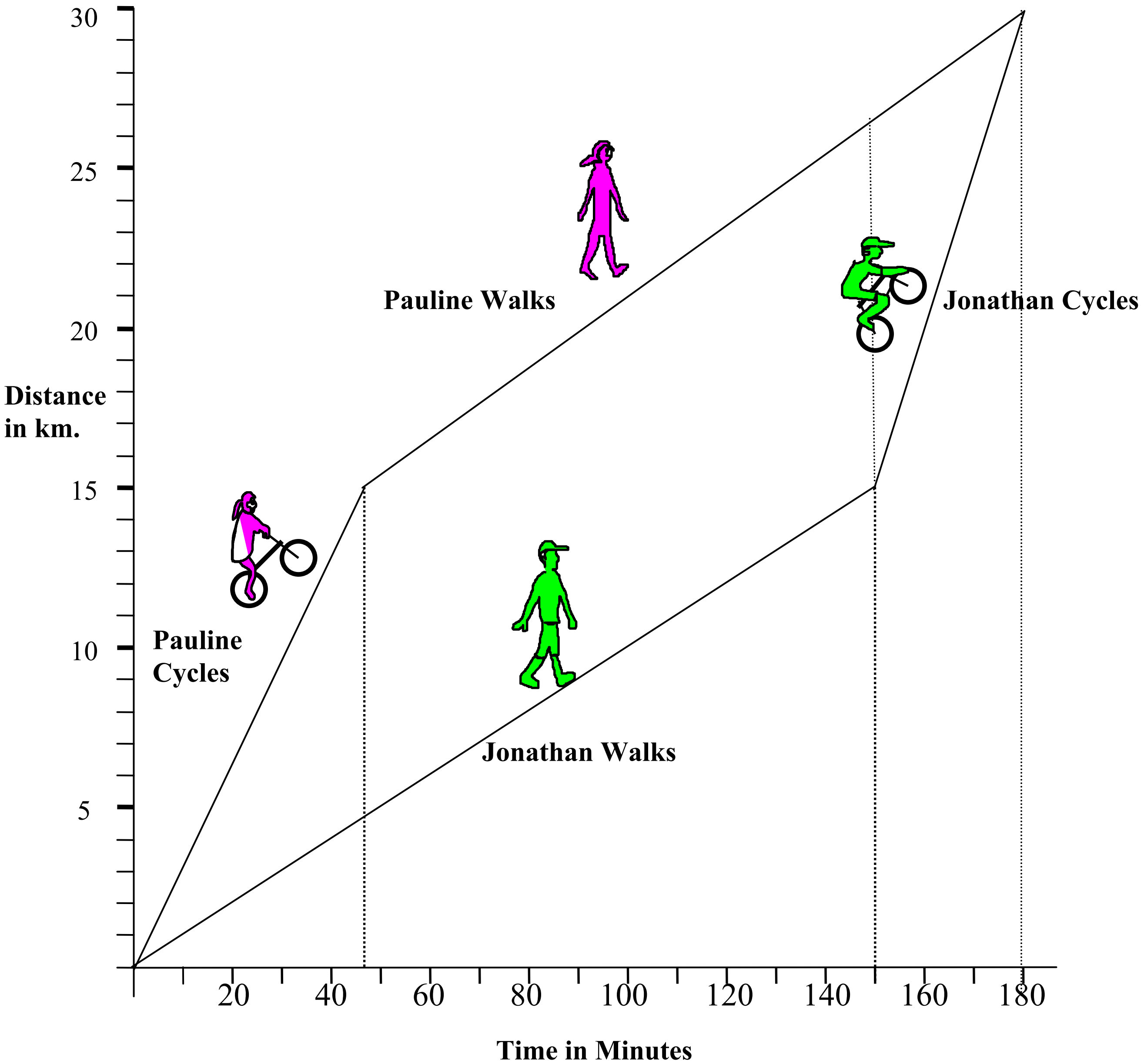
TRAVEL GRAPHS

1. The journeys of Pauline and Jonathan



Pauline and Jonathan set out together to go from Dundrum to Newry, 30 km away. Firstly, Pauline cycled and Jonathan walked. When Pauline had cycled half the distance, she left the bicycle by the roadside at Stang for Jonathan to use when he reached it. Pauline completed the journey by walking. The graph above illustrates their journeys.

(a) They left Dundrum at 10.00 a.m.
At what time did they arrive in Newry?

(b) For how long did Pauline cycle?

(c) What was Pauline's average cycling speed in km/hr?
(d) For how long was the bicycle lying beside the road at Stang?
(e) When Pauline left the bicycle, how far had Jonathan walked?
(f) How far were Pauline and Jonathan apart after $2\frac{1}{4}$ hours?
(g) What was Jonathan's average cycling speed in km/hr?
(h) What was Pauline's average speed in km/hr. for the whole journey?

ANSWERS

(a) The whole time is **180 mins** = 180 ÷ 60 = **3 hrs**. Then

	Hrs.	Mins
	10	00
+	3	00
	13	**00**

= **1.00 p.m.**

(b) **45 minutes**.

(c) Pauline cycled a distance of 15 km in 45 mins. Then:

Distance = **15 km.**

Time = $\frac{45}{60} = \mathbf{\frac{3}{4}}$ **hr**.

Speed = Distance ÷ Time

$= 15 \div \frac{3}{4}$

$= \frac{15}{1} \times \frac{4}{3}$

$= \frac{60}{3}$

= **20 km/hr**.

(d) Pauline left the bicycle at Stang **45** minutes after leaving Dundrum and Jonathan reached the bicycle **150** mins after leaving Dundrum.

Therefore, the time that the bicycle was left by the roadside is

$$\begin{array}{r} 150 \\ -\ 45 \\ \hline 105 \end{array}$$ minutes = **1 hour 45 minutes**.

(e) Pauline left the bicycle after 45 minutes.

After 45 minutes, Jonathan had walked approximately

4.4 km.

(f) $2\frac{1}{4}$ hrs = $2\frac{1}{4} \times 60$ = **135 minutes**.

The **distance** between them after 135 minutes is

25 km - 13.5 km

= **11.5 km**.

(g) Jonathan cycled a distance of 15 km in 30 mins

= **30 km/hr**.

(h) Pauline travelled a total distance of **30 km** in 3 hrs.

Speed = Distance ÷ Time

= 30 ÷ 3

= **10 km/hr.**

SECTION 2 - Polygons

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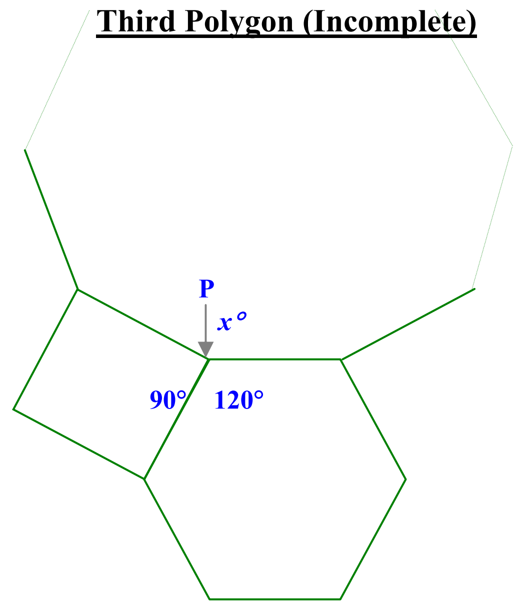
2. A patio area is to be covered with three different flagstones, each of which is a regular polygon of side **1m**.
The three flagstones fit together exactly at a point.
One is a square and another is a hexagon.

Find:
(a) the number of sides in the third polygon
and
(b) the length of railing required to fence in the patio.

The patio area looks like this:



(a) The three polygons fit together at the point **P** on the diagram.

$x°$ is the interior angle in the third polygon and we need to find it before we can proceed.

The size of each **interior angle** in the **square** is **90°** and the size of each **interior angle** in the **regular hexagon** is **120°**.
Since the **three** angles at **P** add up to **360°**, the size of the third angle is:

$$x = 360° - (90° + 120°)$$
$$\Rightarrow x = 360° - 210°$$
$$\Rightarrow x = 150°.$$

The size of each **exterior angle** in the third polygon is, therefore: **180° - 150°** = **30°**.

$\therefore$ the **number** of **sides** is:

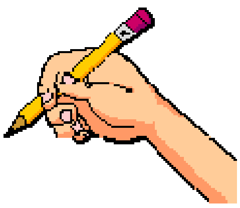
$$\frac{360}{30} = 12.$$

(b) Clearly, there are **two sides** of each polygon 'buried' **inside** the patio area.

Then, the **perimeter** consists of **two** sides of the **square**, **four** sides of the **hexagon** and **ten** sides of the **duodecagon** (i.e. a twelve-sided polygon).

The total length of railing required to fence in the patio is, therefore:

2m + 4m + 10m = **16m**.



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SECTION 2: **Similar Shapes**

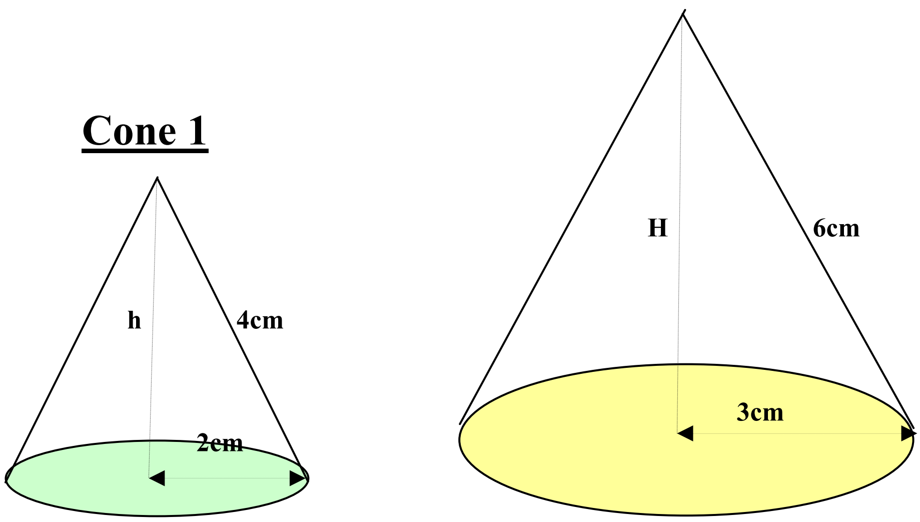
Two shapes are **similar** if their **corresponding linear dimensions** are in the **same proportion**.

It is clear, from this definition, that:

(i) **All squares** are **similar**.
(ii) **All cubes** are **similar**.
(iii) **All circles** are **similar**.
(iv) **All spheres** are **similar**.
(v) **All regular polygons** are **similar** to **regular polygons** with the **same number** of **sides**.

However, other shapes, like cones, pyramids and cylinders are similar **only if** two **base** dimensions and two **height** dimensions are in the **same proportion**.

E.g.(i) **Cone 2**



Since	**Cone 1**	:	**Cone 2**			
Radius	**2**	**:**	**3**			
Slant Height	**4**	**:**	**6**	=	**2 : 3**,	(i.e. the same proportion)

these two cones are **similar**.

This means that **any length** on **Cone 2** is $\frac{3}{2}$ × the **corresponding length** on **Cone 1** or, conversely, **any length** on **Cone 1** is $\frac{2}{3}$ × the **corresponding length** on **Cone 2**.

It follows that:

h : H is in the ratio **2 : 3**.

and **H : h** is in the ratio **3 : 2**.

The **area ratio** for **Cone 1 : Cone 2** is $2^2 : 3^2$ and the **area ratio** for **Cone 2 : Cone 1** is $3^2 : 2^2$.

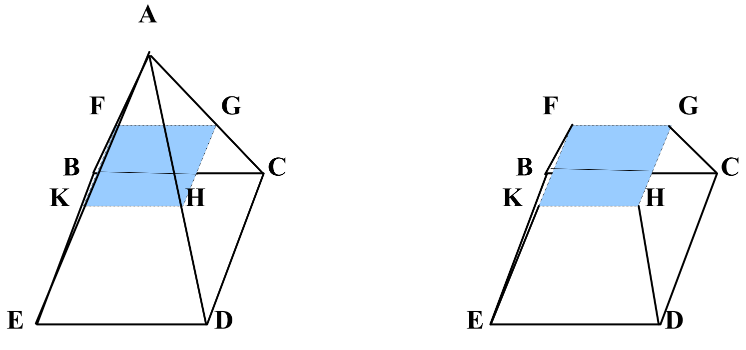
The **volume ratio** for **Cone 1 : Cone 2** is $2^3 : 3^3$ and the **volume ratio** for **Cone 2 : Cone 1** is $3^3 : 2^3$.

Clearly, then, if **Cone 1** has **circumference** x, **Cone 2** has **circumference** $\frac{3}{2}x$. **(N.B. circumference is a <u>length</u>, i.e. linear measure.)**

Also, if **Cone 2** has **surface area** y, then **Cone 1** must have **surface area** $\frac{4}{9}y$.

And, if **Cone 1** has **volume** z, then **Cone 2** has **volume** $\frac{27}{8}z$.

<u>Worked Example</u>:



<u>frustum</u>

F, **G**, **H** and **K** are the **mid-points** of the sloping edges **AB**, **AC**, **AD** and **AE** (respectively) of the pyramid on the previous page, of which **BCDE** is the horizontal base.

The top is cut off this pyramid along the plane **FGHK**, leaving the **frustum** on the right.

(a) If the **area** of the base **BCDE** is **540cm^2**, find the **area** of the **top** of the **frustum**.

(b) If the **volume** of the pyramid **ABCDE** is **0.162m^3**, find the **volume** of the **frustum**, in **cm^3**.

ANSWER:

∵ **F**, **G**, **H** and **K** are the **mid-points** of **AB**, **AC**, **AD** and **AE**, the **plane FGHK** is **parallel** to the **plane BCDE**, and is, therefore, **horizontal**.

Clearly, then, we have two **similar** pyramids, **AFGHK** and **ABCDE** where the **ratios** are as follows:

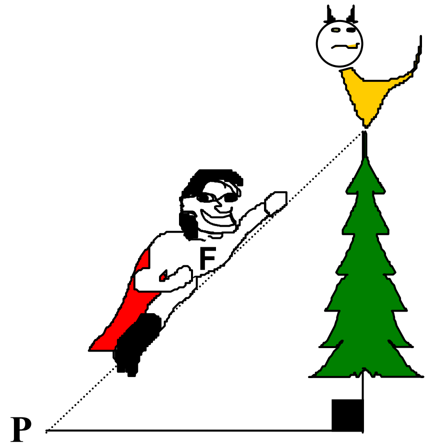
Pyramid	**AFGHK**	:	**ABCDE**		
Linear Ratio	**1**	**:**	**2**		
Area Ratio	**1^2**	**:**	**2^2**	=	**1 : 4**
Volume Ratio	**1^3**	**:**	**2^3**	=	**1 : 8**.

(a) **Area** of **BCDE** = **540 ÷ 4** = **135cm^2**.

(b) **Volume** of **frustum** = $\frac{7}{8}$ of **162000cm^3** = **141750cm^3**.

Angle of Elevation

Example:

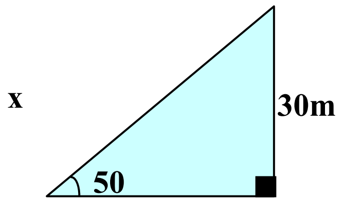


Flyman flies from a point **P** on the horizontal ground to rescue a cat at the top of a tree, **30 m** tall. If the **angle** of **elevation** from **P** to the tree-top is **50°**, find:

(a) (i) the **distance** travelled by Flyman, and
(ii) the **distance** from **P** to the **foot** of the tree.

(b) If Flyman takes **0.1 seconds** to reach the cat, find his **average speed** in **km / hr**.

Answer:



(a)(i) $\sin 50° = \frac{30}{x}$

$\Rightarrow x = \frac{30}{\sin 50^\circ}$

$\Rightarrow$ **39.162m**.

(ii) $\sqrt{39.162^2 - 30^2} =$ **25.173m**.

(b) **39.162…m** in **0.1 seconds**
$\Rightarrow$ **1409832 m** in **3600 seconds**
$\Rightarrow$ **1409.832 km** in **1 hour**
$\Rightarrow$ **1410km/hr**.

Sample 4: GCSE – Additional Level

Section 7 - Mechanics

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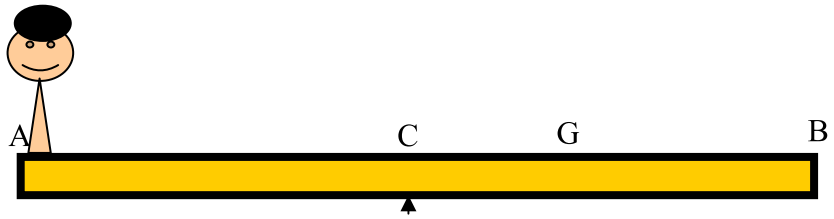
5. A **non-uniform** plank AB of length 8 metres and mass M kg rests in equilibrium on supports at each end of the plank. The reactions at A and B are 15g N and 35g N respectively. The centre of mass of the plank is at G where AG = d metres as shown in the figure below.

Calculate

(i) the mass M of the plank,
(ii) the distance d of the centre of mass from A.

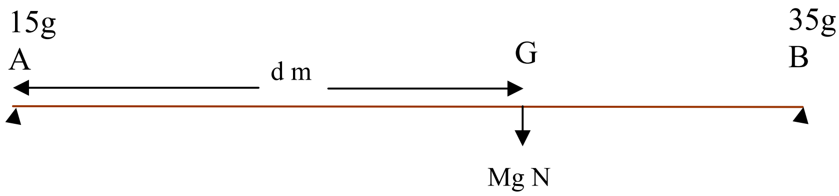
One of the supports is now removed and the other is placed at the centre C of the plank. A child of mass m kg stands on the plank at A and the plank again rests in equilibrium in a horizontal position as shown in the figure below.



(iii) Find the mass m of the child.

Worked Answer:

5.



(i) The plank is in equilibrium if the forces acting upwards balance those acting downwards:

$$\Rightarrow \quad \mathbf{Mg} = \mathbf{15g + 35g}$$

$$\Rightarrow \quad \mathbf{Mg} = \mathbf{50g}$$

$$\Rightarrow \quad \mathbf{M} = \mathbf{50kg.}$$

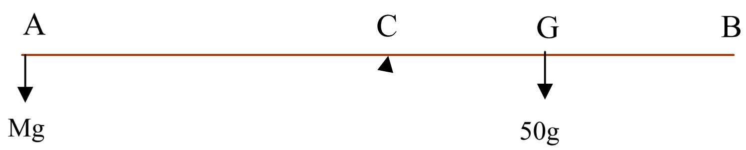
(ii) Take moments at (**A**) :

$$\mathbf{50g \times d} = \mathbf{35g \times 8}$$

$$\mathbf{d} = \frac{\mathbf{35g \times 8}}{\mathbf{50g}}$$

$$\Rightarrow \quad \mathbf{d} = \mathbf{5.6\ m}.$$

(iii)



Take moments at (**C**):

$$\mathbf{mg \times 4} = \mathbf{50g \times (5.6 - 4)}$$

$$\mathbf{4mg} = \mathbf{50g \times 1.6}$$

$$\Rightarrow \quad \mathbf{m} = \frac{\mathbf{50g \times 1.6}}{\mathbf{4g}}$$

$$\Rightarrow \quad \mathbf{m} = \mathbf{20kg}.$$

SECTION 6 - Differentiation and Integration

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Worked Example - Differentiation

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Find, **from first principles**, the derived functions of:

1. $y = x^2 - 2x + 1$.

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METHOD:

1. $y = x^2 - 2x + 1$.

Consider two points, (x,y) and $(x + h, y + k)$ on the curve of

$y = x^2 - 2x + 1$.

Then $y + k = (x + h)^2 - 2(x + h) + 1$

$\Rightarrow \quad y + k = x^2 + 2xh + h^2 - 2x - 2h + 1$ … (i)

and $\quad y = x^2 - 2x + 1$ … (ii).

(i) – (ii)…$k = 2xh + h^2 - 2h$.

$\Rightarrow \quad \frac{k}{h}$ (the **gradient** of the **chord** joining the two points) $= 2x + h - 2$.

$(h \neq 0)$.

$\therefore$ the limit of the gradient as h tends to 0, the derived function, $\frac{dy}{dx}$ to the curve,

$y = x^2 - 2x + 1$ is $\underline{2x - 2}$.

Worked Example - Integration

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5. Given that $\frac{dy}{dx} = x^2 - 3x + 1$ for a certain curve y and that the point $(1, -1)$ lies on the curve, find the equation of y.

METHOD:

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5. $$\frac{dy}{dx} = x^2 - 3x + 1$$

$$\Rightarrow \quad y = \frac{x^3}{3} - \frac{3x^2}{2} + x + c$$

$x = 1, y = -1$:

$$\Rightarrow \quad -1 = \frac{1^3}{3} - \frac{3(1)^2}{2} + 1 + c$$

$$\Rightarrow \quad -1 = \frac{1}{3} - \frac{3}{2} + 1 + c$$

$$\Rightarrow \quad c = -\frac{5}{6}$$

$\Rightarrow \quad y = \frac{x^3}{3} - \frac{3x^2}{2} + x - \frac{5}{6}$ is the equation of the curve y.

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Differentiation

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E.g. (ii) Differentiate $y = \frac{1}{\sqrt{x^2 - 4}}$.

Let $u = x^2 - 4 \Rightarrow \frac{du}{dx} = 2x$

and $y = u^{-\frac{1}{2}} \Rightarrow \frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$.

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(-\frac{1}{2}u^{-\frac{3}{2}}\right)(2x) = -x\,u^{-\frac{3}{2}}$$

$$= -x(x^2 - 4)^{-\frac{3}{2}} \quad \text{or} \quad -\frac{x}{\sqrt{(x^2-4)^3}}.$$

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<u>SAMPLE 7</u>: GCE ADVANCED LEVEL

SECTION 3 – Coordinate Geometry – The Circle

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Worked Example 2:

A circle C has centre **O(8, 6)** and radius **10cm**. The point **P** lies on its circumference to the right of the **y – axis** and **OP** is parallel to the **x - axis**.

(a) Find the coordinates of the point **P**.

Another point **Q(- 2, 6)** lies on C.

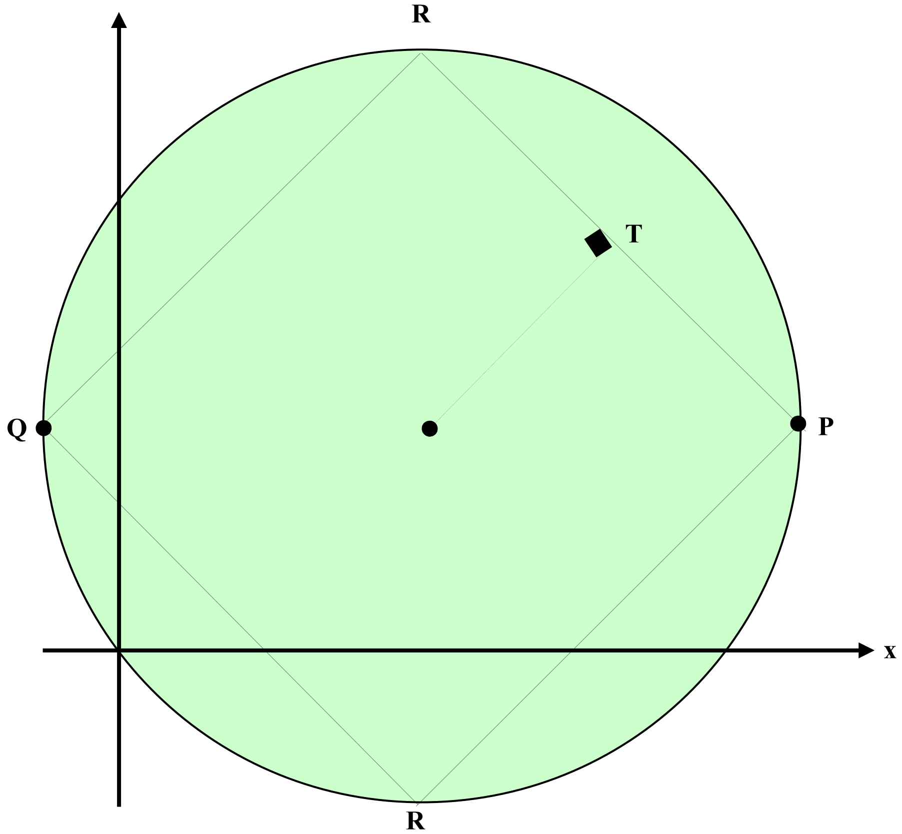
(b) Prove that **PQ** is a diameter of C.

A third point **R** lies on the circumference of C and the triangle **PQR** is isosceles.

(c) Find (i) the coordinates of **R** and
(ii) the length of **PR**, in its simplest surd form.

(d) A perpendicular is drawn from **O** onto **PR**, meeting **PR** at the point **T**.
Find the length of **OT**, giving reasons for your answer.

Method:



(a) $\because$ **OP** is parallel to the **x – axis**, **P** has its coordinate **y = 6**.

Radius **10cm** $\Rightarrow$ **P = (8 + 10, 6)**

$\Rightarrow$ **P = (18, 6)**.

(b) $\mathbf{PQ = 18 - (-2) = 20cm = 2 \times 10cm}$ (i.e. **2 × radius**).

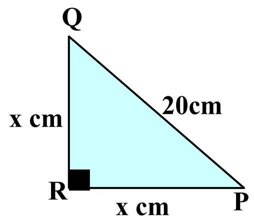
$\therefore$ **PQ** is a diameter of C.

(c)(i) $\because$ Δ **PQR** is isosceles **OR** is perpendicular to **PQ**, and is, therefore, parallel to the **y – axis**,

$\Rightarrow$ $\mathbf{R = (8, 6 \pm 10) = (8, 16)}$ or $\mathbf{(8, -4)}$.

(ii) $\because$ **PQ** is a diameter of C, angle $\mathbf{PRQ = 90°}$

(angle in semicircle is **90°**)



By **Pythagoras's Theorem**:

$$\mathbf{PQ^2 = x^2 + x^2 = 2x^2}$$

$$\Rightarrow \mathbf{20^2 = 2x^2}$$

$$\Rightarrow \mathbf{400 = 2x^2}$$

$$\Rightarrow \mathbf{200 = x^2}$$

$$\therefore \mathbf{\sqrt{200} = x}$$

$$\Rightarrow \mathbf{10\sqrt{2} = x}$$

$$\therefore \mathbf{PR = 10\sqrt{2}}.$$

(d) **OT** bisects **PR** (A line through the centre perpendicular to a chord bisects the chord.)

$$\therefore \mathbf{PT = \frac{1}{2} PR = 5\sqrt{2}}.$$

$\because$ Δ **PQR** is isosceles and angle $\mathbf{PRQ = 90°}$, angle $\mathbf{OPT = 45°}$.

$\because$ angle $\mathbf{OTP = 90°}$, angle $\mathbf{TOP = 45°}$ also (sum of angles in triangle is 180°).

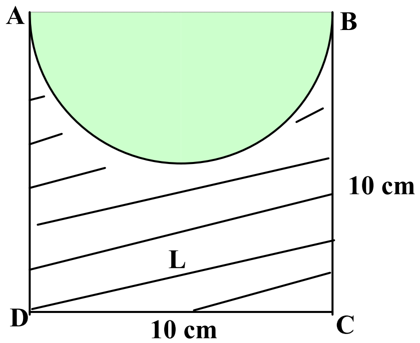
$\therefore$ Δ **TOP** is isosceles, making $\mathbf{OT = PT}$

$$\therefore \mathbf{OT = 5\sqrt{2}}.$$

SAMPLE 8: GCE ADVANCED LEVEL

SECTION 2 – Centre of Mass

Worked Example 2:



A uniform lamina **L** is formed by taking a uniform square sheet of material **ABCD**, of side **10 cm**, and removing the semi-circle with diameter **AB** from the square, as shown in the figure above.

(a) Find, in cm to 2 decimal places, the distance of the centre of mass of the lamina **L** from the mid-point of **AB**.

[The centre of mass of a uniform semi-circular lamina, radius **a**, is at a distance $\frac{4a}{3\pi}$ from the centre of the bounding diameter.]

The lamina is freely suspended from **D** and hangs at rest.

(b) Find, in degrees to one decimal place, the angle between **CD** and the vertical.

(Edexcel GCE Mechanics – M2, June, '02.)

Method:

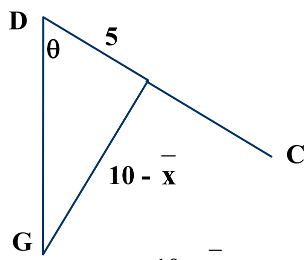
(a)

Shape	**Square**	**Semi-circle**	**Lamina L**
Relative Masses	$s^2 = 100$	$\frac{1}{2}\pi r^2 = \frac{25}{2}\pi$	$100 - \frac{25}{2}\pi$
Distance of centre of mass from AB	5	$\frac{4r}{3\pi} = \frac{20}{3\pi}$	$\bar{x}$

Moments about **AB**: $100 \times 5 - 12\frac{1}{2}\pi \times \frac{20}{3\pi} = (100 - 12\frac{1}{2}\pi)\ \bar{x}$

$\Rightarrow \quad \bar{x} = $ **6.86 cm**.

(b)



$\text{Tan } \theta = \frac{10 - \bar{x}}{5}$

$\Rightarrow \quad \text{Tan } \theta = \frac{10 - 6.86}{5}$

$\Rightarrow \quad \theta \approx$ **32.1°**.

<u>SAMPLE 9</u>: GCSE ORDINARY – SECTION 15 AND ALGEBRA BOOK –SECTION 4

Simultaneous Equations

<u>Note from the Author</u>:

As a state examiner, I am painfully aware of the difficulty posed by algebra. With that in view, I wrote this book.
In order to be able to tackle algebra effectively, you have to know the foundations and each layer that is built on top.
The book is designed to allow you to start from the beginning of the subject, progressing to more difficult material.

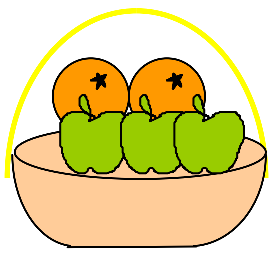
SECTION 4 – Simultaneous Equations

If we have **two** equations, each containing **2 unknown quantities**, it is possible to find a **solution** that satisfies **both equations simultaneously**, which means '**at the same time**'.
Equations of this type are called **'Simultaneous Equations'**.

Eg. (i) If **2 oranges** and **3 apples** together cost **95p** and **1 orange** and **3 apples** together cost **70p**, find the cost of:
(a) 1 orange and
(b) 1 apple.

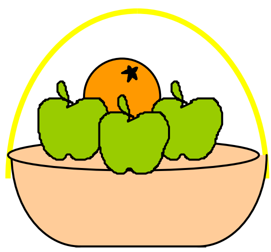
If we put the **two lots** of fruits into **separate baskets**, we have:

<u>Cost = 95p</u>



Basket 1

<u>Cost = 70p</u>



Basket 2

Notice that there are **3 apples** in each basket. If we take the **difference in the contents** of the baskets, it must be **equal** to the **difference** in the **costs** of the contents.
This gives:

1 orange = 95p - 70p
i.e. 1 orange costs **25p**.

Now, from Basket **2**, **1 orange** plus **3 apples** cost **70p**. Therefore, **3 apples** must cost **70p - 25p**,

i.e. 3 apples cost **45p**.
Therefore, **1 apple** costs **15p**.

Answer: **(a)** **1 orange** costs **25p**; **(b)1 apple** costs **15p**.

Algebraic Fractions – Worked Examples

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Eg.(iii) **Simplify** the following:

$$\frac{\mathbf{2x+2}}{\mathbf{12x-6}} \times \frac{\mathbf{6x-3}}{\mathbf{6x+6}}$$

Factorising all numerators

and denominators gives:

$$\frac{\mathbf{2(x+1)}}{\mathbf{6(2x-1)}} \times \frac{\mathbf{3(2x-1)}}{\mathbf{6(x+1)}}$$

Cancelling gives:

$$\frac{{}^{1}\cancel{2}\cancel{(x+1)}^{1}}{{}_{1}^{2}\cancel{6}\cancel{(2x-1)}_{1}} \times \frac{{}^{1}\cancel{3}\cancel{(2x-1)}}{\cancel{6}\cancel{(x+1)}}$$

$$= \frac{\mathbf{1}}{\mathbf{6}}, \quad \text{in its } \textbf{simplest} \text{ form.}$$

Ex.(iv) **Simplify** the following:

$$\frac{\mathbf{2x^2-x}}{\mathbf{4x+2}} \div \frac{\mathbf{4}}{\mathbf{2x^2+x}}$$

Invert divisor and ×:

$$\frac{\mathbf{2x^2-x}}{\mathbf{4x+2}} \times \frac{\mathbf{2x^2+x}}{\mathbf{4}}$$

Factorising all numerators

and denominators gives:

$$\frac{x(2x-1)}{2(2x+1)} \times \frac{x(2x+1)}{4}$$

Cancelling gives:

$$\frac{x(2x-1)}{2\cancel{(2x+1)}_{1}} \times \frac{x\cancel{(2x+1)}^{1}}{4}$$

$$= \frac{\mathbf{x^2(2x-1)}}{\mathbf{8}} \quad \left(\left(\text{or } \frac{2x^3-x^2}{8}\right), \text{ in its } \textbf{simplest} \text{ form.}\right.$$

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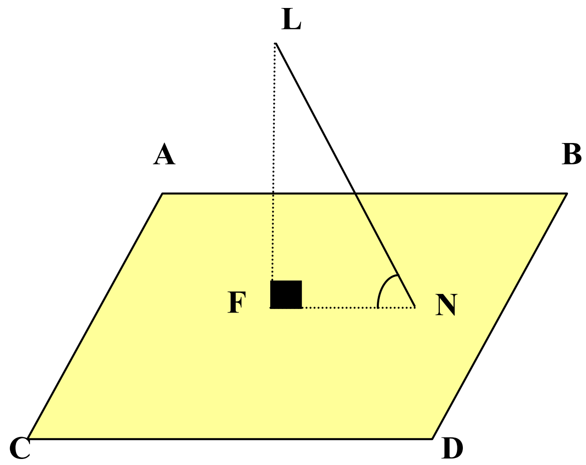
THREE-DIMENSIONAL PROBLEMS

(i) To find the angle between a line and a plane:

Drop a **perpendicular** from a **point** on the **line** onto the **plane** and join the '**foot**' of this **perpendicular** to the **point** where the line **meets** the **plane**.

In the diagram below, the line **LN** meets the plane **ABCD** at **N**.
The **angle** between **LN** and the plane **ABCD** is ∠ **LNF**.



(ii) To find the angle between two planes:

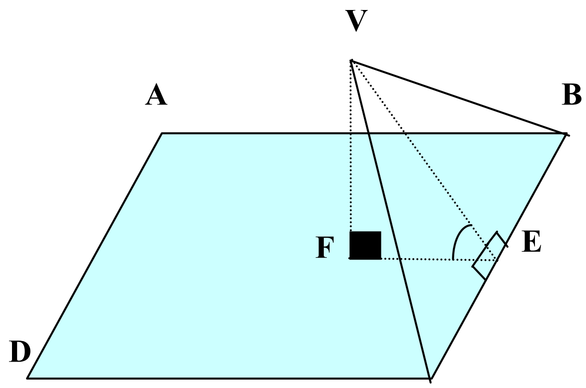
Draw a **perpendicular** on **each plane** onto the **common line** of intersection, to meet at a **point** on the common line.

The **angle** between these **two perpendiculars** is the **angle between** the **two planes**.

In the diagram below, the plane **VCB** meets the plane **ABCD** along the **common line BC**.

VE is drawn perpendicular to **BC** on the plane **VBC** and **FE** is drawn perpendicular to **BC** on the plane **ABCD**.

The **angle** between the two planes is ∠ **VEF**.

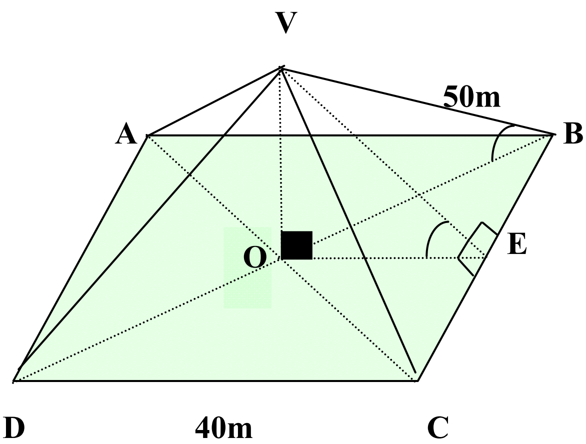


Worked Example on 3-dimensional Trigonometry

The diagram below shows a **square pyramid**, base **ABCD** and vertex **V**, which is **vertically** above **O**. If the square base has **edge 40m** and **VB** is **50m**, find:

(i) the **angle** between the sloping edge **VB** and the base **ABCD**;

(ii) the **angle** between the sloping face **VBC** and the base **ABCD**.

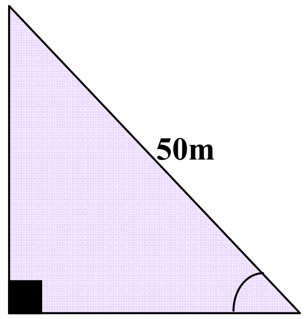


(i) The **angle** between **VB** and **ABCD** is $\angle$ **VBO**.

Firstly, we must find the length of **OB**:

$$\mathbf{OB} = \frac{1}{2}\sqrt{40^2 + 40^2} = \mathbf{28.2843m}$$ **(by Pythagoras's Theorem).**

Now, triangle **VOB** looks like this:



Using the **cos** ratio, we have:

$$\text{Cos}\angle \text{VBO} = \frac{28.2843}{50}$$

$$\therefore \quad \text{Cos}\angle \text{VBO} = 0.565686$$

$$\Rightarrow \quad \angle \text{VBO} = 55.55°.$$

(ii) If we draw a perpendicular from **V** onto **BC**, on plane **VBC**, **VE** bisects **BC** **(since VBC is an isosceles triangle).**

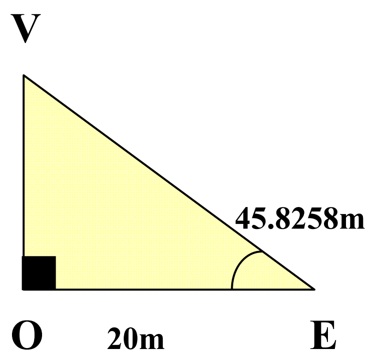
OE is perpendicular to **BC**, on plane **ABCD**.

$\therefore$ $\angle$ **VEO** is the **angle** between planes **VBC** and **ABCD**.

Firstly, we must find the length of **VE**, in triangle **VBC**:

$$\mathbf{VE} = \sqrt{50^2 - 20^2} = \mathbf{45.8258m}$$ **(by Pythagoras' Theorem).**

Now, triangle **VOE** looks like this:



Using the **cos** ratio, we have:

$$\textbf{Cos } \angle \textbf{VEO} = \frac{\textbf{20}}{\textbf{45.8258}}.$$

$$\therefore \quad \textbf{Cos } \angle \textbf{VEO} = \textbf{0.436435}$$

$$\Rightarrow \quad \angle \textbf{VEO} = \textbf{64.1°}.$$

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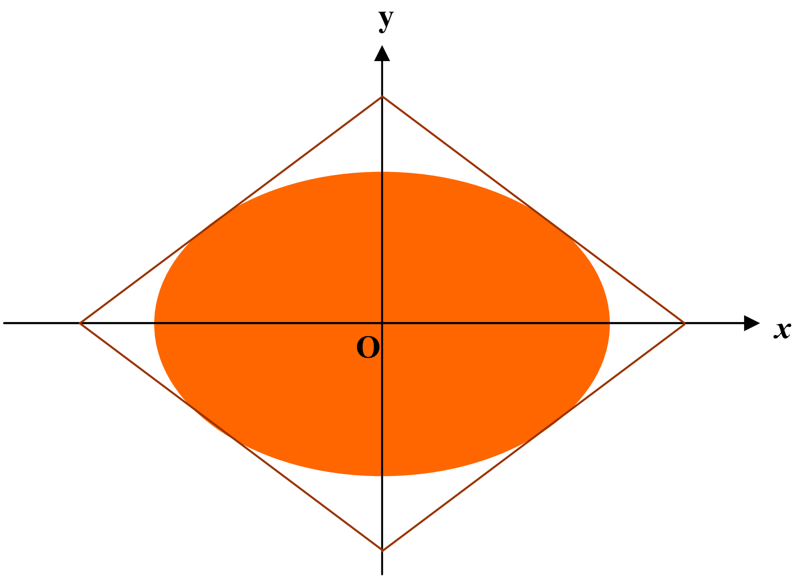
3.



A table top, in the shape of a parallelogram, is made from two types of wood. The design is shown in the figure above. The area inside the ellipse is made from orange wood, and the surrounding area is made from white wood.

The ellipse has parametric equations:

$$x = 5\cos\theta, \quad y = 4\sin\theta, \quad 0 \le \theta < 2\pi.$$

The parallelogram consists of four line segments, which are tangents to the ellipse at the points where $\theta = \alpha$, $\theta = -\alpha$,
$\theta = \pi - \alpha$, $\theta = -\pi + \alpha$.

(a) Find an equation of the tangent to the ellipse at $(5\cos\alpha, 4\sin\alpha)$ and show that it can be written in the form

$$5y\sin\alpha + 4x\cos\alpha = 20.$$

(b) Find by integration the area enclosed by the ellipse.

(c) Hence show that the area enclosed between the ellipse and the parallelogram is

$$\frac{80}{\sin 2\alpha} - 20\pi.$$

(d) Given that $0 < \alpha < \frac{\pi}{4}$, find the value of α for which the areas of the two types of wood are equal. (Edexcel GCE - Pure Mathematics P3 – January, 2002.)

Method:

(a) $x = 5\cos\theta \Rightarrow \frac{x}{5} = \cos\theta$ and $y = 4\sin\theta \Rightarrow \frac{y}{4} = \sin\theta$.

$\because \quad \sin^2\theta + \cos^2\theta = 1, \quad \frac{x^2}{25} + \frac{y^2}{16} = 1.$

Implicit differentiation gives: $\frac{2x}{25} + \frac{y}{8}\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{8}{y}\left(\frac{-2x}{25}\right) = \frac{-16x}{25y}.$$

$$x = 5\cos\alpha, \quad y = 4\sin\alpha \Rightarrow \frac{dy}{dx} = \frac{-80\cos\alpha}{100\sin\alpha} = \frac{-4\cos\alpha}{5\sin\alpha}.$$

$$y - 4\sin\alpha = \frac{-4\cos\alpha}{5\sin\alpha}(x - 5\cos\alpha).$$

$$5y\sin\alpha - 20\sin^2\alpha = -4x\cos\alpha + 20\cos^2\alpha$$

$$\Rightarrow \quad 5y\sin\alpha + 4x\cos\alpha = 20(\sin^2\alpha + \cos^2\alpha) = 20.$$

$\therefore \quad 5y\sin\alpha + 4x\cos\alpha = 20$ is the equation of the tangent.

Q.E.D.

(b) $4\int_0^5 y\,dx$ will give the area of the ellipse. (**N.B.** **a**, the semi-major axis is **5**.)

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$\Rightarrow \quad y^2 = 16\left(1 - \frac{x^2}{25}\right) = \frac{16(25 - x^2)}{25}.$$

$$\Rightarrow \quad y = \frac{4}{5}\sqrt{25 - x^2}.$$

$\therefore$ **Area of ellipse**

$$= \frac{16}{5}\int_0^5 \sqrt{25 - x^2}\,dx$$

$$= \frac{16}{5}\int_0^5 \sqrt{25\left(1 - \frac{x^2}{25}\right)}\,dx$$

$$= 16\int_0^5 \sqrt{1 - \frac{x^2}{25}}\,dx.$$

(Use **integration by substitution** here.)

Let $\frac{x}{5} = \sin u \Rightarrow \frac{dx}{du} = 5\cos u \Rightarrow dx = 5\cos u\, du.$

$x = 0 \Rightarrow u = 0; \qquad x = 5 \Rightarrow u = \frac{\pi}{2}.$

$$\therefore \quad 16\int_0^5 \sqrt{1 - \frac{x^2}{25}}\,dx \quad \Rightarrow \quad 16\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 u}\,(5\cos u)du$$

$$\Rightarrow \quad 80\int_0^{\frac{\pi}{2}} \sqrt{\cos^2 u}\,(\cos u)du$$

$$\Rightarrow \quad 80\int_0^{\frac{\pi}{2}} \cos^2 u\,du.$$

$\text{Cos } 2u = \cos^2 u - \sin^2 u = \cos^2 u - 1 + \cos^2 u = 2\cos^2 u - 1,$

$\Rightarrow \quad \cos^2 u = \frac{1}{2}(\cos 2u + 1).$

Now we have: $80\int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos 2u + 1)du$

$$\Rightarrow \quad 40\int_0^{\frac{\pi}{2}} (\cos 2u + 1)du$$

$$\Rightarrow \quad 40\left[\frac{\sin 2u}{2} + u\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad 40\left\{\left[\frac{\pi}{2}\right] - [0]\right\}$$

$\Rightarrow \quad 20\pi$ square units.

(c) $\quad 5y \sin\alpha + 4x\cos\alpha = 20.$

On x – axis, $y = 0 \Rightarrow x = \frac{5}{\cos\alpha}$

$\Rightarrow \frac{5}{\cos\alpha} \times 2 = \frac{10}{\cos\alpha} =$ **longer diagonal** of parallelogram.

On y – axis, $x = 0 \Rightarrow y = \frac{4}{\sin\alpha}$

$\Rightarrow \frac{4}{\sin\alpha} \times 2 = \frac{8}{\sin\alpha} =$ **shorter diagonal** of parallelogram.

Area of rhombus $= \frac{1}{2}$ product of diagonals

$$\Rightarrow \frac{1}{2}\left(\frac{10}{\cos\alpha} \times \frac{8}{\sin\alpha}\right) = \frac{1}{2}\left(\frac{80}{\cos\alpha\sin\alpha}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{80}{\frac{1}{2}\sin 2\alpha}\right)$$

$$\Rightarrow \frac{80}{\sin 2\alpha}.$$

Area of ellipse $= 20\pi$.

$\therefore$ Enclosed area $= \dfrac{80}{\sin 2\alpha} - 20\pi$.

Q.E.D.

(d) $20\pi = \dfrac{80}{\sin 2\alpha} - 20\pi$

$$\Rightarrow 40\pi \sin 2\alpha = 80$$

$$\Rightarrow 2\alpha = \sin^{-1}\left(\frac{80}{40\pi}\right)$$

$$\Rightarrow 2\alpha \approx 0.69 \text{ radians.}$$

$$\therefore \alpha \approx 0.345 \text{ radians.}$$