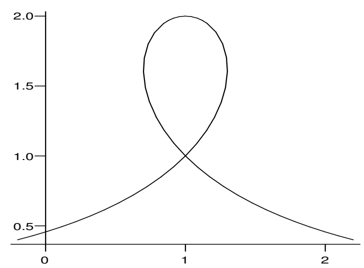
Sample Test 3 − Solutions

1. Sketch the following parametric curve and find the equation of the tangent at the point of self intersection

$$x = \frac{1+t+t^2-t^3}{1+t^2}, \quad y = \frac{2}{1+t^2}.$$

Solution
From the graph, it appears that they cross at the point $(1, 1)$.



Two determine the times where they cross we choose y (its easier) and set it to 1

$$y = 1 \quad \Rightarrow \quad \frac{2}{1+t^2} = 1 \quad \Rightarrow \quad t^2 = 1 \quad \Rightarrow \quad t = \pm 1.$$

Substituting both $t = -1$ and $t = 1$ into x shows both are 1 so yes, $(1, 1)$ is the point the curve crosses itself. Next we find derivatives

$$\frac{dx}{dt} = -\frac{t^4+4t^2-1}{(t^2+1)^2}, \quad \frac{dy}{dt} = -\frac{4t}{(t^2+1)^2},$$

and dividing gives

$$\frac{dy}{dx} = \frac{4t}{t^4+4t^2-1}.$$

At $t = -1$, $\frac{dy}{dx} = -1$ and at $t = 1$, $\frac{dy}{dx} = 1$. So the tangents are

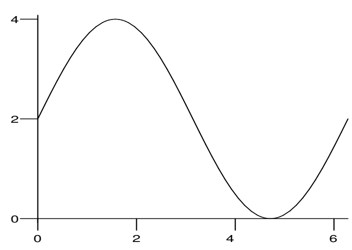
$$y - 1 = -1(x-1), \qquad y - 1 = 1(x-1).$$

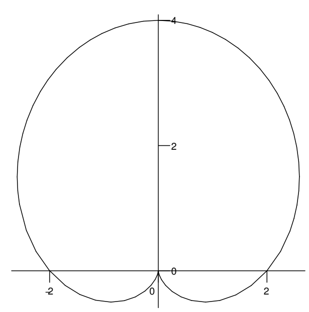
2. Graph the following polar equations

$$r = 2 + 2\sin\theta, \quad r = 2\sin 2\theta, \quad r^2 = 2\sin 2\theta.$$

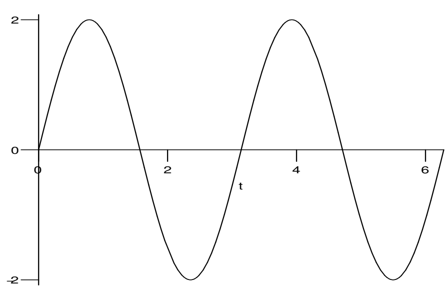
Solutions

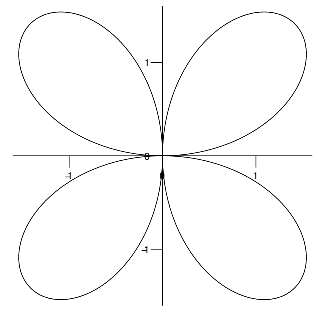
$r = 2 + 2\sin\theta$,



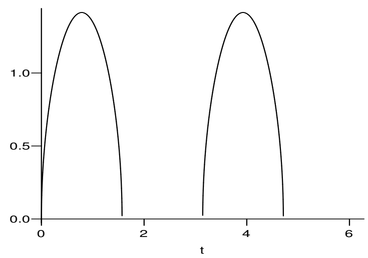


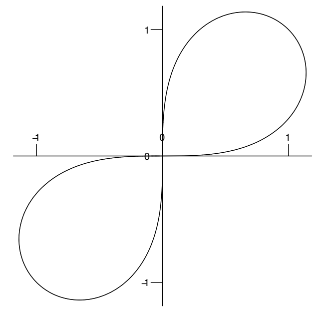
$r = 2\sin 2\theta$,





$r^2 = 2\sin 2\theta$





3. Find the area inside one leaf of the rose described by

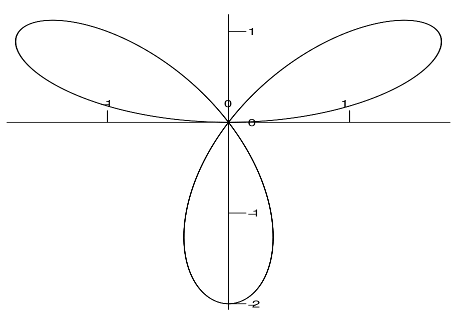
$$r = 2\sin 3\theta.$$

Solution
Here we use

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

From the picture below, we find that we sweep out the area when $\theta = 0 \to \frac{\pi}{3}$, so these are the limits of integration. Thus,

$$A = \frac{1}{2}\int_{0}^{\frac{\pi}{3}} (2\sin 3\theta)^2\, d\theta = \frac{\pi}{3}$$



4. Find the area of the following:

(i) inside $r = 2 + 2\sin\theta$,
(ii) inside the outer loop and outside the inner loop of $r = 1 - 2\sin\theta$,
(iii) outside $r = \cos 2\theta$ and inside $r = \sin 2\theta$ on $\left[0, \frac{\pi}{2}\right]$.

Solutions

(i) $r = 2 + 2\sin\theta$ The picture is above

$$A = \frac{1}{2}\int_0^{2\pi} (2 + 2\sin\theta)^2\, d\,\theta = 6\pi.$$

(ii) inside the outer loop and outside the inner loop of $r = 1 - 2\sin\theta$,

$$InnerLoop \quad \frac{2}{2}\int_{\pi/6}^{\pi/2} (1 - 2\sin\theta)^2\, d\,\theta = \pi - \frac{3\sqrt{3}}{2}$$

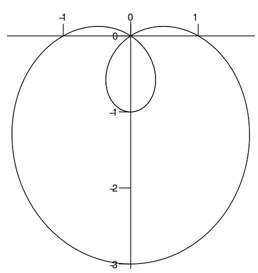
$$OuterLoop \quad \frac{2}{2}\int_{5\pi/6}^{3\pi/2} (1 - 2\sin\theta)^2\, d\,\theta = 2\pi + \frac{3\sqrt{3}}{2}$$

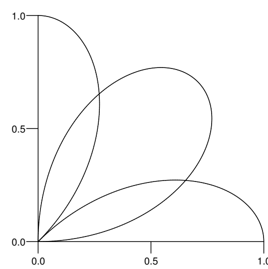
$$A = 2\pi + \frac{3\sqrt{3}}{2} - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$$

(iii) outside $r = \cos 2\theta$ and inside $r = \sin 2\theta$ on $\left[0, \frac{\pi}{2}\right]$.

In the first quadrant, the curves intersect at $\theta = \pi/8$ and sweeps out half the area between $\theta = \pi/8$ and $\theta = \pi/4$. The area is given by

$$A = \frac{2}{2}\int_{\pi/8}^{\pi/4} \sin^2 2\theta - \cos^2 2\theta\, d\,\theta = \frac{1}{4}.$$

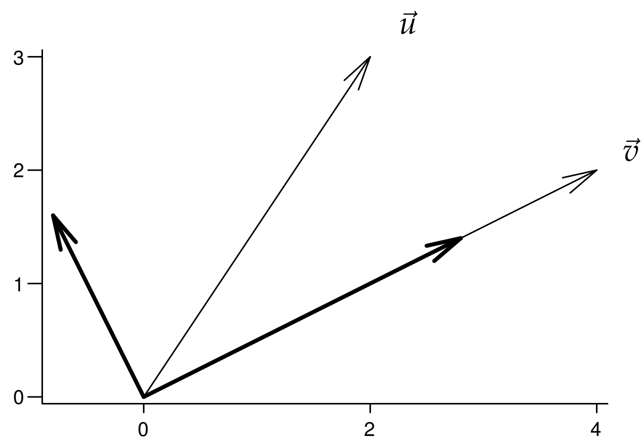




Graphs for 4 (ii) and 4 (iii)

5. Find the projection of the vector $\vec{u}$ onto $\vec{v}$ where $\vec{u} =< 2, 3 >$, and $\vec{v} =< 4, 2 >$. Sketch both vectors, the projected vector and the orthogonal complement.

In the graph, the vectors $\vec{u}$ and $\vec{v}$ are shown



$$\vec{u}\cdot\vec{v} = 8+6 = 14, \quad \vec{v}\cdot\vec{v} = 16+4 = 20,$$
$$proj_{\,\vec{v}}\,\vec{u} \;=\; \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\,\vec{v} = \frac{7}{10} < 4, 2 >$$

The orthogonal complement is given by

$$\vec{u} - proj_{\,\vec{v}}\,\vec{u} =< 2, 3 > -\frac{7}{10}\; < 4, 2 > = \;\left\langle -\frac{4}{5}, \frac{8}{5} \right\rangle.$$

6. Find the area of the triangle whose vertices are located at the points $P(1, 1, 1)$, $Q(2, 4, 6)$ and $R(-2, 3, 7)$.

Here, we construct the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. These are given by

$$\overrightarrow{u} = \overrightarrow{PQ} =< 1, 3, 5 >, \quad \overrightarrow{v} = \overrightarrow{PR} =< -3, 2, 6 > .$$

The cross product

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} i & j & k \\ 1 & 3 & 5 \\ -3 & 2 & 6 \end{vmatrix} =< 8, -21, 11 >,$$

and so the area is given by

$$A = \frac{1}{2}\,\|\overrightarrow{u} \times \overrightarrow{v}\| = \frac{1}{2}\sqrt{8^2 + 21^2 + 11^2} = \frac{\sqrt{626}}{2}.$$

7. (i) Find the equation of the plane that contains the vector $< 1, 2, 4 >$ and the points $(1, 1, 1)$ and $(-2, 3, 7)$.
   (ii) Find the equation of the plane that contains the points $(1, 3, 5)$, $(2, -1, 2)$ and $(0, 4, 6)$.

(i) We first construct a vector between the two points, this is $< -3,2,6 >$. Next, cross the two vectors

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 4 \\ -3 & 2 & 6 \end{vmatrix} = < 4, -18, 8 > .$$

The equation of the plane is given by

$$2(x-1) - 9(y-1) + 4(z-1) = 0.$$

(ii) Label the three points $P(1,3,5)$, $Q(2,-1,2)$ and $R(0,4,6)$. find two vectors that connects two pairs, i.e. $\overrightarrow{PQ} =< 1,-4,-3 >$ and $\overrightarrow{PR} =< -1,1,1 >$. The cross product will give the normal

$$\overrightarrow{n} = \begin{vmatrix} i & j & k \\ 1 & -4 & -3 \\ -1 & 1 & 1 \end{vmatrix} = < -1,2,-3 > .$$

The equation of the plane is given by

$$(x-1) - 2(y-3) + 3(z-5) = 0.$$

8. (i) Find the equation of the line that passes through the points $(1,2,4)$ and $(-2,3,7)$.
   (ii) Find the equation of the line perpendicular to the plane $x + 2y - 3z = 6$ passing through the point $(1,-1,3)$.

(i) The line will follow the vector $< -3,1,3 >$ so the equation of the line is

$$x = 1 - 3t, \quad y = 2 + t, \quad z = 4 + 3t.$$

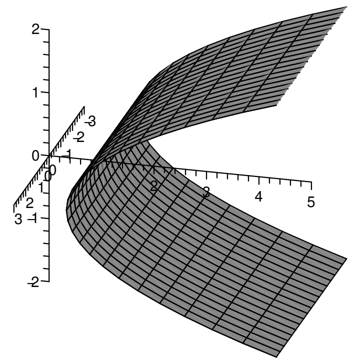
(ii) The line will follow the normal vector $< 1,2,-3 >$ so the equation of the line is

$$x = 1 + t, \quad y = -1 + 2t, \quad z = 3 - 3t.$$

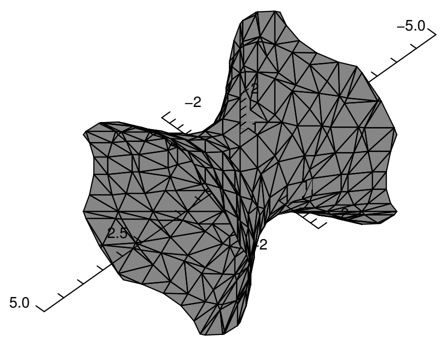
9. Sketch and name the following surfaces

$$(i)\ \ y - z^2 = 1, \qquad (ii)\ \ -x^2 + y^2 + z^2 = 1, \qquad (iii)\ \ x^2 - y + z^2 = 0.$$

(i) parabolic cylinder



(ii) hyperboloid of 1 sheet



(iii) paraboloid