# Robustness of Learning in Games With Heterogeneous Players

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***Abstract*—We consider stochastic learning dynamics in games and present a novel notion of robustness to heterogeneous players for a stochastically stable action profile. A standard assumption in these dynamics is that all the players are homogeneous, and their decision strategies can be modeled as perturbed versions of myopic best or better response strategies. We relax this assumption and propose a robustness criteria, which characterizes a stochastically stable action profile as robust to heterogeneous behaviors if a small fraction of heterogeneous players cannot alter the long-run behavior of the rest of the population. In particular, we consider confused players who randomly update their actions, stubborn players who never update their actions, and strategic players who attempt to manipulate the population behavior. We establish that radius–coradius based analysis can provide valuable insights into the robustness properties of stochastic learning dynamics for various game settings. We derive sufficient conditions for a stochastically stable profile to be robust to a confused, stubborn, or strategic player and elaborate these conditions through carefully designed examples. Then we explore the role of network structure in our proposed notion of robustness by considering graphical coordination games and identifying network topologies in which a single heterogeneous player is sufficient to alter the population's behavior. Our results will provide foundations for future research on designing networked systems that are robust to players with heterogeneous decision strategies.**

***Index Terms*—Game theory, heterogeneous agents, Markov processes, stochastic systems.**

## I. Introduction

An IMPORTANT objective in evolutionary game theory is to understand how collective behaviors evolve when independent players with bounded rationality repeatedly interact with each other [1]–[3]. A variety of learning behaviors have been presented in the literature that can be classified as variations of imitation or playing best or better response to the actions of other players (see [4]–[6] for a detailed account of various behavioral rules). We consider stochastic learning dynamics in which players update their actions according to learning behavior such as best/better response but make errors on rare occasions for exploring the action space. These dynamics are popular because they assume bounded rationality and have equilibrium selection properties. Moreover, stochastic learning dynamics have applications in designing multiagent systems in engineering applications as discussed in [7]–[11]. An example of stochastic learning dynamics is log-linear learning (LLL), in which the probability of selecting an action is proportional to its utility [2] and [12].

An important assumption in the standard setup of these dynamics is that all the players are homogeneous in the sense that they play myopic best or better response with a high probability [1], [13], [14]. We claim that this assumption can be overly restrictive for population settings comprising a large number of players. To establish our claim, we relax this assumption and ask the following question: If a small number of heterogeneous players, whose decision strategies cannot be modeled as noisy best response, are included in the population, what will be the impact on the long-run behavior of the rest of the population? In particular, we consider the following three fundamental behaviors for the heterogeneous players:

1) confused player who updates his actions uniformly at random;
2) stubborn player who never updates his action;
3) strategic player who is not myopic and can adjust his influence on players to manipulate the population's behavior to his advantage.

To quantify the impact of heterogeneous players, we present a novel notion of *robustness of stochastically stable action profiles*. We say that a stochastically stable profile is robust to a particular type of heterogeneous behavior if replacing a subset of players in the population with heterogeneous players of that type does not alter the long-run behavior of the rest of the population. Using radius–coradius analysis from [15] and [16], we explicitly derive scenarios in which even a single heterogeneous player can alter the entire population's behavior. The fact that even a single heterogeneous player can impact the global population behavior provides a strong motivation for a detailed analysis of

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stochastic learning dynamics in various heterogeneous settings. Initial results were presented in [17] in which we proposed the notion of robustness and analyzed the setup with confused and stubborn players.

Robustness analysis of stochastically stable profiles is an important research topic in the literature on learning in games. For instance, in [18] and [19], the impacts of slowly varying environments and noisy measurements on the stochastically stable behavior under LLL were analyzed. In [20], the authors investigated the robustness of stochastically stable action profiles against specific structural properties of different stochastic learning dynamics such as player revision protocols or tie-breaking rules. In [21], it was shown that the stochastically stable equilibria under the imitation dynamics of [13] were not robust to the player interaction patterns. However, we are interested in scenarios in which a small number of heterogeneous players with different decision behaviors are included in a population of myopic players, and our objective is to analyze the robustness of the stochastically stable behavior of the rest of the population.

Heterogeneity in various aspects of decision rules in stochastic learning dynamics has been considered in the literature. In [22], a notion of degree of rationality was introduced based on the levels of iterative reasoning that a player can process for developing conjectures about other players. Players with a higher level of rationality in [22], which were termed as clever agents in [23], could incorporate sophisticated models for opponents' behavior and could best respond to these sophisticated models. A similar setup was considered in [24], in which one rational player was included in the population who knew that all the other players were myopic planners, and could plan over future to manipulate the population's behavior. These works provide a motivation for our definition of a strategic player but do not consider confused or stubborn players. Moreover, the results in [23] were presented for Young's bargaining model [25] and the analysis in [24] was for the setup in which myopic players follow fictitious play with limited memory [26]. In [27], the author analyzed the impact of heterogeneous behaviors on an asymmetry property, which was presented in [28], in coordination games. Similarly, the authors in [29] analyzed coordination games in which players were heterogeneous with respect to their payoffs and preferences. In [30]–[32], the impact of adversarial players on population behavior under various information settings was analyzed for graphical coordination games over generalized ring networks.

*Contribution Statement:* We present a framework for analyzing the robustness of stochastically stable behaviors against heterogeneous players for general normal form games for a class of noisy best response dynamics. Our framework for analyzing the robustness of stochastically stable profiles is based on radius–coradius (Rd-CR) criteria as presented in [16], which is an important contribution since our framework is applicable to a class of finite normal form games in which stochastically stable profiles satisfy this criterion. Rd-CR result was initially presented in [15] for noisy best response dynamics with mistake model and was later extended to a generalized version of LLL in [16]. Therefore, although we consider standard LLL as presented in [2], our results can easily be extended to the class of dynamics discussed in [15] and [16]. The article can be divided into two parts. In Section III, we present qualitative conditions for scenarios in which a single player of a particular type can change the behavior of the rest of the population. These results are supported by carefully designed examples in which the impacts of our conditions are highlighted and discussed for deeper insights. In Section IV, we consider graphical coordination games, which is one of the most important game setups and has been studied extensively, particularly in the context of innovation diffusion in social networks [1], [33]–[35].

Coordination games have been a focus of existing literature on stochastic dynamics with heterogeneous players such as [29] and [36]–[39]. Similarly, some of the previous works on robustness have focused entirely on coordination games for specific networks like random networks [24] and generalized ring networks [31] and [30]. We consider graphical coordination games in a population setting for several important network topologies such as path graph, ring graph, 2-D grid, and wheel network and determine whether these topologies are robust to heterogeneous decision strategies or not. We analyze the robustness of these topologies and identify which of these topologies are robust to a confused, stubborn, or strategic player. We also consider the setup in which, at each decision time, the network is generated randomly according to the Erdős–Rényi (ER) graph model.

*Outline:* Section II defines notations and provides the related background discussion on stochastic learning dynamics and resistance tree analysis. Section III presents our notion of robustness and derives sufficient conditions using the radius–coradius result. Section IV considers graphical coordination games over networks and analyzes the robustness of various network topologies. Finally, Section V concludes the article.

## II. Background

In this section, we define the notations used throughout the article and present the background material on stochastic learning dynamics.

### A. Notation

The distance between any two vectors $u$ and $v$ in $\mathbb{R}^n$ is the Hamming distance

$$d(u, v) = |\{p \,|\, u_p \neq v_p\}|$$

where $u_p$ and $v_p$ are the $p^{th}$ elements in vectors $u$ and $v$, respectively. We consider finite state Markov chains with state space $S$. Let $P_0$ be the transition matrix of an unperturbed Markov chain and let $P_\epsilon$ represent a family of perturbed Markov chains, where $\epsilon$ is the perturbation parameter. We will refer to a Markov chain by its transition matrix. A perturbed Markov chain $P_\epsilon$ is a regular perturbation of an unperturbed chain $P_0$ if the following properties are satisfied.

1) $P_\epsilon$ is ergodic for sufficiently small perturbations $\epsilon$.
2) For any state pair $x$ and $y$ in $S$, $\lim_{\epsilon \to 0} P_\epsilon(x, y) = P_0(x, y)$, where $P_\epsilon(x, y)$ and $P_0(x, y)$ are the transition probabilities from $x$ to $y$ for perturbed and unperturbed Markov chains, respectively.

3) For any state pair $x$ and $y$ in $S$ and for any $\epsilon > 0$, a resistance function $R(x, y)$ exists such that

$$P_\epsilon(x, y) > 0 \Rightarrow 0 < \lim_{\epsilon \to 0} \frac{P_\epsilon(x, y)}{\epsilon^{R(x,y)}} < \infty.$$

Here, $R(x, y)$ is the resistance in transition from $x$ to $y$.

A state $x$ in $S$ is stochastically stable if and only if $\lim_{\epsilon \to 0} \pi_\epsilon(x) > 0$, where $\pi_\epsilon$ is the stationary distribution of $P_\epsilon$. A path $\omega^S$ is a sequence of distinct states $(\omega_1, \omega_2, \ldots, \omega_k)$ such that $\omega_i \in S$, $P_\epsilon(\omega_i, \omega_{i+1}) > 0$, and $d(\omega_i, \omega_{i+1}) = 1$ for all $i \in \{1, 2, \ldots, k-1\}$. We will drop the superscript $S$ when the set in which the path exists is clear from the context. We denote a path between any two states $x$ and $y$ in $S$ as $\omega^S_{x,y}$ such that $\omega_1 = x, \omega_k = y$. Given a subset $A$ of $S$, a path $\omega$ belongs to $A$ if $\omega_i \in A$ for all $i \in \{1, 2, \ldots, |\omega|\}$, where $|\omega|$ is the path length.

The set of all paths between $x$ and $y$ is $\Omega(x, y)$. For any two sets $A$ and $B$ in $S$, $\Omega(A, B)$ is the set of all paths starting from states in $A$ and terminating on states in $B$, i.e.,

$$\Omega(A, B) = \{\omega_{x,y} \text{ for all } x \in A \text{ and } y \in B\}.$$

The resistance of a path $\omega$ is

$$R_{\text{path}}(\omega) = \sum_{i=1}^{|\omega|-1} R(\omega_i, \omega_{i+1}). \tag{1}$$

For any state pair $x$ and $y$ such that either $P_\epsilon(x, y) = 0$ or there exist multiple paths from $x$ to $y$, the resistance from $x$ to $y$ is

$$R_{\min}(x, y) = \min\{R_{\text{path}}(\omega_{x,y}) \,\forall\, \omega_{x,y} \in \Omega(x, y)\}. \tag{2}$$

Thus, $R_{\min}(x, y)$ is the minimum resistance between $x$ and $y$. In a regularly perturbed Markov chain, there always exists a bounded resistance path between any two states in $S$.

### B. Game Setup

We consider a standard setup of normal form games with a finite set of players $S_p = \{1, 2, \ldots, n\}$ such that each player $i$ has a finite set of actions $A_i = \{1, 2, \ldots, m_i\}$ and has preferences over the set of joint action profiles $\mathcal{A}$ defined through utility functions $U_i : \mathcal{A} \to \mathbb{R}$, where $\mathcal{A} = A_1 \times A_2 \times \cdots \times A_n$. Given a joint action profile $a \in \mathcal{A}$, we represent it with respect to some player $i$ as $a = (a_i, a_{-i})$, where $a_i$ is the action of player $i$ in $a$ and $a_{-i}$ represents the actions of all the other players. Here, $a_{-i}$ belongs to the set $\mathcal{A}_{-i} = A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n$. We also represent an action profile $a$ with respect to a subset $H \subset S_p$ as $a = (a_H, a_{-H})$, where $a_H$ and $a_{-H}$ are the actions of the players in the sets $H$ and $S_p \backslash H$, respectively. Given any $a_{-i}$ in $\mathcal{A}_{-i}$, the *best response set* of $i$ is

$$B_i(a_{-i}) = \{a_i \in A_i \,|\, U_i(a_i, a_{-i}) \geq U_i(a'_i, a_{-i}) \,\forall\, a'_i \in A_i\}.$$

An action profile $\alpha^* = (a^*_1, a^*_2, \ldots, a^*_n)$ is a Nash equilibrium (NE) if and only if $a^*_i$ belongs to the best response set $B_i(a^*_{-i})$ for every $i$. Thus, an action profile is a NE if no player has any incentive to unilaterally change his action. The neighborhood of an action profile $a$ is

$$\mathcal{N}(a) = \{a' \in \mathcal{A} \,|\, d(a, a') = 1\} \tag{3}$$

i.e., $\mathcal{N}(a)$ is the set of all action profiles in which exactly one player is playing an action that is different from his action in $a$.

The player-specific neighborhood of $a$ is

$$\mathcal{N}_i(a) = \{a' \in \mathcal{A} \,|\, a'_i \in A_i \backslash a_i \text{ and } a'_{-i} = a_{-i}\}. \tag{4}$$

A game is a potential game if there exists a global potential function $\phi : \mathcal{A} \to \mathbb{R}$ such that for any two action profiles $a = (a_i, a_{-i})$ and $a' = (a'_i, a_{-i})$ that differ in the action of one player only, the following condition holds:

$$U_i(a) - U_i(a') = \phi(a) - \phi(a').$$

Thus, a game is a potential game if local utilities of all the players are aligned with some global potential function.

Let $\omega^{\mathcal{A}}_{a,a'} = (\omega_1, \ldots, \omega_k)$ be a path from action profile $a$ to $a'$ having total resistance as defined in (1). Then, the contribution of a player $h \in S_p$ in the resistance of this path is

$$R_h(\omega^{\mathcal{A}}_{a,a'}) = \sum_{j \in I_h(\omega^{\mathcal{A}}_{a,a'})} R(\omega_j, \omega_{j+1}), \tag{5}$$

where

$$I_h(\omega^{\mathcal{A}}_{a,a'}) = \{j \in \{1, \ldots, k-1\} \,|\, w_j = (a_h, a_{-h}) \text{ and } w_{j+1} = (a'_h, a_{-h}) \text{ for any } a_h \text{ and } a'_h \text{ in } A_h\}$$

i.e., $I_h(\omega^{\mathcal{A}}_{a,a'})$ is the set of indices for the path $\omega^{\mathcal{A}}_{a,a'}$ that correspond to player $h$ updating his action. Here, $A_h$ is the action set of player $h$.

### C. Stochastic Learning Dynamics

In stochastic learning dynamics, each player uses a combination of exploration and exploitation for selecting actions. We consider the setup in which players update their actions at discrete time instances. In this setup, a player assumes that all the other players repeat their actions from the previous time step. Then, he selects a noisy version of best/better response to the assumed actions of the other players. In the noisy best response dynamics, a player plays an action from his best response set with high probability. However, with a small but nonzero probability, he randomly selects an action from his action set. In this work, we will consider LLL as a representative dynamics from the class of stochastic learning dynamics.

*Log-linear-learning*, as presented in [2], is an example of noisybest response dynamics. Let $a(t-1) = (a_i(t-1), a_{-i}(t-1))$ be the action profile at time $t-1$. Then the steps involved in decision making at $t$ are as follows.

1) A player $i$ is randomly selected from $S_p$ such that every player has a nonzero probability of being selected.
2) The other players repeat their actions, i.e., $a_{-i}(t) = a_{-i}(t-1)$.
3) Player $i$ selects an action $a_i$ from $A_i$ with probability

$$p^{\text{LLL}}_i(a_i, a_{-i}(t)) = \frac{e^{-\frac{1}{T}[U_i(a^*_i, a_{-i}(t)) - U_i(a_i, a_{-i})]}}{Z_i(a_{-i})},$$
$$Z_i(a_{-i}) = \sum_{\bar{a}_i \in A_i} e^{-\frac{1}{T}[U_i(a^*_i, a_{-i}) - U_i(\bar{a}_i, a_{-i})]} \tag{6}$$

where $Z_i(a_{-i})$ is a normalizing constant and $a^*_i$ is an action from the best response set $B_i(a_{-i}(t))$.

In (6), parameter $T$ determines the level of noise in decision making. When $T$ approaches infinity, players randomly select actions from their action sets with uniform distribution. However, as $T$ approaches zero, players select actions from their best response sets with a probability approaching one.

In LLL, action profile at time $t$ only depends on the action profile at time $t-1$ and decision at time $t$. Therefore, the evolution of action profiles under LLL can be modeled as a Markov chain over the set of joint action profiles $\mathcal{A}$. Let $P^{\text{LLL}}$ be the transition matrix for the Markov chain induced by LLL. The transition probability between any two action profiles $a$ and $a'$ is

$$P^{\text{LLL}}(a,a') = \frac{1}{n}\begin{cases} 0 & d(a,a')>1,\\ p_i^{\text{LLL}}(a_i,' a_{-i}) & a' \in N_i(a). \end{cases}$$

Here, the resistance in transition from $a$ to $a'$ is

$$R^{\text{LLL}}(a,a') = \begin{cases} U_i(a_i^*, a_{-i}) - U_i(a') & a_i' \neq a_i, a_{-i}' = a_{-i},\\ \infty & \text{otherwise} \end{cases} \tag{7}$$

where $a_i^* \in B_i(a_{-i})$. We will consider LLL for our analysis in this work, and, therefore, the resistance function will be

$$R(a,a') = R^{\text{LLL}}(a,a')$$

throughout the article.

***1) Radius–Coradius Analysis:*** Resistance tree analysis, as presented in [1], completely characterizes stochastic stability for a wide class of stochastic learning dynamics. However, computing stochastically stable states through this approach is computationally intensive since it requires evaluating resistances of all the possible trees rooted at all the states. To address this issue, an alternative approach was presented in [15] for a particular noisy best response dynamics in which players select a noisy action with uniform distribution. In [16], the approach was extended to a generalized version of LLL, in which the probability of a noisy action is proportional to its utility. In this approach, two quantities, namely radius (Rd) and coradius (CR) are computed for states that are candidates for being stochastically stable. Then a simple comparison between Rd and CR provides a *sufficient condition* for stochastic stability.

Next, we define the terms involved in radius–coradius (Rd-CR) based analysis from [16] and a brief discussion on its significance.

*Definition II.1* Consider a Markov chain over the set of joint action profiles $\mathcal{A}$.

1) The *basin of attraction* of an action profile $a$, $\text{BA}(a)$, is the set of all action profiles $a'$ in $\mathcal{A}$ such that there exists a path of zero resistance from $a'$ to $a$.
2) The *recurrent class* of a profile $a$, $L(a)$, is the set of all profiles $a'$ such that $a$ and $a'$ are connected to each other through paths of zero resistances.
3) The *radius* of an action profile $a$ is
$$\text{Rd}(a) = \min\{R_{\min}(a,a') \,|\, a' \in \text{BA}^c(a)\} \tag{8}$$
where $\text{BA}^c(a) = \mathcal{A}\backslash\text{BA}(a)$ is the complement of the set $\text{BA}(a)$ and $R_{\min}(a,a')$ is the resistance as defined in (2). Thus, $\text{Rd}(a)$ is the minimum resistance of leaving $\text{BA}(a)$.
4) The *coradius* of an action profile $a$ is
$$\text{CR}(a) = \max\{R_{\min}(a,' a) \,|\, a' \in \text{BA}^c(a)\}. \tag{9}$$

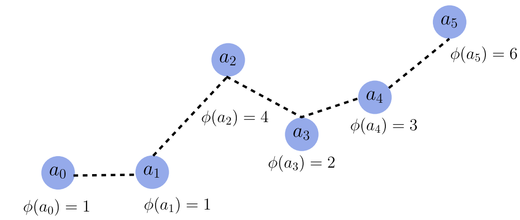


Fig. 1. Induced Markov chain under LLL for an identical interest game with five players and two actions each $\{0,1\}$. The game is a potential game and $\phi(a_i)$ is the potential of the state $a_i$.

Based on the definitions in (8) and (9), the radius of $a$ is a measure of how easy it is to leave $a$ and coradius of $a$ is a measure of how difficult it is to reach $a$ if the Markov chain is initialized randomly at a profile outside of $\text{BA}(a)$. Given any subset $B$ of $\mathcal{A}$, the definitions of radius and coradius can be extended to $B$ as follows:

$$\begin{aligned} \text{Rd}(B) &= \min\{\text{Rd}(a) \,|\, a \in B\}, \text{ and}\\ \text{CR}(B) &= \min\{\text{CR}(a) \,|\, a \in B\}. \end{aligned} \tag{10}$$

Using the concepts of radius and coradius, the following criteria for stochastically stable states in LLL was presented in [16, Prop. 2].

*Proposition 1 ([16]):* Let $a$ be an action profile in $\mathcal{A}$ that satisfies $\text{Rd}(a) > \text{CR}(a)$. Then, stochastically stable states are exactly those in $L(a)$.

The condition in Proposition 1 is a sufficient condition for an action profile to be stochastically stable. Moreover, if an action profile $a \in \mathcal{A}$ satisfies the Rd-CR criteria, then it was also established in the proof of [16, Prop. 2] that no action profile outside of the equivalent class $L(a)$ can be stochastically stable. In this work, we will use this result extensively for verifying stochastic stability of various states.

***2) Illustrative Example:*** We present a simple example to illustrate the implications of the basin of attraction, radius, and coradius in stochastic stability analysis. The basic setup of our example is presented in Fig. 1. We consider a game with five players in which each player has two actions $\{0,1\}$. The state of the system is the number of players playing action 0. Thus, the state space is $S = \{a_0, a_1, \ldots, a_5\}$, where $a_k$ is the state with $k$ players playing 0. The links between the states correspond to valid transitions under a stochastic learning dynamics. The game is an identical interest game in which all the players have the same utility at a particular state, which leads to a potential game setup. Here, $\phi(\cdot)$ is the global payoff function, which is also a potential function for identical interest games.

We assume that the players update their actions according to LLL. Based on the payoff structure in Fig. 1, the set of NE is $\{a_2, a_5\}$. Moreover, for potential games, potential function maximizers are stochastically stable, which imply that $a_5$ is the unique stochastically stable state. We will also arrive at this solution using Rd-CR result. The one step resistance between any two states under LLL is given in (7). The resulting resistance matrix is

$$R = \begin{bmatrix} 0 & 0 & \infty & \infty & \infty & \infty \\ 3 & 3 & 0 & \infty & \infty & \infty \\ \infty & 3 & 0 & 2 & \infty & \infty \\ \infty & \infty & 0 & 2 & 1 & \infty \\ \infty & \infty & \infty & 4 & 3 & 0 \\ \infty & \infty & \infty & \infty & 3 & 0 \end{bmatrix}$$

where $R(a_i, a_j)$ is the resistance between $a_i$ and $a_j$ for $i$ and $j$ in $\{0, 1, \ldots, 5\}$. The basin of attractions of the two NEs are

$$\mathrm{BA}(a_2) = \{a_0, a_1, a_2, a_3\} \text{ and } \mathrm{BA}(a_5) = \{a_4, a_5\}.$$

Since $\phi(a_2) > \phi(a_4)$, the best response from $a_3$ is to transition to $a_2$ instead of $a_4$. Therefore, $R(a_3, a_2) = 0$, whereas $R(a_3, a_4) = \phi(a_2) - \phi(a_4) = 1$. The only path leaving $\mathrm{BA}(a_2)$ is $(a_2, a_3, a_4)$ and the only path leaving $\mathrm{BA}(a_5)$ is $(a_5, a_4, a_3)$. Therefore

$$\mathrm{Rd}(a_2) = 3 \text{ and } \mathrm{Rd}(a_5) = 7.$$

The resistance of the path $(a_5, a_4, a_3, a_2)$, which is the only path from $a_5$ to $a_2$, is 7. Similarly, the resistance of the path $(a_2, a_3, a_4, a_5)$, which is the only path from $a_2$ to $a_5$, is 3. Thus

$$\mathrm{CR}(a_2) = 7 \text{ and } \mathrm{CR}(a_5) = 3.$$

Since $\mathrm{Rd}(a_5) > \mathrm{CR}(a_5)$, state $a_5$ is the only stochastically stable state.

## III. Robustness in Games With Heterogeneous Players

In this section, we start by presenting our new notion of robustness for stochastic learning dynamics. These learning dynamics are often employed in population settings that comprise a large number of players. In such scenarios, assuming a population of homogeneous players may be overly restrictive because human populations generally have idiosyncratic individuals with strong tendencies to defy standard decision-making practices. Such individuals may or may not have an impact on the decisions of the rest of the population. Similarly, in the engineering applications of multiagent systems, some agents may be faulty or are compromised by an adversarial attack.

We assume a population in which a set of heterogeneous players has a different decision strategy than the rest of the players. Having a heterogeneous player in the population raises several interesting questions regarding the long-run behavior of the population under stochastic learning dynamics. For instance, can a small group of heterogeneous players affect the long-run behavior of the entire population, and how to quantify and analyze this impact? To investigate the impact of player heterogeneity, we consider three types of behaviors.

1) Confused player: Randomly updates his actions.
2) Stubborn player: Never updates his action.
3) Strategic player: Can update his actions strategically to alter the stochastically stable behavior of the population.

We refer to the setups with and without heterogeneous players as the heterogeneous and standard setups, respectively. Let $\Theta$ be the set of possible heterogeneous behaviors. In this work, we consider

$$\Theta = \{\mathrm{cnf}, \mathrm{stb}, \mathrm{str}\}$$

where $\mathrm{cnf}$, $\mathrm{stb}$, and $\mathrm{str}$ refer to confused, stubborn, and strategic behaviors, respectively.

*Definition III.1:* Let $\mathcal{A}_{\mathrm{ss}} \subset \mathcal{A}$ be the set of stochastically stable action profiles for a stochastic learning dynamics in the standard setup, and let $s$ belong to $\mathcal{A}_{\mathrm{ss}}$.

1) Suppose all the players in a subset $H \subset S_p$ are replaced with $\theta \in \Theta$ players, and let $\mathcal{A}_{\mathrm{ss}}^{\theta,H}$ be the set of stochastically stable action profiles in the heterogeneous setup. Then, $s = (s_H, s_{-H})$ is robust to $\theta$ players in $H$ if there exists an $s'$ in $\mathcal{A}_{\mathrm{ss}}^{\theta,H}$ such that $s'_{-H} = s_{-H}$.
2) A stochastically stable action profile $s \in S$ is robust to $\theta \in \Theta$ players if $s$ is robust to $\theta$ players in any subset $H$ of $S_p$.
3) A stochastic learning dynamics is robust to $\theta \in \Theta$ players if all stochastically stable profiles in the set $S$ are robust to $\theta$ players.

Thus, a stochastically stable action profile under the standard setup is robust to a heterogeneous behavior if replacing any subset of players $H$ with heterogeneous players of that type cannot affect the behavior of other players in the population. To analyze the impact of player heterogeneity on the long-run behavior, we thoroughly investigate the three types of heterogeneous players in the set $\Theta$.

In our analysis, we restrict our attention to LLL and a simple scenario in which a single player, say player $h$, is replaced with a heterogeneous player, i.e., $H = \{h\}$ for some $h \in S_p$. We establish that even a single heterogeneous player can significantly alter the long-run behavior of the entire population under certain conditions. Then, we present several insightful examples that provide a better understanding of the conditions in which a single heterogeneous player can change the population's behavior.

### A. Confused Player: Random Action Updates

We begin with the case of a confused player who randomly updates his action whenever given the opportunity. To keep the analysis simple, we assume that for all $s \in \mathcal{A}_{\mathrm{ss}}$, the equivalence class $L(s)$ is a singleton. The main results related to the addition of a confused player are as follows.

1) Under certain scenarios, a stochastically stable profile $s$ is not robust to a single confused player.
2) There exist scenarios in which $s$ is robust if player $h$ is confused but is not robust if some other player $h'$ is confused.
3) There exist scenarios in which an action profile $a$ that was not stochastically stable under the standard setup may become stochastically stable after replacing a player with a confused player.

*Proposition 2:* Let $s$ be a stochastically stable action profile in the standard case such that $\mathrm{Rd}(s) > \mathrm{CR}(s)$. Then, $s$ is robust to a confused player $h \in S_p$ if

$$\mathrm{Rd}(L^{\mathrm{cnf}}(s,h)) > \mathrm{CR}(L^{\mathrm{cnf}}(s,h))$$

$U_c(c_1) = 10$

| | $b_1$ | $b_2$ | $b_3$ |
|---|---|---|---|
| $a_1$ | 10 | 6 | 7 |
| $a_2$ | 6 | 0 | 0 |
| $a_3$ | 0 | 0 | 9 |

$U_c(c_1) = 5$

| | $b_1$ | $b_2$ | $b_3$ |
|---|---|---|---|
| $a_1$ | 10 | 0 | 0 |
| $a_2$ | 0 | 1 | 1 |
| $a_3$ | 0 | 1 | 1 |

Fig. 2. Game with three players $\{a, b, c\}$. Players $a$ and $b$ select the rows and columns of the matrices and player $c$ selects between left and right matrix. Players $a$ and $b$ have identical interest with utilities given in the matrices. The utility of $c_1$ is 10 and $c_2$ is 5, i.e., $c_1$ is the dominant action for $c$.

*where*

$$L^{\mathrm{cnf}}(s,h) = \bigcup_{s' \in N_h(s)} s,' \tag{11}$$

*and $N_h(s)$ is the neighborhood of $s$ defined in (4).*

Since a confused player randomly updates his actions, all the transitions that involve $h$ have zero resistance, i.e.,

$$R^{\mathrm{cnf}}(a,a') = \begin{cases} 0 & a' \in N_h(a) \\ R(a,a') & a' \notin N_h(a). \end{cases}$$

Thus, by replacing $h$ with a confused player, transitions between $s$ and any member of the set $N_h(s)$ have zero resistance, which can change the radius and coradius of $s_{-h}$. Therefore, the Rd-CR based sufficient condition for stochastic stability will be defined on the set $L^{\mathrm{cnf}}$ instead of a single state in the case of a confused player.

We illustrate the implications of the result in Proposition 2 through an example for which the matrix form is presented in Fig. 2. Consider a game with three players $S_p = \{a, b, c\}$. In the standard setup, players $a$ and $b$ have identical interests, i.e., their utilities are identical for all action profiles. For player $c$, action $c_1$ strictly dominates $c_2$. Thus, the game has two pure NEs in the standard setup, which are $\alpha_1^* = (a_1, b_1, c_1)$ and $\alpha_2^* = (a_3, b_3, c_1)$. To check for stochastically stable states, we need to compute radius and coradius for both of the NEs. The basin of attraction of $\alpha_1^*$ contains all the states except $\alpha_2^*$. The minimum resistance path from $\alpha_1^*$ to $\mathrm{BA}^c(\alpha_1^*)$ is

$$\omega_{\alpha_1^*,\alpha_2^*} = ((a_1,b_1,c_1),\ (a_1,b_3,c_1),(a_3,b_3,c_1)). \tag{12}$$

The minimum resistance path entering $\mathrm{BA}(\alpha_1^*)$ from outside is $((a_3, b_3, c_1), (a_1, b_3, c_1))$. Therefore

$$\mathrm{Rd}(\alpha_1^*) = 3 \text{ and } \mathrm{CR}(\alpha_1^*) = 2.$$

Since $\mathrm{Rd}(\alpha_1^*) > \mathrm{CR}(\alpha_1^*)$, equilibrium $\alpha_1^*$ is stochastically stable based on the Rd-CR criteria.

Next, we study the impact of replacing one of the players with a confused player. We will have $s = (a_1, b_1, c_1)$ in the two cases below.

*Case 1: Player c is confused.*

If player $c$ is confused, i.e., $h = c$, then the transitions between the entries from the left matrix to the right matrix have zero resistance. To verify the robustness of the stochastically stable state $s$ with $c$ as confused player, we apply the result of Proposition 2. We start with the set $L^{\mathrm{cnf}}(s,c)$. The neighborhood $N_c(s)$ has

| | $c_1$: $b_1$ | $c_1$: $b_2$ | $c_2$: $b_1$ | $c_2$: $b_2$ |
|---|---|---|---|---|
| $a_1$ | 10, 10, 10 | 6, 5, 3 | 10, 2, 6 | 6, 5, 5 |
| $a_2$ | 7, 5, 3 | 4, 5, 6 | 7, 6, 5 | 8, 8, 8 |

Fig. 3. Matrix form representation of a three-player game with $S_p = \{a, b, c\}$. Player $a$ selects rows, player $b$ selects columns, and player $c$ selects left or right matrix.

one member only, which is $(a_1, b_1, c_2)$. Thus

$$L^{\mathrm{cnf}}(s,c) = \{(a_1,b_1,c_1),\ (a_1,b_1,c_2)\}.$$

To compute the radius and coradius of $L^{\mathrm{cnf}}(s,c)$, we observe that the minimum resistance path leaving $L^{\mathrm{cnf}}(s,c)$ is still $\omega_{\alpha_1^*,\alpha_2^*}$ in (12). Similarly, the easiest access to $L^{\mathrm{cnf}}(s,c)$ is also through $(a_1, b_1, c_1)$. Therefore

$$\mathrm{Rd}(L^{\mathrm{cnf}}(s,c)) = 3 \text{ and } \mathrm{CR}(L^{\mathrm{cnf}}(s,c)) = 2.$$

Since $\mathrm{Rd}(L^{\mathrm{cnf}}(s,c)) > \mathrm{CR}(L^{\mathrm{cnf}}(s,c))$, the set $L^{\mathrm{cnf}}(s,c)$ is stochastically stable and the action profile $s$ is robust if $c$ is confused.

*Case 2: Player b is confused.*

If player $b$ is confused, then

$$L^{\mathrm{cnf}}(s,b) = \{(a_1,b_1,c_1),(a_1,b_2,c_1),(a_1,b_3,c_1)\}.$$

In this case, player $c$ will still play action $c_1$ with high probability. A minimum resistance path from $L^{\mathrm{cnf}}(s,b)$ to $\alpha_2^*$ will be $((a_1,b_3,c_1),(a_3,b_3,c_1))$, which has a resistance of zero. Similarly, a minimum resistance path from $\alpha_2^*$ to $L^{\mathrm{cnf}}(s,b)$ will be $((a_3,b_3,c_1),(a_3,b_1,c_1),(a_1,b_1,c1))$, which again has a resistance of zero. Therefore, in the case with $b$ as confused player

$$\mathcal{A}_{\mathrm{ss}}^{\mathrm{cnf},b} = L^{\mathrm{cnf}}(s,b) \cup \{(a_3,b_1,c_1),(a_3,b_2,c_1),(a_3,b_3,c_1)\}.$$

Since $s_{-b} = (a_1, c_1)$ belongs to $\mathcal{A}_{\mathrm{ss}}^{\mathrm{cnf},b}$, we say that $s$ is robust if $b$ is confused. However, the size of the set of stochastically stable strategies $\mathcal{A}_{\mathrm{ss}}^{\mathrm{cnf},b}$ has significantly increased.

*Proposition 3:* Let $s$ be an action profile that is not stochastically stable under the standard setup. Then, there can exist a player $h \in S_p$ such that replacing it with a confused player can result in $L^{\mathrm{cnf}}(s,h)$ to become stochastically stable.

*Proof:* By replacing player $h$ with a confused player, $\mathrm{Rd}(L^{\mathrm{cnf}}(s,h))$ in the heterogeneous setup cannot be greater that $\mathrm{Rd}(s)$ under the standard setup because a confused player can only reduce the resistance of a path. Therefore, for $L^{\mathrm{cnf}}(s,h)$ to be stochastically stable, we need to show that it is possible to reduce the coradius such that the Rd-CR criteria is satisfied. We establish our claim through a simple example.

Consider a three-player game with each player having two actions. The matrix form representation of the game is given in Fig. 3.

The game has two NEs, which are $\alpha_1^* = (a_1, b_1, c_1)$ and $a_2^* = (a_2, b_2, c_2)$. We assume that the players are updating their actions according to LLL. The basin of attraction of $\alpha_1^*$ includes all the action profiles except $(a_2, b_2, c_2)$ and $(a_1, b_2, c_2)$. The paths that determine the $\mathrm{Rd}(\alpha_1^*)$ and $\mathrm{CR}(\alpha_1^*)$ are

$$\omega_{\alpha_1^*,\alpha_2^*} = ((a_1, b_1, c_1), (a_2, b_1, c_1), (a_2, b_1, c_2), (a_2, b_2, c_2)),$$

$$\omega_{\alpha_2^*,\alpha_1^*} = ((a_2, b_2, c_2), (a_2, b_1, c_2), (a_1, b_1, c_2), (a_1, b_1, c_1)).$$

Then, the radius and coradius of $\alpha_1^*$ are

$$\mathrm{Rd}(\alpha_1^*) = 3 \text{ and } \mathrm{CR}(\alpha_1^*) = 2$$

which implies that $\alpha_1^*$ is stochastically stable and $\alpha_2^*$ is not stochastically stable. For $\alpha_2^*, \mathrm{Rd}(\alpha_2^*) = R_{\text{path}}(\omega_{\alpha_2^*,\alpha_1^*}) = 2$ and $\mathrm{CR}(\alpha_2^*) = R_{\text{path}}(\omega_{\alpha_1^*,\alpha_2^*}) = 3$.

Now suppose that player $a$ is replaced with a confused player, i.e., $h = a$. Then

$$L^{\text{cnf}}(\alpha_1^*, a) = \{(a_1, b_1, c_1), (a_2, b_1, c_1)\}, \text{ and}$$

$$L^{\text{cnf}}(\alpha_2^*, a) = \{(a_2, b_2, c_2), (a_1, b_2, c_2)\}.$$

The radius and coradius of $L^{\text{cnf}}(\alpha_2^*, a)$ under the heterogeneous setup are the following:

$$\mathrm{Rd}(L^{\text{cnf}}(\alpha_2^*, a)) = 2,\ \mathrm{CR}(L^{\text{cnf}}(\alpha_2^*, a)) = 0.$$

Thus, replacing $a$ with a heterogeneous player reduced the coradius of $\alpha_2^*$ and resulted in $L^{\text{cnf}}(\alpha_2^*, a)$ to be stochastically stable, which verifies the proposition statement. ∎

The important takeaway from this section is that the presence of even a single confused player can significantly alter the long-term behavior of the entire population.

### B. Stubborn Player: No Action Updates

A stubborn player is the one who never updates his action irrespective of the number of revision opportunities he receives. Consequently, having a stubborn player restricts the state space over which the Markov chain induced by a stochastic learning dynamics evolves. Let $\mathcal{A}$ be the set of joint action profiles in the standard setup. Replacing player $h$ with a stubborn player restricts $\mathcal{A}$ to $\mathcal{A}^{\text{stb}}(h)$, where

$$\mathcal{A}^{\text{stb}}(h) = \{a \in \mathcal{A} \mid a = (a^{\text{stb}}, a_{-h}) \text{ for all } a_{-h} \in \mathcal{A}_{-h}\}.$$

Here, $a^{\text{stb}}$ is the action of the stubborn player. Because of a stubborn player, the resistance between action profiles is updated as follows:

$$R^{\text{stb}}(a, a') = \begin{cases} R(a, a') & \text{if } a_h = a'_h = a^{\text{stb}}, \\ \infty & \text{otherwise.} \end{cases}$$

*Proposition 4:* Suppose player $h$ is replaced with a stubborn player having action $a^{\text{stb}}$ and let $s = (s_h, s_{-h})$ be a stochastically stable action profile under the standard setup.

1) There exist scenarios in which $s$ is not robust to a stubborn player $h$.
2) Even if $s_h = a^{\text{stb}}$, there exist conditions in which $s = (s_h, s_{-h})$ is not robust to a stubborn player $h$.

*Proof:* To prove the statement, we present two conditions in which $s$ will not be robust to the addition of a stubborn player, i.e., we will provide sufficient conditions in which $(a^{\text{stb}}, s_{-h})$ will not be stochastically stable.

*Condition 1:* Suppose $s_h \neq a^{\text{stb}}$, i.e., player $h$ had a different action in the stable profile under the standard setup. If there exists an action profile $a = (a^{\text{stb}}, a_{-h})$ such that the Hamming distance $d(s_{-h}, a_{-h}) = 1$, $R((a^{\text{stb}}, s_{-h}), (a^{\text{stb}}, a_{-h})) = 0$, and $R_{\min}((a^{\text{stb}}, a_{-h}), (a^{\text{stb}}, s_{-h})) > 0$, then $(a^{\text{stb}}, s_{-h})$ is not stochastically stable. If this condition is satisfied, then $(a^{\text{stb}}, s_{-h})$ cannot be stochastically stable because there will be a zero resistance path from $(a^{\text{stb}}, s_{-h})$ to $(a^{\text{stb}}, a_{-h})$, whereas all the paths from $(a^{\text{stb}}, a_{-h})$ to $(a^{\text{stb}}, s_{-h})$ will have nonzero resistance. In this condition, player $h$ had an important role in the standard setup since the resistance from $(s_h, s_{-h})$ to $(a^{\text{stb}}, s_{-h})$ was large enough to keep $(s_h, s_{-h})$ stochastically stable. However, the stubborn behavior of $h$ reduced this resistance to zero, which shifted the behavior of the population away from $s_{-h}$.

*Condition 2:* Let $s_h = a^{\text{stb}}$, i.e., $s = (a^{\text{stb}}, s_{-h})$ be stochastically stable in the standard setup. Even in this case, we cannot guarantee that $(a^{\text{stb}}, s_{-h})$ will be stochastically stable under the heterogeneous setup. Consider the matrix game in Fig. 4 with three players. The game has two NEs, which are $\alpha_1^* = (a_1, b_1, c_1)$ and $\alpha_2^* = (a_1, b_1, c_1)$. The minimum resistance paths between $\alpha_1^*$ and $\alpha_2^*$ are

$$\omega_{\alpha_1^*,\alpha_2^*} = ((a_1, b_1, c_1), (a_2, b_1, c_1), (a_2, b_1, c_2)) \text{ and}$$

$$\omega_{\alpha_2^*,\alpha_1^*} = ((a_2, b_1, c_2), (a_2, b_2, c_2), (a_2, b_2, c_1), (a_2, b_1, c_1), (a_1, b_1, c_1)).$$

The radius and coradius of $\alpha_1^*$ are

$$\mathrm{Rd}(\alpha_1^*) = R_{\text{path}}(\omega_{\alpha_1^*,\alpha_2^*}) = 3 \text{ and}$$

$$\mathrm{CR}(\alpha_1^*) = R_{\text{path}}(\omega_{\alpha_2^*,\alpha_1^*}) = 1.$$

Since $\mathrm{Rd}(\alpha_1^*) > \mathrm{CR}(\alpha_1^*)$, $\alpha_1^*$ is stochastically stable.

Next, we replace player $b$ with a stubborn player with $a^{\text{stb}} = b_1$. Note that $b_1$ is the action of $b$ in the stochastically stable profile, i.e., $s_h = a^{\text{stb}}$, as in condition 2. With player $b$ fixed at $b_1$, the game is now restricted to the first columns of the left and right matrices. For this restricted game, the minimum resistance paths between $\alpha_1^*$ and $\alpha_2^*$ are

$$\omega^{\text{stb}}_{\alpha_1^*,\alpha_2^*} = ((a_1, b_1, c_1), (a_2, b_1, c_1), (a_2, b_1, c_2)) \text{ and}$$

$$\omega^{\text{stb}}_{\alpha_2^*,\alpha_1^*} = ((a_2, b_1, c_2), (a_1, b_1, c_2), (a_1, b_1, c_1))$$

and the resulting radius and coradius of $\alpha_2^*$ are

$$\mathrm{Rd}(\alpha_2^*) = R_{\text{path}}(\omega^{\text{stb}}_{\alpha_2^*,\alpha_1^*}) = 6 \text{ and}$$

$$\mathrm{CR}(\alpha_2^*) = R_{\text{path}}(\omega^{\text{stb}}_{\alpha_1^*,\alpha_2^*}) = 3.$$

Thus, in the heterogeneous case with player $b$ as a stubborn player fixed at $b_1$, the stochastically stable profile switched to $\alpha_2^*$, i.e., $\alpha_1^*$ was not robust to the replacement of $b$ with a stubborn player. The key observation here is that the minimum resistance exit path from $\alpha_2^*$ required player $b$ to transition from $b_1$ to $b_2$.

$c_1$

| | $b_1$ | $b_2$ |
|---|---|---|
| $a_1$ | $10, 10, 10$ | $6, 5, 3$ |
| $a_2$ | $7, 6, 2$ | $4, 5, 6$ |

$c_2$

| | $b_1$ | $b_2$ |
|---|---|---|
| $a_1$ | $2, 6, 5$ | $8, 5, 5$ |
| $a_2$ | $8, 8, 8$ | $7, 7, 5$ |

Fig. 4. Matrix form representation of a three-player game with $S_p = \{a, b, c\}$.

This transition only had a resistance of one, but the resulting action profile $(a_2, b_2, c_2)$ was in the basin of attraction of $\alpha_1^*$. When player $b$ restricted itself to $b_1$, this low resistance path was no longer available. The remaining options to leave the basin of attraction of $\alpha_2^*$ were either a transition by player $a$ to $a_1$ or a transition by player $c$ to $c_1$, and both of these transitions had a resistance of 6. ■

Conditions 1 and 2 are not the only conditions under which a stochastically stable profile is not robust to a stubborn player. However, these conditions establish the fact that even a single stubborn player can be sufficient to alter the behavior of the entire population. Even if $s_h = a^{\mathrm{stb}}$, the behavior of the rest of the population can be impacted by the stubborn nature of the heterogeneous player.

## C. Strategic Player

In this section, we assume that player $h$ is replaced with a strategic player interested in achieving some desired behavior. In the case of confused or stubborn players, the heterogeneous players could not control their impact over the long-run behavior of the system because their strategies were independent of the population state. However, strategic players are fundamentally different from the other two types because they can adapt their strategies to steer the global behavior toward their desired behavior. Thus, the potential impact of a strategic player should be more serious than the impact of a stubborn or a confused player.

The desired behavior of a strategic player will depend on the details of the game setup. For instance, in $2 \times 2$ coordination games, in which the risk dominant NE is stochastically stable, the desired behavior of the strategic player may be to move the population away from the stable profile or to steer the population toward the other NE [30], [31]. Similarly, in the Nash bargaining game considered in [23], the objective of strategic players can be to increase their shares as compared to the NE. Thus, in general, we can say that a strategic adversary's desired behavior is to either stop the population from reaching an equilibrium behavior or to steer the population toward specific behavior.

In Propositions 5 and 6, we present a set of sufficient conditions that are based on the qualitative description of the path resistances and highlight the requirements on strategic adversaries for achieving their objective. Then, in Section IV-B, we explore two specific models for strategic players in the context of graphical coordination games. In the first model, we assume that the strategic adversary is rational and can plan over the future as was modeled in [24]. In the second model, we explore the role of network connectivity of a strategic adversary on the robustness of coordination games over ER random networks.

*Proposition 5:* Let $s$ be a stochastically stable profile under the standard setup and let player $h$ be replaced with a strategic adversary. If any of the statements below are true, $s$ is not robust to a strategic player $h$.

1) For an action $a_h \neq s_h$ of player $h$, there exist profiles $a = (a_h, s_{-h})$ and $a' = (a_h, a_{-h})$ such that $d(s_{-h}, a_{-h}) = 1$ and $R(a, a') = 0$ but $R_{\min}(a,' a) > 0$.
2) There exists an action profile $a \in \mathcal{A}$ such that for each $\omega \in \Omega(a, s)$, there is an index $j \in \{0, 1, \ldots, |\omega - 1|\}$ such that $\omega_j = (a_h, a_{-h})$, $\omega_{j+1} = (a_h', a_{-h})$, and $d(a_{-h}, s_{-h}) > 1$.

*Proof:* The argument for statement 1) is similar to the argument in condition 1 of Proposition 4. However, in this case, we argue that the strategic adversary will switch from $s_h$ to $a_h$ with probability one, i.e., he will reduce $R(s, a) = 0$. From $a$, there is a zero resistance path to $a'$, but all the paths from $a'$ to $a$ have nonzero resistance because of which $s$ will not be robust to strategic behavior of $h$.

In statement 2), the set $\Omega(a, s)$ is the set of all paths from $a$ to $s$. If in each $\omega \in \Omega(a, s)$, the strategic player $h$ is involved, his strategy will be to not update his action and increase the resistance of all the paths in $\Omega(a, s)$ to infinity. Moreover, at the transition that involves the adversary, the action profile of rest of the population $a_{-h}$ is at least one Hamming distance away from the stochastically stable action profile $s_{-h}$. Consequently, there will be no path from $a_{-h}$ to $s_{-h}$ which implies that $s$ will not be robust to strategic behavior of $h$. ■

*Proposition 6:* Let $\alpha^*$ be an NE in the standard setup such that $\alpha^* \notin \mathcal{A}_{\mathrm{ss}}$. Let player $h$ be replaced with a strategic player. The strategic player can steer the global behavior to $\alpha^*$, i.e., make $\alpha^*$ stochastically stable, if either of the following statements are true.

1) Let $\Omega^{\mathrm{CR}}(\alpha^*, \mathrm{BA}^c(\alpha^*))$ be the set of all paths $\omega$ from $\alpha^*$ to $\mathrm{BA}^c(\alpha^*)$ such that $R_{\mathrm{path}}(\omega) < \mathrm{CR}(\alpha^*)$. In each $\omega \in \Omega^{\mathrm{CR}}(\alpha^*, \mathrm{BA}^c(\alpha^*))$, there exists a transition from $\omega_j$ to $\omega_{j+1}$ such that $\omega_j = (a_h, a_{-h}) \in \mathrm{BA}(\alpha^*)$, $\omega_{j+1} = (a_h,' a_{-h})$, where $a_h'$ is any action in $A_h$, i.e., player $h$ has to update his action to leave $\mathrm{BA}(\alpha^*)$.
2) For each $a \in \mathrm{BA}^c(\alpha^*)$, the set $\Omega(a, \alpha^*)$ is the set of all paths from $a$ to $\alpha^*$. There exists an $\omega \in \Omega(a, \alpha^*)$ such that $R_{\mathrm{path}}(\omega) - R_h(\omega) < \mathrm{Rd}(\alpha^*)$ for all $a \in \mathrm{BA}^c(\alpha^*)$.

*Proof:* In statement 1), $\Omega^{\mathrm{CR}}(\alpha^*, \mathrm{BA}^c(\alpha^*))$ is the set of all paths from $\alpha^*$ to outside the basin of attraction and have resistance less than or equal to $\mathrm{CR}(\alpha^*)$. If in each $\omega \in \Omega^{\mathrm{CR}}(\alpha^*, \mathrm{BA}^c(\alpha^*))$, strategic adversary is required to update his action to leave $\mathrm{BA}(\alpha^*)$, his strategy would be to not update his action. As a result of this strategy, the resistance of all the paths in $\Omega^{\mathrm{CR}}(\alpha^*, \mathrm{BA}^c(\alpha^*))$ will become infinity and $\mathrm{Rd}(\alpha^*)$ will be guaranteed to be greater than $\mathrm{CR}(\alpha^*)$.

The condition in 2) states that for each $a$ that does not belong to $\mathrm{BA}(\alpha^*)$, there exists at least one path $\omega \in \Omega(a, \alpha^*)$ such that $R_{\mathrm{path}}(\omega) - R_h(\omega) < \mathrm{Rd}(\alpha^*)$, where $R_h(\omega)$ is the contribution of player $h$ in the resistance of the path as defined in (5). If the strategic adversary decides to update his actions involved in $\omega$ with probability one, he will reduce his contribution $R_h(\omega)$ to zero and the resistance of the path will be less than $\mathrm{Rd}(\alpha^*)$. If this condition is satisfied for all $a \in \mathrm{BA}^c(\alpha^*)$, we can guarantee $\mathrm{CR}(\alpha^*) < \mathrm{Rd}(\alpha^*)$, which implies that $\alpha^*$ will be stochastically stable. ■

The conditions in Proposition 6 appear restrictive for population settings, which is expected because only one strategic player is assigned the task of changing the behavior of the entire population. These conditions signify the degree of influence that a single strategic player should possess in the network in order to

| | $A$ | $B$ |
|---|---|---|
| $A$ | $1+\alpha, 1+\alpha$ | $0, 0$ |
| $B$ | $0, 0$ | $1, 1$ |

Fig. 5. Payoff matrix of a $2 \times 2$ coordination game.

steer the population behavior toward his desired behavior. Using Rd-CR results, the conditions state that the strategic player can lead the population to a desired profile $\alpha^*$ if he has the ability to change the radius and/or the coradius of $\alpha^*$ to the desired values. Condition 1) represents the scenario in which the strategic player can increase the resistance for leaving the basin of attraction of the desired profile, which, in turn, will increase the radius of $\alpha^*$ from its coradius. Condition 2) represents the scenario in which for every action profile $a$ outside the basin of attraction of $\alpha^*$, there exists at least one path $\omega_{a,\alpha^*}$ such that the strategic player can reduce the resistance of this path below $\text{Rd}(\alpha^*)$, which will reduce the coradius of $\alpha^*$ from its radius and will lead to the desired result. Thus, if either of the two conditions in the proposition statement is satisfied, the strategic player has the capability to achieve his objective and steer the global behavior to $\alpha^*$.

## IV. Network Topology and Robustness

In the previous section, we presented sufficient conditions for a single heterogeneous player to alter the behavior of the rest of the population. In this section, we extend our analysis to population settings comprising a large number of players. In particular, we consider the setup of coordination games played over a network of $N$ players. First, in Section IV-A, we will analyze the robustness against stubborn and confused players in the case of path graph, cycle graph, grid graph, Peterson graph, and a class of networks with diameter $D > 1$. Then, in Section IV-B, we will consider strategic players and investigate their robustness properties for wheel graph and ER random networks.

In a standard setup of a two-player coordination game, each player has two actions $A$ and $B$ and the payoff matrix is as shown in Fig. 5. In this game setup, $(A, A)$ and $(B, B)$ are the two NEs and $\alpha \in (0, 1)$ is the added benefit of coordinating on action $A$ as compared to $B$. Thus, $(A, A)$ is Pareto optimal as well as risk dominant. In any $2 \times 2$ coordination game, the risk dominant NE, which in our case is $(A, A)$, is stochastically stable as shown in [1]–[3]. We will consider the setup in which the game is played over a network that is represented by a graph $G(V, E)$. In the network scenario, each player $i \in S_p$ plays the game against all the players in his neighborhood set $\mathcal{N}^G(i)$, where $\mathcal{N}^G(i)$ is the set of all vertices adjacent to vertex $i$ in the network, i.e.,

$$\mathcal{N}^G(i) = \{j \in S_p \,|\, (i, j) \in E\}. \tag{13}$$

Let $\eta_A(i)$ and $\eta_B(i)$ be the fraction of players with actions $A$ and $B$ in the neighborhood set $\mathcal{N}^G(i)$. These fractions will depend on time $t$ as well, but we will suppress time dependence for notational convenience. The utility function of player $i$ is

$$U_i(A, a_{-i}) = \eta_A(i)(1 + \alpha) \text{ and } U_i(B, a_{-i}) = \eta_B(i) \tag{14}$$

where $\eta_B(i) = 1 - \eta_A(i)$. Then, given $a_{-i}$

$$\begin{aligned} &A \in B_i(a_{-i}) \text{ if } \eta_A(i) > \frac{1}{2+\alpha}, \text{ and} \\ &B \in B_i(a_{-i}) \text{ if } \eta_B(i) > \frac{1+\alpha}{2+\alpha}. \end{aligned} \tag{15}$$

Since $\alpha$ belongs to the open interval (0,1), $\eta_A(i)$ and $\eta_B(i)$ satisfy the following bounds:

$$\frac{1}{3} < \eta_A(i) < \frac{1}{2} \text{ and } \frac{1}{2} < \eta_B(i) < \frac{2}{3}$$

for all $i$. In the network setup, the population state at any time corresponds to the number of players with actions $A$ and $B$. We will use the notations $\mathbf{1}_A$ and $\mathbf{1}_B$ to represent the states in which all the players play actions $A$ and $B$, respectively. Moreover, starting from $\mathbf{1}_A$, let $\mathbf{1}_A^k$ represent the situation when $k$ players have switched to action $B$ while the remaining players are still playing action $A$. Similarly, starting from $\mathbf{1}_B$, let $\mathbf{1}_B^k$ represent the situation when $k$ players have switched to action $A$ while the remaining players are still playing action $B$.

This setup of network coordination games was considered in [36] for best response dynamics and various conditions were derived on network structure that will cause risk dominant or risk dominated equilibrium to spread as a contagion. In this work, we consider LLL, which is asynchronous noisy best response dynamics, and investigate certain fundamental network structures and verify the robustness of the stochastically stable profile $\mathbf{1}_A$ after including a heterogeneous player of the three types.

### A. Stubborn and Confused Players

***1) Path Network:*** A path network of $N$ players has $N - 1$ total edges such that all the inner players have two neighbors and the two end players have one neighbor only. According to the condition in (15), action $A$ is the best response of a player if the fraction of his neighbors playing $A$ is greater than $1/(2 + \alpha)$. Thus, in a path network, a player would choose action $A$ with zero resistance as long as one of his neighbors is playing action $A$. On the other hand, a player has action $B$ as his best action if the fraction of neighbors playing action B is greater than $(1 + \alpha)/(2 + \alpha)$. Therefore, a player would switch to action $B$ if both of his neighbors are already playing action $B$. The end players have one neighbor only, which implies that placing a stubborn player next to an end player will change his behavior. A confused player will be unable to change the action of the end player from $B$ to $A$ because of the higher payoff of $A$. Thus, a path network is robust to a single stubborn player if the stubborn player is not placed in the neighborhood of the end players. Moreover, a path network is robust to a confused player.

***2) Cycle Network:*** A cycle network of $N$ players is a two-regular network and is constructed by connecting the end players of a path network. The conditions for an action to be best response for a given player are the same as presented in the case of path network. Since there are no end players in a cycle

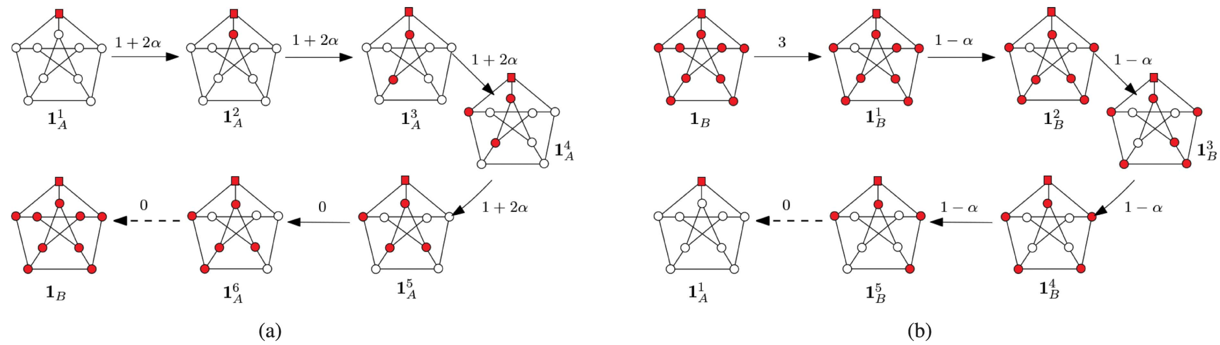


Fig. 6. Minimum resistance paths between the states $\mathbf{1}_A$ and $\mathbf{1}_B$ in the Peterson graph. The shaded nodes are the players with action $B$, the clear nodes are the players with action $A$, and the top shaded square is the stubborn player playing action $B$. (a) Minimum resistance path from $\mathbf{1}_A^1$ to $\mathbf{1}_B$. (b) Minimum resistance path from $\mathbf{1}_B$ to $\mathbf{1}_A^1$.

network, this network is robust to a single stubborn or a confused player.

***3) Grid Network:*** In a 2-D grid network, all the inner players have four neighbors. Thus, the condition in (15) implies that $B$ will be the best response if more than half of the neighboring players are already playing $B$. Thus, a player in a 2-D grid network would choose $B$ with zero resistance if three of his neighbors are already playing $B$, which renders this network robust to the presence of a single confused player.

The robustness of path, cycle, and grid networks can easily be established through Rd-CR criteria or the resistance tree analysis. We have omitted the detailed analysis of these cases because of its simplicity. Next, we will present examples of networks that are not robust to a stubborn player. Our analysis will be based on the Rd-CR criteria as presented in Proposition 1. For coordination games over networks, action profiles $\mathbf{1}_A$ and $\mathbf{1}_B$ are the two candidates for stochastically stable states. Let $\omega_{\mathbf{1}_A,\mathbf{1}_B}$ and $\omega_{\mathbf{1}_B,\mathbf{1}_A}$ be the minimum resistance paths from $\mathbf{1}_A$ to $\mathbf{1}_B$ and $\mathbf{1}_B$ to $\mathbf{1}_A$, respectively. Then radius and coradius of $\mathbf{1}_A$ and $\mathbf{1}_B$ are

$$\mathrm{Rd}(\mathbf{1}_A) = R_{\text{path}}(\omega_{\mathbf{1}_A,\mathbf{1}_B}), \quad \mathrm{Rd}(\mathbf{1}_B) = R_{\text{path}}(\omega_{\mathbf{1}_B,\mathbf{1}_A}),$$
$$\mathrm{CR}(\mathbf{1}_A) = R_{\text{path}}(\omega_{\mathbf{1}_A,\mathbf{1}_B}), \quad \mathrm{CR}(\mathbf{1}_B) = R_{\text{path}}(\omega_{\mathbf{1}_B,\mathbf{1}_A}).$$

Since the radius of $\mathbf{1}_A$ is the coradius of $\mathbf{1}_B$ and the radius of $\mathbf{1}_B$ is the coradius of $\mathbf{1}_A$, proving one of these states as stochastically stable implies that the other one is not stochastically stable.

***4) Peterson Graph:*** Peterson graph is a special undirected graph with 10 nodes and 15 edges as shown in Fig. 6. The network is such that each node has three neighbors. This graph is popular in graph theory since it serves as example and counter example for various network phenomena.

*Proposition 7: Peterson graph is not robust to a stubborn player for* $\alpha < 1/4$.

*Proof:* We consider Peterson graph as shown in Fig. 6 in which the top player, which is represented by a shaded square, is replaced with a stubborn player that always plays $B$, i.e., $a^{\text{stb}} = B$. To prove that the network is not robust, we need the minimum resistance paths $\omega_{\mathbf{1}_B,\mathbf{1}_A^1}$ and $\omega_{\mathbf{1}_A^1,\mathbf{1}_B}$ and show that $R_{\text{path}}(\omega_{\mathbf{1}_B,\mathbf{1}_A^1}) > R_{\text{path}}(\omega_{\mathbf{1}_A^1,\mathbf{1}_B})$ for $\alpha < 1/4$. Minimum resistance paths from $\mathbf{1}_A^1$ to $\mathbf{1}_B$ and $\mathbf{1}_B$ to $\mathbf{1}_A^1$ are shown in Fig. 6(a) and (b), respectively. Based on these paths, the radius and coradius of $\mathbf{1}_B$ are

$$\mathrm{Rd}(\mathbf{1}_B) = 7 - 4\alpha, \quad \mathrm{CR}(\mathbf{1}_B) = 4 + 8\alpha.$$

The maximum value of $\alpha$ for which radius of $\mathbf{1}_B$ remains greater than its coradius turns out to be $1/4$. We want to highlight here that there exist many paths of least resistance between $\mathbf{1}_B$ and $\mathbf{1}_A^1$. However, it can be easily verified that the paths considered in Fig. 6 are indeed the least resistance paths. ■

***5) Wheel Network:*** Wheel network is constructed by adding a node to a cycle graph such that the additional node is connected to all the players on the cycle. We will refer to the additional node as the central node and the cycle nodes as the peripheral nodes. Thus, the central node has $N - 1$ neighbors and a peripheral node has three neighbors. An important aspect of this network is that the players can be divided into two categories based on their importance. The central player has a global influence over the network, whereas any peripheral node has a local influence only.

*Proposition 8:* Wheel network is robust to a peripheral stubborn player but not robust to a central stubborn player.

*Proof:* The impact of a peripheral stubborn player is localized to his two neighbors on the periphery and the central node. Since the central node has $N - 1$ neighbors, the impact on the stubborn player is of limited nature, particularly for large values of $N$. Although the stubborn player does reduce the resistance of his peripheral neighbors in switching from $A$ to $B$, this impact on resistance is not sufficient to change the stochastically stable states. For action $B$ to become the best response of a peripheral player, at least two of his three neighbors should play $B$. Thus, introducing a single stubborn player on the periphery cannot induce a change in the global behavior.

Next, we consider the case of central stubborn player with action $B$. Starting from $\mathbf{1}_A^1$, each peripheral node has one neighbor (central player) with action $B$ and two players with action $A$. Thus, the resistance faced by a peripheral player in switching his action from $A$ to $B$ is

$$R_{\min}\left(\mathbf{1}_A^1, \mathbf{1}_A^2\right) = 2(1 + \alpha) - 1 = 1 + 2\alpha.$$

After the first peripheral player switches to $B$, each of his peripheral neighbors now have two neighbors playing action

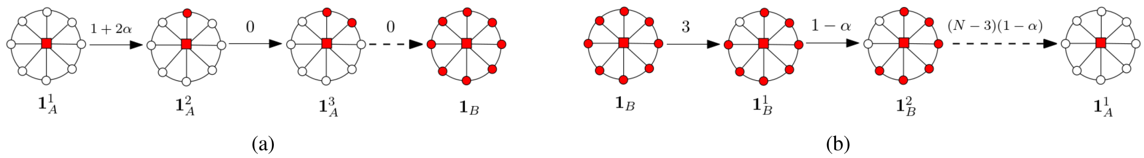


Fig. 7. Minimum resistance paths between the states $\mathbf{1}_A^1$ and $\mathbf{1}_B$ in the wheel network. The shaded nodes are the players with action $B$, the clear nodes are the players with action $A$, and the central shaded square is the stubborn player playing action $B$. (a) Minimum resistance path from $\mathbf{1}_A$ to $\mathbf{1}_B$. (b) Minimum resistance path from $\mathbf{1}_B$ to $\mathbf{1}_A$.

$B$, which implies that $B$ is now their best response. Thus, after the switch of the first peripheral node, action $B$ spreads with zero resistance as shown in Fig. 7(a), which implies that

$$R_{\text{path}}\left(\omega_{\mathbf{1}_A^1,\mathbf{1}_B}\right) = 1 + 2\alpha.$$

If the initial configuration is $\mathbf{1}_B$ with the central stubborn player, the first peripheral node will have a resistance of 3 for choosing action $A$. After the first switch to $A$, all the subsequent nodes will have a resistance of $1-\alpha$ as shown in Fig. 7(b). Thus

$$R_{\text{path}}\left(\omega_{\mathbf{1}_B,\mathbf{1}_A^1}\right) = 3 + (N-2)(1-\alpha).$$

Since $\text{Rd}(\mathbf{1}_B) = R(\omega_{\mathbf{1}_B,\mathbf{1}_A^1})$ and $\text{CR}(\mathbf{1}_B) = R(\omega_{\mathbf{1}_A^1,\mathbf{1}_B})$, the Rd-CR criteria implies that $\mathbf{1}_B$ is stochastically stable if the central player is stubborn. ■

*Proposition 9:* Wheel network is robust to a confused player.

*Proof:* Recall that a confused player selects an action from his action set uniformly at random. In the coordination game, the confused player will select $A$ or $B$ with equal probability. If the confused player is on the periphery, then his response will be limited to his immediate neighbors and the central player. In this case, the confused player cannot alter the behavior of the rest of the population.

If the confused player is placed at the central node, then he influences the behavior of all the players. We can verify through Rd-CR analysis that the confused player will not change the stochastically stable state $\mathbf{1}_A$. Intuitively, since the confused player selects his actions with uniform probability, the impact of $a^{\text{cnf}} = A$ will be higher as compared to $a^{\text{cnf}} = B$ because $A$ has a higher payoff. Thus, the steady-state behavior of the rest of the players will not be impacted by a confused player and we can declare wheel network to be robust to a confused player. ■

***6) Network With Diameter $D > 1$:*** Next, we present a class of networks for any diameter $D$ that is not robust to a stubborn player.

*Proposition 10:* Given a positive integer $D > 1$, there exists a graph with diameter $D$ such that the graph is not robust to one stubborn player for $0 < \alpha < 1$.

*Proof:* For a given $D$, we construct the graph as follows: Consider two perfect binary trees $X$ and $Y$, each of height $D-1$, with root nodes $x$ and $y$, respectively. Let the leaf nodes of $X$ be denoted by $\ell_i^x$ where $i \in \{1, 2, \ldots, 2^{D-1}\}$. Moreover, $\ell_i^x$ and $\ell_{i+1}^x$ have a common parent if $i$ is odd. Similarly, we denote the leaf nodes of $Y$ by $\ell_i^y$. In each tree, we create edges between leaf nodes having a common parent. In other words, if $i$ is odd, we create edges between $\ell_i^x$ and $\ell_{i+1}^x$ in $X$ and between $\ell_i^y$ and $\ell_{i+1}^y$ in $Y$. Then, we also add an edge between root nodes $x$ and $y$. Additionally, consider another node $s$ and create edges between $s$ and $\ell_i^x$, $\forall i$. Similarly, we add edges between $s$ and $\ell_i^y$, $\forall i$. A general construction is shown in Fig. 7(a). Note that the graph obtained will have $2^{D+1}-1$ nodes and diameter $D$.

We assume that the central player $s$ is stubborn with $a^{\text{stb}} = B$ and we need to compute minimum resistance paths $\omega_{\mathbf{1}_B,\mathbf{1}_A^1}$ and $\omega_{\mathbf{1}_A^1,\mathbf{1}_B}$ between $\mathbf{1}_A^1$ and $\mathbf{1}_B$. In Fig. 8(b) and (c), we present these minimum resistance paths for the case of $D=3$. For a general analysis, we start with the state $\mathbf{1}_A^1$. For each $i$, $\ell_i^x$ player (similarly, $\ell_i^y$ player) is connected to the stubborn player $s$ in addition to his parent and sibling players. Thus, the neighborhood of each player $\ell_i^x$ (similarly, $\ell_i^y$) has two players with action $A$ and one player with action $B$. For $\omega_{\mathbf{1}_A^1,\mathbf{1}_B}$, we require one player in each pair $\ell_i^x$, $\ell_{i+1}^x$ (similarly, $\ell_i^y$, $\ell_{i+1}^y$), where $i$ is odd, to play the noisy response with resistance $1+2\alpha$ and switch to $B$. From a population state in which each pair of nodes $\ell_i^x$, $\ell_{i+1}^x$ (similarly, $\ell_i^y$, $\ell_{i+1}^y$), where $i$ is odd, contains one player with action $B$ and the other with action $A$, action $B$ will spread in the population with zero resistance. Since there are a total of $2^{D-1}$ such node pairs in the graph, the total resistance of a path from $\mathbf{1}_A^1$ to $\mathbf{1}_B$ will be

$$R_{\text{path}}\left(\omega_{\mathbf{1}_A^1,\mathbf{1}_B}\right) = (1+2\alpha)2^{D-1}.$$

Next, we start with the state $\mathbf{1}_B$. Since the central player is stubborn with $a^{\text{stb}} = B$, action $A$ will be a noisy response for all $\ell_i^x$ and $\ell_i^y$. Within each pair of nodes $\ell_i^x$ and $\ell_{i+1}^x$ (similarly, $\ell_i^y$ and $\ell_{i+1}^y$), where $i$ is odd, the resistance of the player selecting action $A$ first will be 3, and the resistance of the player selecting action $A$ second will be $1-\alpha$. Thus, the total resistance for a pair of players $\ell_i^x, \ell_{i+1}^x$ (similarly, $\ell_i^y$ and $\ell_{i+1}^y$), where $i$ is odd, to switch from $B$ to $A$ will be $4-\alpha$. Once all the players $\ell_i^x$ and $\ell_i^y$ have switched their actions, action $A$ will spread in the population with zero resistance. Thus, the resistance of a minimum resistance path from $\mathbf{1}_B$ to $\mathbf{1}_A^1$ is

$$R_{\text{path}}\left(\omega_{\mathbf{1}_B,\mathbf{1}_A^1}\right) = (4-\alpha)2^{D-1}.$$

Comparing the two resistances, the Rd-CR criteria imply that $\mathbf{1}_B$ is stochastically stable, which concludes the proof. ■

## *B. Strategic Player*

In this section, we present a setup for evaluating the impact of a strategic player on the population behavior with particular focus on the following.

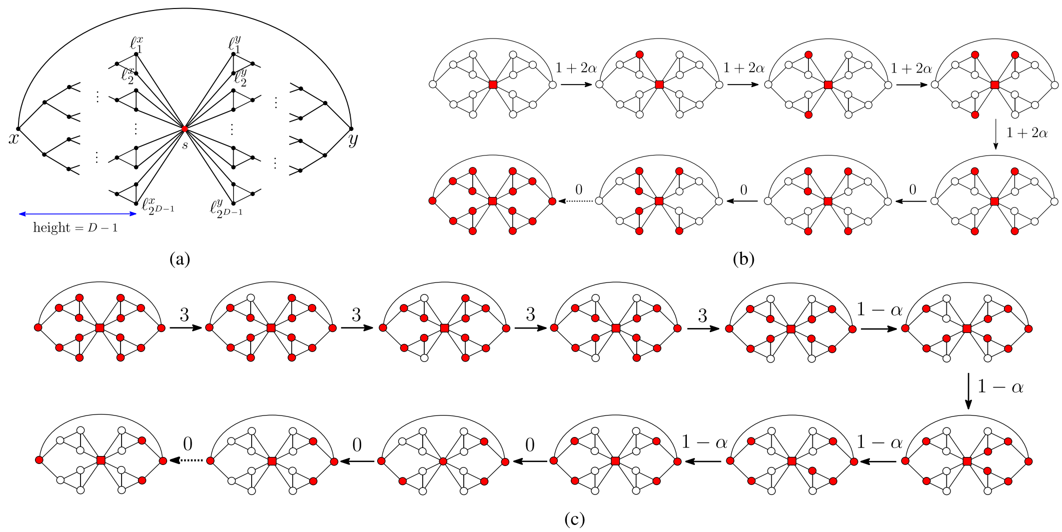


Fig. 8. Minimum resistance paths between the states $\mathbf{1}_A^1$ and $\mathbf{1}_B$ in the proposed network with diameter $D$ and $N = 2^{D+1} - 1$. The shaded nodes are the players with action $B$, the clear nodes are the players with action $A$, and the central shaded square is the stubborn player playing action $B$. (a) Network configuration. (b) Minimum resistance path from $\mathbf{1}_B$ to $\mathbf{1}_A$. (c) Minimum resistance path from $\mathbf{1}_B$ to $\mathbf{1}_A$.

| | $A$ | $B$ | $C$ |
|---|---|---|---|
| $A$ | $6,6$ | $0,5$ | 0,0 |
| $B$ | $5,0$ | $7,7$ | 3,5 |
| $C$ | $0,0$ | $5,3$ | 8,8 |

Fig. 9. Matrix form representation of a two-player and three-action game.

1) We compare the impact of a strategic player who is a myopic planner with a strategic player who can plan a strategy over the game horizon.
2) We analyze the significance of resources and network influence required by a strategic player to achieve his objective.

For the first item, we will consider a network coordination game with three actions in which a myopic strategic adversary will not succeed, but a fully rational player, who can plan a strategy over the game horizon, will succeed. For the second item, we will present a random network setup in which a strategic player can succeed if he has a relatively higher degree of influence over the network.

***1) Wheel Network With a Strategic Player:*** We consider a wheel network in which the central player is replaced with a strategic player. Moreover, the players are now engaged in a three-action game with payoff matrix in Fig. 9. In the standard setup, the game has three Nash equilibria: $\{\mathbf{1}_A, \mathbf{1}_B, \mathbf{1}_C\}$. Applying the Rd-CR result, we can easily verify that $\mathbf{1}_C$ is the unique stochastically stable state in the standard setup.

Suppose that the objective of the strategic player is to shift the action of all the players to $B$. If the strategic player is a myopic planner, his strategy will be to select an action which will maximally improve the utility of action $B$ for the rest of the players in one step. If $a^{\rm str} = B$, where $a^{\rm str}$ is the action of the strategic player, then the neighborhood of every peripheral player will have one player with action $B$ and two players with action $C$, i.e., $a_{-i} = (B, C, C)$. This neighborhood configuration will result in

$$U_i(B, a_{-i}) = U_i(B, B) + U_i(B, C) + U_i(B, C) = 13.$$

However, if $a^{\rm str} = A$, then $a_{-i} = (A, C, C)$ for the peripheral players and

$$U_i(B, a_{-i}) = 5 + 6 = 11.$$

Thus, the strategy of a myopic player will be $a^{\rm str} = B$ for all times. However, the stochastically stable profile $\mathbf{1}_C$ is robust to the central strategic player if $a^{\rm str} = B$.

If the strategic player is fully rational, i.e., plans his long-run strategy, then he will select $a^{\rm str} = A$ for all times. Then, each peripheral player will have one neighbor with action $A$ and two neighbors with action $C$, i.e., for any player $i$, the neighbors' action profile is $a_{-i} = (A, C, C)$ and the resulting utility function is

$$U_i(A, a_{-i}) = 6,\ U_i(B) = 11,\ U_i(C, a_{-i}) = 16.$$

The resistance for any player to switch to $B$ will be equal to 5. After the first peripheral player switches to $B$, its immediate neighbors will have $a_{-i} = (A, B, C)$ and the resulting utility function will be

$$U_i(A, a_{-i}) = 6,\ U_i(B, a_{-i}) = 15,\ U_i(C, a_{-i}) = 13.$$

Thus, for the immediate neighbors of the peripheral player who selected $B$, switching to $B$ will have zero resistance. Therefore, action $B$ will spread throughout the peripheral network with a total resistance of 5.

Once all the peripheral players are switched to $B$ and the central strategic player is playing $A$, a player who wants to switch to $C$ will face a resistance of 9 because

$$U_i(A, a_{-i}) = 0, U_i(B, a_{-i}) = 19, \text{ and } U_i(C, a_{-i}) = 10$$

where $a_{-i} = (A, B, B)$ for all the peripheral players. The minimum resistance path from all $B$ to all $C$ will have a higher

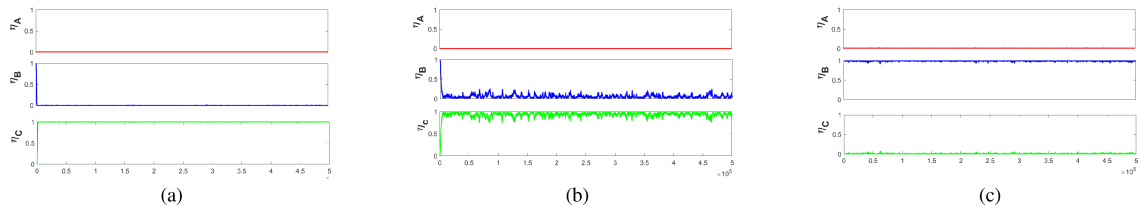

Fig. 10. Population behavior for wheel network with the game structure in Fig. 9 and a strategic player placed at the center node. Simulation parameters are $N = 100$, noise $T = 0.07$, and simulation iterations $5 \times 10^5$. (a) and (b) represent scenarios with standard setup, $a^{\text{str}} = B$, and $a^{\text{str}} = A$, respectively. Vertical axes represent fraction of players playing $A$ ($\eta_A$), $B$ ($\eta_B$), and $C$ ($\eta_C$). (a) Standard setup. (b) Myopic strategic player with action $B$. (c) Rational strategic player with action $A$.

resistance as compared to the minimum resistance path from all $C$ to $B$ if the central strategic player always plays $A$. Therefore, a fully rational strategic player can achieve his objective in this scenario although his strategy of selecting $a^{\text{str}}$ results in an initial decrease in utility.

We verified these results through MATLAB simulations, and the results are presented in Fig. 10. In the simulations, we considered a wheel network over $N = 100$ players for the game setup in Fig. 9 and the game was simulated for $5 \times 10^5$ iterations with $T = 0.07$. In Fig. 10(a)–(c), the vertical axes correspond to $\eta_A$, $\eta_B$, and $\eta_C$, which are the fractions of players with actions $A$, $B$, and $C$, respectively. We started the simulations with the standard setup and Fig. 10(a) shows that $\mathbf{1}_C$ is stochastically stable. Then, we placed a strategic player at the central node with $a^{\text{str}} = B$. Fig. 10(b) shows that the network is robust to strategic player with $a^{\text{str}} = B$. Finally, the strategic player switched his action to $a^{\text{str}} = A$ and Fig. 10(c) shows that the network is not robust to the strategic player in this case.

*2) ER Networks:* The standard setup in graphical coordination games assumes a fixed network structure, which implies that the neighborhood sets of all the players remain the same. However, when modeling interactions in a large population setting, random sampling of players at each time instant is also a well-studied model in game theory literature (see for instance [1] and [24]). Next, we consider ER-network model, which is suitable to model random interactions among players. In this section, we will refer back to the two-player coordination game presented in Fig. 5.

In an ER random network model, player interactions are determined by the model parameter $p$, where $p$ is the probability of an undirected edge between any two players in $S_p$. Thus, $p$ determines the degree of connectivity of the network. Once the network is setup, a player is randomly selected to update his action while all the other players repeat their actions from the previous time step. The randomly selected player observes the actions of his current neighbors and responds according to LLL by selecting an action with probability in (6). In the ER network model, the parameter $p$ can be interpreted as the degree of influence that the players have on each other. For the coordination game played over ER network, we can show through the Rd-CR criteria that the population state $\mathbf{1}_A$ is stochastically stable for the standard coordination game in Fig. 5.

In our setup, we introduce a strategic adversary by replacing a random player $h \in S_p$ with a strategic player whose objective is to drive the system to $\mathbf{1}_B$, i.e., change the stochastically stable state. To achieve his objective, the strategic player has to influence the behaviors of the others. Since the strategic player himself is not impacted, his influence is modeled by directed edges from the strategic player to the other players. If the strategic player has the same level of influence as the other players in the population, then it will not be possible for him to alter the behavior of the entire population when the number of players $N$ is large enough. Therefore, we assume that the strategic player can have a higher level of influence than the rest of the population. This higher influence is modeled by having $p_h$ to be greater than $p$, where $p_h$ is the probability that the strategic player has a directed connection to a player $i$ for all $i \in S_p \backslash h$. We present a sufficient condition on $p_h$ that will ensure that the network is not robust to the strategic player.

*Proposition 11: An ER random network is not robust to a strategic player if*

$$\frac{p_h}{p} > \frac{(N-2)\alpha}{2} \text{ and } p_h \leq 1. \tag{16}$$

*Proof:* We assume that the strategy of the strategic player is to always play action $B$, i.e., $a^{\text{str}} = B$. In the random network setup, the utility function of any player, say player $i$, is the expected utility. Let $n_A$ and $n_B$ be the number of homogeneous players other than $i$ playing actions $A$ and $B$, respectively, such that $n_A + n_B = N - 2$. Then, the utility function of player $i$ is

$$\begin{aligned} U_i(A, a_{-i}) &= p\frac{n_A}{N-1}(1+\alpha), \\ U_i(B, a_{-i}) &= p\frac{n_B}{N-1} + p_h\frac{1}{N-1}. \end{aligned} \tag{17}$$

Consider the case when all players are initially playing action $B$. Then, the resistance of going from $\mathbf{1}_B$ to $\mathbf{1}_B^1$ can be computed as follows. Suppose player $i$ switches action from $B$ to $A$ when all the other players including the strategic player are playing $B$, i.e., $n_A = 0$ and $n_B = N - 2$. Then, from (17)

$$R\left(\mathbf{1}_B, \mathbf{1}_B^1\right) = \frac{p(N-2) + p_h}{N-1}.$$

In general, for any $k \geq 0$, the resistance of going from $\mathbf{1}_B^k$ to $\mathbf{1}_B^{k+1}$, when $n_A = k$ and $n_B = N - 2 - k$, is

$$R\left(\mathbf{1}_B^k, \mathbf{1}_B^{k+1}\right) = \max\left\{0, \frac{p(N-2-k) + p_h}{N-1} - \frac{p(1+\alpha)k}{N-1}\right\}. \tag{18}$$

On the other hand, when all players except the strategic player are playing $A$ and the initial network configuration is $\mathbf{1}_A^1$, the resistance of going from $\mathbf{1}_A^k$ to $\mathbf{1}_A^{k+1}$, where $k \geq 1$, is either

$$R\left(\mathbf{1}_A^k, \mathbf{1}_A^{k+1}\right) = \frac{p(N-(k+1))(1+\alpha)}{N} - \frac{p_h + p(k-1)}{N} \tag{19}$$

or the resistance is zero if the right-hand side in the above equation is nonpositive. Note that in (18) and (19), there is an offset in the value of $k$ because of the effect of strategic adversary when starting from $\mathbf{1}_A^1$. To prove the stochastic stability of the state $\mathbf{1}_B$ in the presence of a strategic player, we need to show that the overall resistance of going from $\mathbf{1}_B$ to $\mathbf{1}_A^1$ is greater than the resistance of going from $\mathbf{1}_A^1$ to $\mathbf{1}_B$, i.e.,

$$R_{\text{path}}\left(\omega_{\mathbf{1}_B, \mathbf{1}_A^1}\right) > R_{\text{path}}\left(\omega_{\mathbf{1}_A^1, \mathbf{1}_B}\right).$$

If $p_h$ is equal to $p$, i.e., the influence of strategic player is the same as any other player, then the above condition can never be true. However, if $p_h$ is allowed to be greater than $p$, then under certain conditions, we can find $p_h$ that will satisfy the desired condition. A sufficient condition for $\mathbf{1}_B$ to be stochastically stable is that the resistance of each transition in the path $\omega_{\mathbf{1}_B, \mathbf{1}_A^1}$ is greater than the resistance of the corresponding transition in the path $\omega_{\mathbf{1}_A^1, \mathbf{1}_B}$, i.e.,

$$\begin{aligned} R\left(\mathbf{1}_B \to \mathbf{1}_B^1\right) &> R\left(\mathbf{1}_A^1 \to \mathbf{1}_A^2\right), \\ R\left(\mathbf{1}_B^1 \to \mathbf{1}_B^2\right) &> R\left(\mathbf{1}_A^2 \to \mathbf{1}_A^3\right), \\ &\vdots \\ R\left(\mathbf{1}_B^{k-1} \to \mathbf{1}_B^k\right) &> R\left(\mathbf{1}_A^k \to \mathbf{1}_A^{k+1}\right). \end{aligned}$$

Using (18) and (19), it is straightforward to prove that all the above inequalities hold if

$$\frac{p_h}{p} > \frac{(N-2)\alpha}{2}.$$

In addition to the above inequality, the condition $p_h \leq 1$ has to be imposed to ensure that $p_h$ is a probability.

The result in Proposition 11 quantifies the degree of influence that the strategic adversary must possess to impact the population behavior. For large $N$, this condition requires the strategic adversary to have significantly higher influence as compared to the rest of the players. To verify the result in the proposition, we simulated a population with $N = 50$ players. At each time, a random network was generated with parameter $p = 6/N - 1$, i.e., each player was connected to six other players on average. The coordination game parameter was $\alpha = 0.3$. The players updated their actions using LLL with $T = 0.25$. A strategic player was included in the population with $p_h = 0.8$, which was selected according to the condition in Proposition 11. The value of this parameter signifies that the strategic player needs to influence majority of the players directly to achieve his objective. The results of the simulation are presented in Fig. 11. The system was initialized at $\mathbf{1}_A^1$ and was simulated for $10^6$ iterations. The vertical axis represents the fraction of players playing action $A$. The simulation clearly depicts that the strategic player was successful in changing the behavior of the population from $\mathbf{1}_A^1$ to $\mathbf{1}_B$.

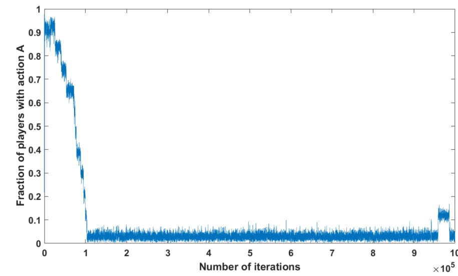


Fig. 11. Evolution of behavior in graphical coordination game over ER random network model with parameters $N = 50$, $\alpha = 0.3$, $p = 6/N - 1$, noise $T = 0.25$. For the strategic player, the parameter $p_h = 0.88$ according to the condition in Proposition 11.

## V. Conclusion

In this article, we presented a new notion of the robustness of stochastic learning dynamics to heterogeneous decision strategies of players in games. By analyzing three types of heterogeneous players, including confused, stubborn, and strategic players, we demonstrated, using our proposed notion of robustness, that the presence of even a single player can alter the behavior of an entire population. We then presented qualitative results and concrete examples of standard game setups and network structures to show the impact of a single heterogeneous agent. The results presented were not restricted to coordination game setup like most of the previous related works.

This article identifies an interesting set of problems in the important domain of stochastic learning dynamics in games. The article also presents a basic framework that should serve as a starting point for future research. Some interesting research directions are as follows:

1) to develop graph theoretic notions to categorize networks that are robust or not robust to heterogeneous players for various game setups;
2) to explore the role of different update rules in stochastic learning dynamics in the context of our notion of robustness;
3) to analyze the impact of multiple heterogeneous players with different behaviors on the population behavior.

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