Part II - Advanced Macroeconomics
Lecture I

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Administration

Nothing changed from Erick/first half

- www.josephsbriggs.com/teaching
  - Syllabus
  - Assignments: Presentation, referee report, research proposal
  - Problem set: interest?
  - Office hours: Tuesday (at JHU, determine time!) and by appointment (Skype)
  - Lecture Notes: Posted at least one hour before lecture, likely night before.
- Schedule:
  - Next 5 weeks: Me
  - Last 3 weeks: Presentations

Agenda

What will we cover in next 5 weeks?

1. Computational techniques
2. Aggregate Risk
3. Lifecycle economies
4. Applications:
    - Public Finance, Life Cycle Taxation
    - Consumer Debt and Default
    - Housing Wealth and the Great Recession

Introductions

- About me
  - Graduated from NYU spring 2016 - macro and finance fields
  - Currently work at Fed Board
  - Research interests:
    - Lifecycle savings and precautionary motives
    - Empirical household finance
    - Overarching theme: Develop data and interpret using lifecycle saving model
- About you
  - ?

Plan for first lecture

- Focus on computation techniques:
  1. Discretizing continuous random variables
  2. Methods for solving consumption saving problems:
  3. Solving models with aggregate risk - Krusell Smith (1998)
- These methods are basic computational tools used in modern macro with heterogeneous agents.
- First two lectures: adapted from Violante NYU notes

Two methods

- Tauchen method (Quick Review)
- Rouwenhorst method (New)

We will now review both quickly. But first, why?

- Continuous distributions computationally costly to evaluate
- Discrete approximations can be good proxies, much faster to evaluate

Set-up

Suppose random variable (e.g., income) follows an AR(1) process:

$$
\begin{aligned}
y_t &= \rho y_{t-1} + \epsilon_t \\
\rho &< 1 \\
\mu_\epsilon &= 0\ ,\ \sigma_\epsilon < \infty \\
\sigma_y^2 &= \sigma_\epsilon^2/(1-\rho^2)
\end{aligned}
$$

Tauchen Method

General idea: discretize space of possible realizations

In practice:

- Set maximum point y_N as a multiple m (e.g., m=3) of unconditional standard deviation:

$$y_N = m\left(\sigma_\epsilon^2/(1-\rho^2)\right)^{.5}$$

- Set minimum point similarly ($y_1 = -y_N$ if symmetric).
- Define $(\tilde{y}_i)_{i=1}^N$ as equispaced points over the interval $[y_1, y_N]$.

Tauchen Method

- Define $d = y_j - y_{j-1}$
- Define $\pi_{j,k} = P(\tilde{y}_t = j | \tilde{y}_{t-1} = k)$. Then

$$\pi_{j,k} = F\left(\frac{y_k + d/2 - \rho y_j}{\sigma_\epsilon}\right) - F\left(\frac{y_k - d/2 - \rho y_j}{\sigma_\epsilon}\right)$$

$$\pi_{j,1} = F\left(\frac{y_1 + d/2 - \rho y_j}{\sigma_\epsilon}\right)$$

$$\pi_{j,N} = 1 - F\left(\frac{y_N - d/2 - \rho y_j}{\sigma_\epsilon}\right)$$

- Obviously, approximation increases with N

Tauchen Method

Accuracy: Given approximation, accuracy can be evaluated numerically.

$$\tilde{y}_t = \tilde{\rho}\tilde{y}_{t-1} + \tilde{\epsilon}_t$$

- $\tilde{\rho} = cov(\tilde{y}_{t-1}, \tilde{y}_t)/var(\tilde{y}_t)$
- $\tilde{\sigma}_{\tilde{\epsilon}} = (1 - \tilde{\rho}^2)var(\tilde{y}_t)$

These statistics can be calculated analytically (given discretization) or compuationally (via simulation). In general, approximaiton is ok for iid variables (i.e., $\rho = 0$), but not great if persistence present (i.e, ρ is close to 1).

- Easily extends to multivariate process

Rouwenhorst Method

- Best method to discretize continuous stochastic process.
- Large performance increases for persistent processes
- Basic idea: Approximate process through a discrete-space Markov chain (sound familiar?)
- Three parameters: p, q (transition probabilities) and ψ (grid limit).
- Recursive definition of Markov chain Θ_n, which we go through now.

Rouwenhorst Method

Define an evenly-spaced, symmetric grid $\{\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_N\}$ with $-\tilde{y}_1 = \tilde{y}_N = \psi$

1. Define Θ_2 as

$$\Theta_2 = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

2. For $N > 2$, given Θ_{N-1} define Θ_N as

$$\Theta_N = p \begin{bmatrix} \Theta_{N-1} & \mathbf{0} \\ \mathbf{0}\prime & 0 \end{bmatrix} + (1-p) \begin{bmatrix} \mathbf{0} & \Theta_{N-1} \\ 0 & \mathbf{0}\prime \end{bmatrix} + (1-q) \begin{bmatrix} \mathbf{0}\prime & 0 \\ \Theta_{N-1} & \mathbf{0} \end{bmatrix} + q \begin{bmatrix} 0 & \mathbf{0}\prime \\ \mathbf{0} & \Theta_{N-1} \end{bmatrix}$$

3. Divide middle rows by 2 so that their elements sum to one.

Rouwenhorst Method

Example, $N = 3$:

$$
\begin{aligned}
\Theta_3 =& p \begin{bmatrix} p & 1-p & 0 \\ 1-q & q & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1-p) \begin{bmatrix} 0 & p & 1-p \\ 0 & 1-q & q \\ 0 & 0 & 0 \end{bmatrix} + \\
& (1-q) \begin{bmatrix} 0 & 0 & 0 \\ p & 1-p & 0 \\ 1-q & q & 0 \end{bmatrix} + q \begin{bmatrix} 0 & 0 & 0 \\ 0 & p & 1-p \\ 0 & 1-q & q \end{bmatrix} \\
=& \begin{bmatrix} p^2 & 2p(1-p) & (1-p)^2 \\ 2p(1-q) & 2pq + 2(1-p)(1-q) & 2(1-p)q \\ (1-q)^2 & 2q(1-q) & q^2 \end{bmatrix}
\end{aligned}
$$

Note:

$$
\begin{aligned}
\sum_j \Theta_{2,j} =& 2p(1-q) + 2pq + 2(1-p)(1-q) + 2(1-p)q \\
=& 2\left(p - pq + pq + (1-p)(1-q) + q - pq\right) = 2\left(p + (1-q) - p(1-q) + q - pq\right) \\
=& 2\left(p + (1-q) - p + pq + q - pq\right) \\
=& 2 \\
\sum_j \Theta_{1,j} =& p^2 + 2p(1-p) + (1-p)^2 = -p^2 + 2p + (1-p)^2 \\
=& 1
\end{aligned}
$$

So after step 3 it is a valid transition matrix.

Rouwenhorst Method

Not derived, but this Markov chain's invariant distribution, $\lambda_i^N = P(\tilde{y}_i)$

$$\lim_{N\to\infty} \lambda_i^N = \binom{N-1}{i-1} s^{i-1}(1-s)^{N-i} \sim Binomial(N-1, 1-s)$$

$$where \quad s = \frac{1-p}{2-p-q}$$

All moments of this distribution can be obtained analytically by manipulating Markov chain. So for any arbitrary distribution you are able to use a minimum distance estimator to choose (p, q, ψ) to fit the distribution. For example, an AR(1):

$$0 = \mathbb{E}[y_t] = \mathbb{E}[\tilde{y}_t] = \frac{(q-p)\psi}{2-p-q} \implies p = q$$

$$\sigma_y^2 = var(y_t) = var(\tilde{y}_t) = \psi\left[1 - 4s(1-s)\left(\frac{N}{N-1}\right)\right] \implies \psi = \sigma_y\sqrt{N-1}$$

$$\rho = corr(y_t, y_{t+1}) = corr(\tilde{y}_t, \tilde{y}_{t+1}) = p + q - 1 \implies p = q = \frac{1+\rho}{2}$$

Finally, because binomial converges to normal distribution, this method works particularly well for log-normal processes.

Model Solution Techniques

Two main solution techniques (briefly review from last week):

1. Policy Function Iteration
2. Endogenous Grid Method (Carroll 2006)
    - Extend to allow for two control variables (New)
    - Extend for non-concave problems (New)

How to evaluate solution accuracy? (New)

- Euler equation errors

General Set-up

- Want to solve for policy functions a, c that satisfy Euler equation

$$u'(c_t) = \beta \mathbb{E}\left[u'(c_{t+1})\right]$$

- In practice, we only need to solve for one policy function, with the other implied by the budget constraint. Substituting budget constraint into discretely approximated Euler equation yields

$$u'(Ra_t + \tilde{y}_t - a'(a_t, \tilde{y}_t)) =$$
$$\beta \sum_{\tilde{y}_{t+1} \in \bar{\tilde{y}}} \pi(\tilde{y}_{t+1}|\tilde{y}_t) u'(Ra'(a_t, \tilde{y}_t) + \tilde{y}_{t+1} - a'(a'(a_t, \tilde{y}_t), \tilde{y}_{t+1})),$$

where a' is the function we need to solve for.

General Solution Method

1. Guess initial policy function a' for all grid points (Note: if PFI, grid over (a_j, y_k) while if EGM grid over y_k)
2. Construct right hand side of Euler equation, denoted as $B(a')$.
3. Solve for $a^\star$:
   - PFI: Use nonlinear equation solver (Linear/Chebishev interpolation) to solve for $a^\star$

$$u'(Ra_j + y_k - a^\star) - B(a^\star) = 0$$

   - EGM: Invert Euler equation to compute next period's policy

$$a^\star = \left[a' - y_k + u'^{,-1}(B(a'))\right] / R$$

4. Check whether this satisfies the borrowing constraint. If not, assume borrowing constraint holds.

General Solution Method

5. Update guess of policy function, e.g.:

$$a'_{i+1} = (1 - \psi)a'_i + \psi a^\star$$

6. Check convergence of policy function within tolerance range.

Note: The EGM is generally much faster because it only requires a single evaluation of expected value in step 3, while PFI requires multiple evaluations. We will focus on EGM for remainder of discussion.

Extension 1: More than one control

- Consider model with two controls:

$$[V(a_t, y_t)] = \max_{ct, l_t} u(c_t, l_t) + \beta \mathbb{E}[V(a_{t+1}, y_{t+1})]$$

$$Ra_{t+1} = h(y_t, l_t) + a_t - c_t$$

$$y_{t+1} \sim \Pi_{y|y_t}$$

- EGM not sufficient - second FOC that pins down labor supply can't be solved for explicitly.

Extension 1: More than one control

- Very high-level overview of solution proposed in Barillas Fernandez-Villaverde (2007)
    1. Fix l_t and solve for optimal c_t using EGM.
    2. Update optimal labor-supply policy function using value function iteration.
    3. Repeat until converged.
- For full details, read Barillas Fernandez-Villaverde (2007)

Extension 2: Non-concave Problems

- EGM assumes that continuation values B are concave.
  - This ensures that there exists a unique $a^{\star}$ that solves

$$a^{\star} = \left[a' - y_k + u'^{,-1}(B(a'))\right] / R$$

- In many interesting problems this isn't necessarily the case:
  - Fixed adjustment costs (durable consumption)
  - Lumpy expenditures
- These introduce a kink in the value function, and a corresponding discrete jump in the marginal utility of assets, so Euler equation is necessary but not sufficient.
- Question: Do we have to use VFI, which is accurate but notoriously slow?

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- These introduce a kink in the value function, and a corresponding discrete jump in the marginal utility of assets, so Euler equation is necessary but not sufficient.
- Question: Do we have to use VFI, which is accurate but notoriously slow?
    - No! Fella (2013)

Extension 2: Non-concave Problems

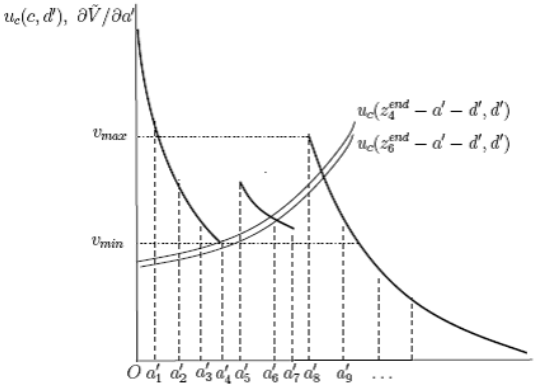

Figure 1: Illustrating the algorithm

Extension 2: Non-concave Problems

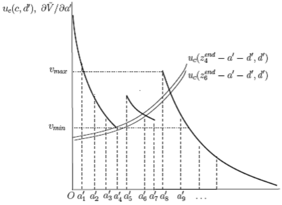

Figure 1: Illustrating the algorithm

- Identify regions of B that are globally concave in next period's assets (denoted G^C, e.g., $a < a_1'$ and $a > a_8'$).
  - G^C is characterized as $a_j' \leq a_{min}'$, where a_{min} is the largest point such that $B(a_j') > B(a_{min}')\ \forall a_j' < a_{min}'$ and $a_j' \geq a_{max}'$, where a_{max} is the smallest point such that $B(a_j') < B(a_{max}')\ \forall a_j' > a_{max}'$
  - For points in this region, the EGM can be applied directly.

Extension 2: Non-concave Problems

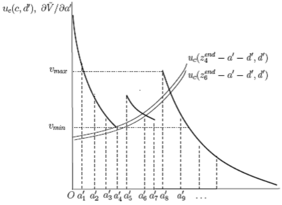

Figure 1: Illustrating the algorithm

- Identify regions of B that are not globally concave in next period's assets (denoted G^{NC}, e.g., $a'_1 \leq a \leq a'_9$).
  - For points in this region, another solution technique must be used.
  - Fella's suggestion: For each point $a'_i \in G^{NC}$ store the candidate point $a^\star(a')$. Check whether the value function

    $$a' = \arg\max_{\hat{a} \in G^{NC}} u(Ra^\star(a') + \tilde{y} - a') + B(a')$$

    If it is, store it in the policy function grid. Otherwise discard it.

Extension 2: Non-concave Problems

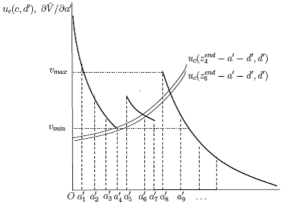

Figure 1: Illustrating the algorithm

- Finally, note the policy function is discontinuous at points a'_i, a'_{i+1} such that a'_i is kept and a'_{i+1} is discarded. One wants a finer grid near these points, and one should not interpolate policy functions between $(a^\star(a'_i, a^\star(a'_{i+k}))$, where a'_{i+k} is the next non-discarded point.
- Otherwise, algorithm proceeds as otherwise specified.

How to Evaluate Accuracy of Approximation?

- All exercises to this point have approximated the solution of problem.
  - Accuracy improves with complexity of approximation, but computation times increase.
  - How do we evaluate when approximation is precise enough?
- Solution: compare Euler equation errors

Euler Equation Errors

$$u'(c_t) - \beta R\mathbb{E}[u'(c_{t+1})] \approx 0$$

- The above will hold exactly at gridpoints, but differ from zero further away from points where exact solutions were obtained.
- Define consumption equivalent error as:

$$u'(c_t(1-\epsilon_t)) = \beta R\mathbb{E}[u'(c_{t+1})]$$

$$\epsilon_t = 1 - \frac{u'^{,-1}(\beta R\mathbb{E}[u'(c_{t+1})])}{c_t}$$

- Error corresponds to the fraction of consumption that our approximated model mis-assigns.
  - Approximation error relative to household welfare is the square of Euler equation error (Santos 2000)

Euler Equation Errors

- Two approaches to evaluate accuracy of approximation:
  1. Simulate long sequence (long enough to approximate invariant distribution) of decisions, then calculate Euler equations at some point.
  2. Use Sobol sequences to explore entire state space.
- See Aruoba Fernandez-Villaverde Rubio -Ramirez (2006) for analysis of Euler equation errors for different solution methods.
- Finally, den Haan Marcet (1994) present formal over-identifying test to evaluate model accuracy.

Bewley-Aiyagari

- Last week - Erick introduced Aiyagari (1994)
- Extension of neoclassical growth model
  - Endowment economy with heterogeneous agents
  - Agents receive idiosyncratic income shocks
  - Incomplete markets - agents restricted to risk-free bond
    - Results in partial consumption insurance.
  - Algorithm for solving model (See Erick's notes for details):
    1. Given r, compute the savings policy function $a'(a, \varepsilon)$.
    2. Given r, compute the stationary distribution $\lambda(a, \varepsilon | r)$ and corresponding density function $f(a, \varepsilon | r)$.
    3. Calculate excess demand and update r until r has converged within desired tolerance.

Aggregate Uncertainty in Bewley-Aiyagari

- This week - Introduce aggregate uncertainty
  - Permits analysis of business cycle dynamics in model with heterogeneous agents.
- Introduces new challenges:
  - Distribution of agents affects evolution of wealth distribution/aggregate capital
  - Distribution of economy becomes state variable, which households must know to forecast future states and prices.
  - Computationally impossible to solve for policies equilibrium allocations as a function of distribution
- Solution: Krusell Smith (1999) present “near-aggregation” result that reasonably approximates economy.
- Next: Present this solution

Consumers

- Unit continuum of infinitely-lived ex-ante identical agents
- Preferences: $u' > 0, u'' < 0$, in DARA class (e.g., CRRA)
- Access to non-contingent bond with interest r

Production

- Representative firm
  - Large number of firms
  - Produce and sell consumption good in perfectly competitive market.
- Production function: Demand Capital K and labor H, produce Y

$$\max_{K,H} Y = zF(K,H)$$

$$e.g., \quad Y = zK^{\alpha}H^{1-\alpha}$$

- Rents capital at rate r

$$r = zF_K(K,H) - \delta$$

- Hires workers at wage rate w

$$w = zF_H(K,H)$$

Labor Supply and Idiosyncratic Risk

- Unit mass of agents inelastically supply unit endowment of labor effort:
- Receive wage w
- Labor productivity shocks, AR(1):

$$log(\varepsilon') = \rho log(\varepsilon) + \epsilon, \;\; \rho < 1, \;\; \epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

- $\pi^\star(\varepsilon)$ - fraction of population with productivity ε.

$$L = \int \varepsilon \pi^\star(\varepsilon) d\varepsilon$$

Introduce Aggregate Risk

New idea this week:

- Assume aggregate shock $z \in \{z_b, z_g\}$, with $z_b < z_g$

$$z' \sim \Pi_{|z}$$

- Simplify labor productivity and allow aggregate shock to affect idiosyncratic productivity transitions
  - $\varepsilon \in \{\varepsilon_b, \varepsilon_g\}$ where $\varepsilon_b < \varepsilon_g$
  - For example, ε_b corresponds to unemployment.
  - $\varepsilon' \sim \Pi_{|z,\varepsilon}$
  - Note: the simplification of productivity is not necessary for results, but eases presentation!

Aggregate Risk

- Rewrite aggregate (Π_z) and idiosyncratic ($\Pi_{|z,\epsilon}$) Markov chains as a single Markov chain that reflects joint evolution of (z, ε). i.e.,

$$\pi(z', \varepsilon' | z, \varepsilon) = Pr(z', \varepsilon' | z, \varepsilon)$$
$$(z', \varepsilon') \sim \Pi_{|(z,\epsilon)}$$

- Note, that there certain patterns that we should expect. e.g.,

$$\pi(z_g, \varepsilon_g | z_g, \varepsilon_g) > \pi(z_b, \varepsilon_b | z_g, \varepsilon_g)$$
$$\pi(z_g, \varepsilon_g | z_b, \varepsilon_b) > \pi(z_b, \varepsilon_g | z_b, \varepsilon_b)$$
$$\pi(z_b, \varepsilon_b | z_g, \varepsilon_g) > \pi(z_g, \varepsilon_b | z_g, \varepsilon_g)$$

Market Clearing

Let $\lambda(a, \varepsilon)$ denote the distribution of households across individual states (a, ε). Then we have market clearing conditions determined by:

$$C = \int_{A \times E} c(a, \varepsilon) d\lambda$$

$$H = \int_{A \times E} \varepsilon d\lambda$$

$$K = \int_{A \times E} a(a, \varepsilon) d\lambda$$

$$C + K' = Y + K(1 - \delta)$$

Note: these are essentially the same market clearing conditions that were presented in last lecture. What is different?

Market Clearing

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$$K = \int_{A\times E} a(a, \varepsilon) d\lambda$$

$$C + K'(\lambda') = Y + K(1 - \delta)$$

Note: these are essentially the same market clearing conditions that were presented in last lecture. What is different?

- Invariant distribution and ability to forecast K'.

State Variables

- Two individual state variables:
  - a
  - ε
- Two aggregate state variables:
  - z
  - λ - Note the difficulty of tracking this as a state variable.

Household Problem

$$v(a, \varepsilon, z, \lambda) = \max_{c, a'} u(c) + \beta \sum_{\varepsilon', z'} v(a', \varepsilon', z', \lambda') \pi(z', \varepsilon' | z, \varepsilon)$$

$$\begin{aligned}
& s.t. \\
c + a' &= w(z, K)\varepsilon + R(z, K)a \\
K &= \int_{A \times E} a(a, \varepsilon) d\lambda \\
a' &\geq 0 \\
\lambda' &= \Psi(z, \lambda, z')
\end{aligned}$$

FOC

$$u'(R(z,K)a + w(z,K)\varepsilon - a'(a,\varepsilon,z,K)) =$$
$$\beta \sum_{\varepsilon',z'} R(z',K')u'(R(z',K')a'(a,\varepsilon,z,K) + w(z',K')\varepsilon' - a''(a'(a,\varepsilon,z,K),\varepsilon',z',K'))\pi(z',\varepsilon'|z,\varepsilon)$$

- Key takeaway: Next period's capital depends on next period's distribution λ', which is a complicated mapping given by $\Psi(z,\lambda,z')$.
- This is why λ has to be a state variable, and the key computational challenge of including aggregate risk!

Recursive Competitive Equilibrium

- A value function v
- Household policy functions (a', c)
- Firm optimal capital and labor (H, K)
- Pricing functions r and W
- Law of motion Ψ

such that

- Given pricing functions (r, w), (c, a') solve the household value function v.
- Given pricing functions (r, w), (H, K) solve the firm's problem.
- Labor, capital, and good markets clear
- For every possible transition $z \to z'$, given a', Ψ solves

$$\lambda' = \Psi(z, \lambda, z')$$

Transition Function

How do we calculate the transition function between states?

$$\begin{aligned}\lambda'(\mathcal{A} \times \mathcal{E}) &= \Psi_{(\mathcal{A}\times\mathcal{E})}(z, \lambda, z') \\ &= \int_{A \times E} P((a, \varepsilon), \mathcal{A} \times \mathcal{E} | z, z') d\lambda\end{aligned}$$

where P is the transition function between two periods given aggregate shock transition $z \to z'$

$$P((a, \varepsilon), \mathcal{A} \times \mathcal{E} | z, z') = \sum_{\varepsilon \in \mathcal{E}} \pi(\varepsilon' | z, \varepsilon, z') \mathbb{1}[a'(a, \varepsilon) \in \mathcal{A}]$$

Approximate Solution

- Key challenge to solving this problem is that λ is infinite dimensional. Obviously need to work with a finite approximation.
  - How do we discretely approximate the distribution? And how do we know when we have a reasonable approximation?
- Krusell Smith (1998) provides answer to both of these questions.
- Key insight: Any distribution is characterized by infinite sequence of moments. Krusell Smith propose approximating distribution with finite set of moments $\bar{m}$. Then

$$\bar{m}' = \Psi_M(z, \bar{m}) = \begin{cases} m_1' = \psi_1(z, \bar{m}) \\ ... \\ m_M' = \psi_M(z, \bar{m}) \end{cases}$$

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- How many moments to include?
- What is appropriate law of motion for these moments ψ_m?

Approximate Solution

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- How many moments to include? - one
- What is appropriate law of motion for these moments ψ_m?- linear

Approximate Solution

- Proposal:

$$\bar{m}' = \psi_1(z, \bar{m})$$
$$ln\ K' = b_z^0 + b_z^1 ln\ K$$

- Then

$$v(a, \varepsilon, z, K) = \max_{c,a'} u(c) + \beta \sum_{\varepsilon', z'} v(a', \varepsilon', z', K')\pi(z', \varepsilon' | z, \varepsilon)$$

$$s.t.$$
$$c + a' = w(z, K)\varepsilon + R(z, K))a$$
$$a' \geq 0$$
$$ln\ K' = b_z^0 + b_z^1 ln\ K$$

- Simply a fixed point problem of type we know how to solve!

Approximate Solution Algorithm

Goal: Solve for coefficients b_0, b_1 that satisfy recursive equilibrium with linear law of motion.

1. Guess β_0, β_1
2. Given law of motion for K', solve Euler equation:

$$u'(R(z,K)a + w(z,K)\varepsilon - a') =$$
$$\beta \sum_{\varepsilon',z'} R(z',K')u'(R(z',K')a'w(z',K')\varepsilon' - a'')\pi(z',\varepsilon'|z,\varepsilon)$$

   using favorite solution method.

3. Simulate economy for N individuals for T periods (e.g., $N = 10000$, $T = 500$) to converge to ergodic distribution. Then simulate a test-sample (e.g., $N = 10000$, $T = 2000$) and calculate the average capital stock $\bar{K}_t = 1/N \sum_i a_{i,t}$ for each period.

Approximate Solution Algorithm

4. Run the following regressions:

$$\ln A' = \hat{b}_z^0 + \hat{b}_z^1 \ln A \text{ for } z \in \{z_b, z_g\}$$

5. Check whether $(\hat{b}_z^0, \hat{b}_z^1) \approx (b_z^0, b_z^1)$ for each z.
   - If not, go back to step 1.
   - If yes, then the law of motion used by agents to forecast prices is consistent with the equilibrium law of motion. In other words, the law of motion that agents use to make decisions is consistent with the aggregate outcomes, and choices a' aggregate to $A' = K'$!

- Is using 1^{st} moment and linear law of motion a good assumption?
  - Krusell Smith check R^2 to evaluate explanatory power. If R^2 too low, could add higher order polynomial terms and/or more moments.

Solution Accuracy

- Krusell Smith's main finding:

$$ln\ K' = \begin{cases} .095 + .962\ ln\ K, & for\ z = z_g \\ .085 + .965\ ln\ K, & for\ z = z_b \\ R^2 = .999998 \end{cases}$$

- So yes, good assumption. Agents make very small forecasting mistakes
- Near aggregation - the evolution of prices and aggregate variables depends nearly entirely on aggregate shock and aggregate capital.

Near Aggregation: Notes

- Why does a linear forecasting rule do such a good job?
  - Policy functions are close to linear except at very low wealth levels!
  - If markets are complete:

$$
\begin{aligned}
a' &= b_z^0 + b_z^1 a + b_z^2 \varepsilon \\
K' &= \int_{A\times E} a' d\lambda \\
K' &= \int_{A\times E} b_z^0 + b_z^1 a + b_z^2 d\lambda \\
K' &= b_z^0 + b_z^1 K
\end{aligned}
$$

  - However, markets aren't complete, so how do we interpret this?

Near Aggregation: Notes

- Intuition: Agents smooth consumption effectively through self-insurance via the risk-free bond in this economy, resulting in a near-constant MPC and near-linear saving policy.

- Example: Yaari (1976)
  - Set $\beta = 1$, $r = 0$.
  - Agents live T periods, face iid shocks each period.
  - $\lim_{T\to\infty} c_t/a_t = k$, where k is the same as an agent in an economy with no risk
  - Take for instance quadratic preferences:

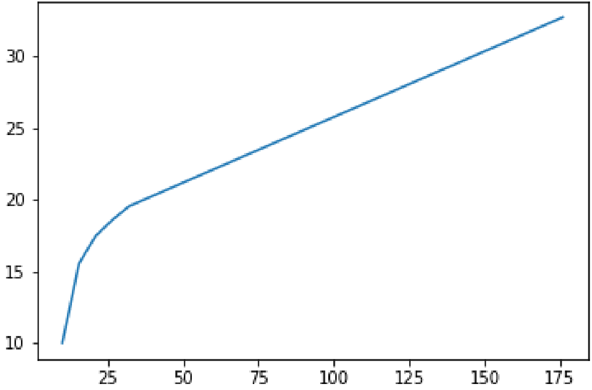
$$c_t = \frac{r}{1+r}\left[a_t + \mathbb{E}_0 \sum_{j=0}^{\infty}\left((1+r)^{-j} y_{t+j}\right)\right]$$

$$\begin{aligned} a_{t+1} &= (1+r)(a_t - c_t + y_t) \\ &= a_t + \mathbb{E}_0 \sum_{j=1}^{\infty}\left((1+r)^{-j} y_{t+j}\right) \end{aligned}$$

Near Aggregation: Notes

- Same intuition holds in this economy:
- Savings policies very curved for low wealth levels, near linear away from borrowing constraint.
- However, agents near borrowing constraint by definition have very little saving, and thus contribute little to aggregate capital stock.
- Aggregate shocks under the Krusell Smith calibration move wealth distribution only slightly, and thus most agents always remain in the part of the state space where policy functions are near linear.

Consumption Function Example



Near Aggregation

Open question:

Can anyone think of any additions/adjustment to the model which might break this result?

Other Approaches to Solving Model

- Krueger-Kubler (2004) — Smolyak collocation approach.
  - Uses a finite number of agents and maintain computational efficiency. Choose polynomial approximation terms and grid points such that computational complexity does not grow exponentially with states.
- Reiter (2010) — linearizes the aggregate state and formulates a linear law of motion for it.
  - Non-iterative, building on linearization.
  - Solves nonlinear individual's problem in steady state
  - Parameterizes the decision rules. All these are then allowed to depend on an aggregate state vector.
  - Movements in aggregate state, and dependence of individuals' parameters on aggregate state, captured by linear system.
- Not covered, but be aware of these other solution approaches!