VANDERBILT UNIVERSITY School of Engineering

Discrete Structures

CS 2212

(Fall 2020)

9 – Sets

Chapter 3

Sets

- So far, we have looked at **Logic and Proof techniques**?
- Then, we applied this machinery to proving statements involving **numbers**.

- Next, we will see another extremely important mathematical object, that is **Sets**.
- What are **sets, set-theoretic operations**, and other **relevant ideas**?

Sets

Set: A set is a collection of objects.

Element: The objects in a set are called elements.

Notation: A set with elements $x_1, \ldots, x_n$ is denoted as $\{x_1, \ldots, x_n\}$

$x \in S$ means x is a member of set S.

$x \notin S$ means x is *not* a member of set S.

N = { 2, 4, 6, 10 }

F = { Watermelon, Strawberry, Banana }

M = { 2, Strawberry, Monkey }

Sets

Empty set: A set with no element is empty set and denoted by ∅ .

Null set: The empty set is also referred to as the null set and can be denoted by { }.

What is {∅}? Is it an empty set?

Singleton: A set with a single element.

Finite set: A finite set has a finite number of elements.

Infinite set: An infinite set has an infinite number of elements.

Cardinality: The cardinality of a finite set A, denoted by |A|, is the number of elements in A.

Sets - Properties

Sets can also be described by **properties** which all of the elements satisfy.

If P is a property, then we use the expression $\{x \mid P\}$ to denote the set of all x that satisfy P.

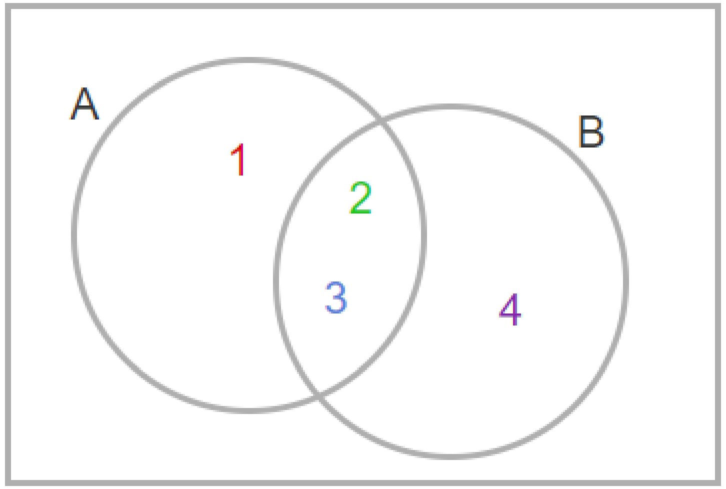
Example:

The set of odd natural numbers can be represented by either of the following.

- $x = \{1, 3, 5, \ldots\}$
- $\{x \mid x = 2k + 1 \text{ for some } k \in N\}$

Venn Diagram

Sets are often represented pictorially with **Venn diagrams**.



$A = \{1, 2, 3\}$

$1 \in A$ $4 \notin A$

$2 \in A$

$3 \in A$

$B = \{2, 3, 4\}$

Subsets

We say that set A is a **subset** of B, denoted $\boldsymbol{A \subseteq B}$, if every element of A is an element of B.

Examples:

- $\mathrm{N} \subseteq \mathrm{Z} \subseteq \mathrm{Q} \subseteq \mathrm{R}$
- $S \subseteq S$ for any set S
- $\emptyset \subseteq S$ for any set S

Subsets

Question: Let A = $\{2k + 7 \mid k \in Z\}$ and
B = $\{4k + 3 \mid k \in Z\}$.
Is A ⊆ B? Justify your answer.

Answer: **No**,

- By definition, in order for A ⊆ B, every value in A must also be in B.
- However, the $9 \in A$ (when k=1), but $9 \notin B$.
- The value $7 \in B$ (when k=1) and the next subsequent value $11 \in B$ (k=2).
- Therefore we can conclude that $9 \notin B$.

Subsets

Prove: If A = $\{2k + 7 \mid k \in Z\}$ and B = $\{4k + 3 \mid k \in Z\}$, then **B ⊆ A**

To show that $B \subseteq A$, we use a direct proof to demonstrate that if an element is in B, then that element must also be in A.

B ⊆ A

Let $x \in B$

$x = 4k + 3$

$x = 4k - 4 + 7$

$x = 2(2k - 2) + 7$

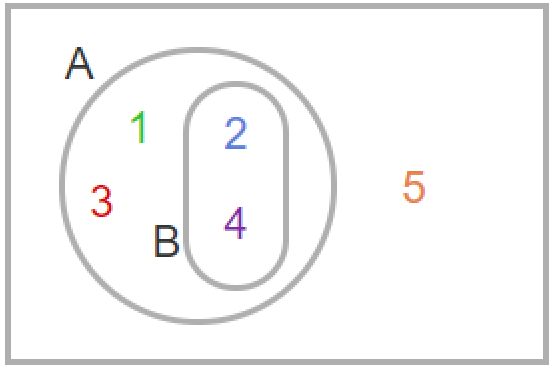
$(2k - 2) \in Z$ so $x \in A$

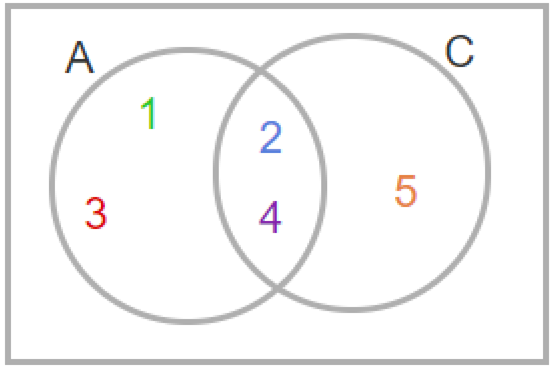
Therefore $B \subseteq A$

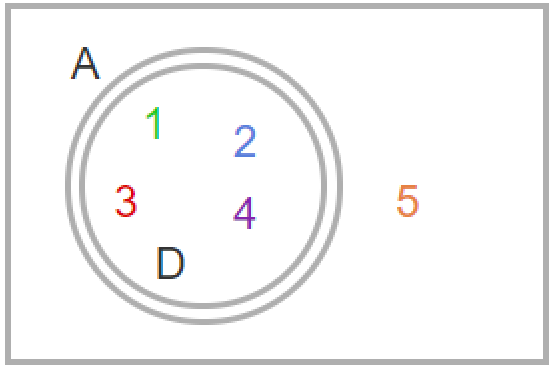
QED

Proper Subset

If $A \subseteq B$ and there is an element of B that is not an element of A (i.e., $A \neq B$), then A is a proper subset of B, denoted as **$A \subset B$**.







$A = \{1, 2, 3, 4\}$

$B = \{2, 4\}$

$B \subseteq A$

$3 \in A \quad 3 \notin B$

$A = \{1, 2, 3, 4\}$

$C = \{2, 4, 5\}$

$5 \in C \quad 5 \notin A$

$C \nsubseteq A$

$A = \{1, 2, 3, 4\}$

$D = \{1, 2, 3, 4\}$

$A \subseteq D, \quad D \subseteq A \Rightarrow A = D$

Equality of Sets

Set equality - We say that sets A and B are equal, that is **A = B**, if they have the same elements.

Order and duplicates do not matter when it comes to sets.

- {a, b, c} = {c, b, a}
- {a, a, b, c} = {a, b, c}

We can also say set A is **equal** to set B (i.e., A=B) **if and only if**

- $A \subseteq B$ and
- $B \subseteq A$

Equality of Sets

Prove: If A = {$2k + 5$ | $k \in Z$} and B = {$2k + 3$ | $k \in Z$}, then **A = B**

By definition, set A is equal to set B (i.e., A = B) iff $A \subseteq B$ and $B \subseteq A$.

$A \subseteq B$

Let $x \in A$

$x = 2k + 5$

$x = 2k + 2 + 3$

$x = 2(k + 1) + 3$

$k + 1 \in Z$ so $x \in B$

Therefore $A \subseteq B$

QED

$B \subseteq A$

Let $x \in B$

$x = 2k + 3$

$x = 2k + 3 + 2 – 2$

$x = 2k -2 + 5$

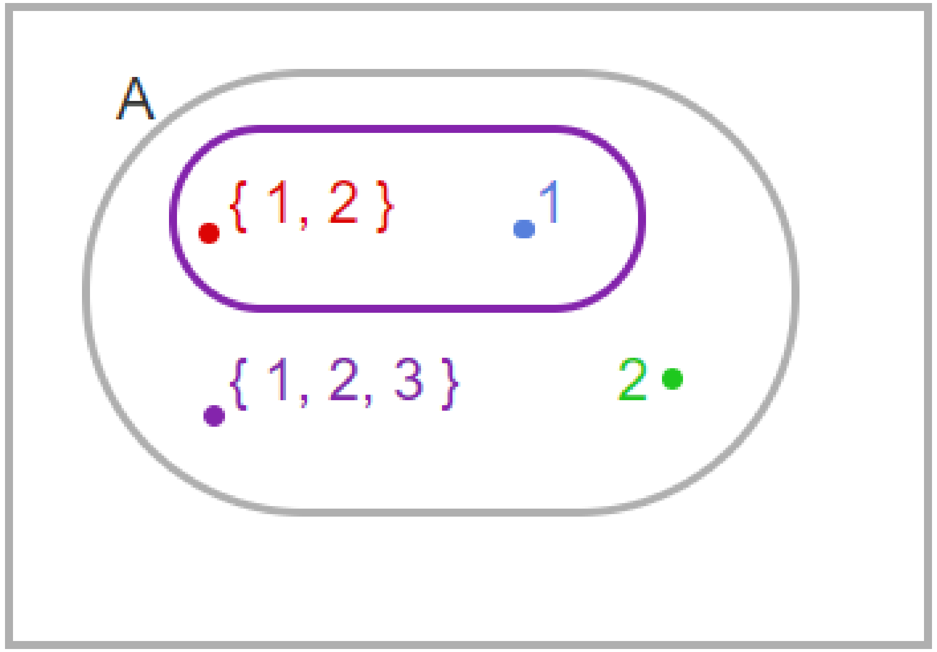
$x = 2(k - 1) + 5$

$k - 1 \in Z$ so $x \in A$

Therefore $B \subseteq A$

QED

Set of Sets



$A = \{ \{ 1, 2 \}, 1, 2, \{ 1, 2, 3 \} \}$

$\{ 1, 2 \} \in A$

$1 \in A$

$|A| = 4$

$2 \in A$

$\{ 1, 2, 3 \} \in A$

$\{ \{ 1, 2 \}, 1 \} \subset A$

Power Set

The power set of a set A, denoted **P(A)**, is the set of all subsets of A.

A = { ○ , □ , △ }

List all subsets:

size 0 ∅ ,

size 1 {○} , {□} , {△} ,

size 2 {○ , □} , {○ , △} , {□ , △} ,

size 3 {○ , □ , △}

P(A) = { ∅ , {○} , {□} , {△} , {○ , □} , {○ , △} , {□ , △} , {○ , □ , △} }

Set of Sets

Let A be a finite set of cardinality n. Then the cardinality of the power set of A is **$|P(A)| = 2^n$**

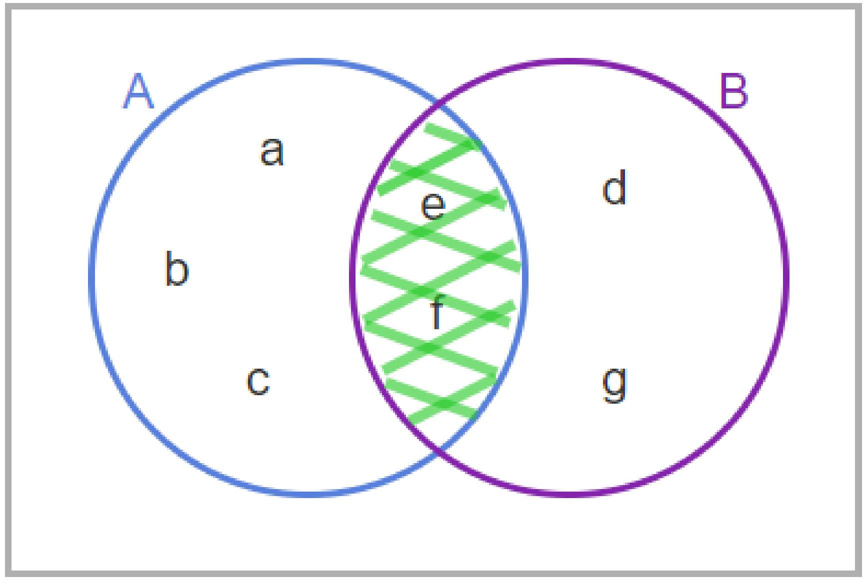
True or False?

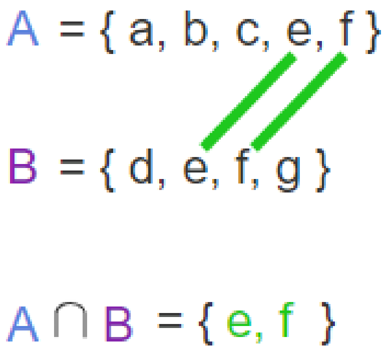
- $|P(X)| = 17$
- $|P(x)| = 0$

Intersection of Sets

The intersection of A and B, denoted **A ∩ B** is the set of all elements that are elements of both A and B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

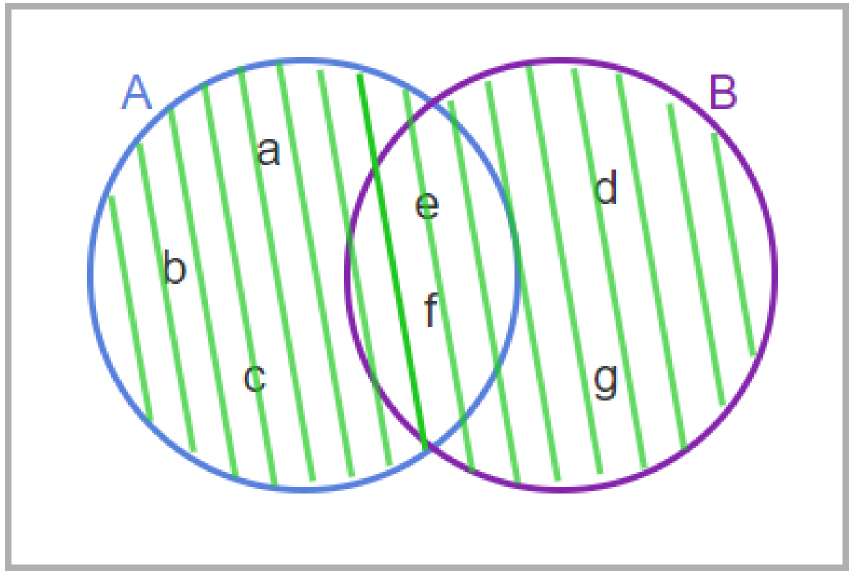




Union of Sets

The union of two sets, A and B, denoted **A ∪ B** is the set of all elements that are elements of A or B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



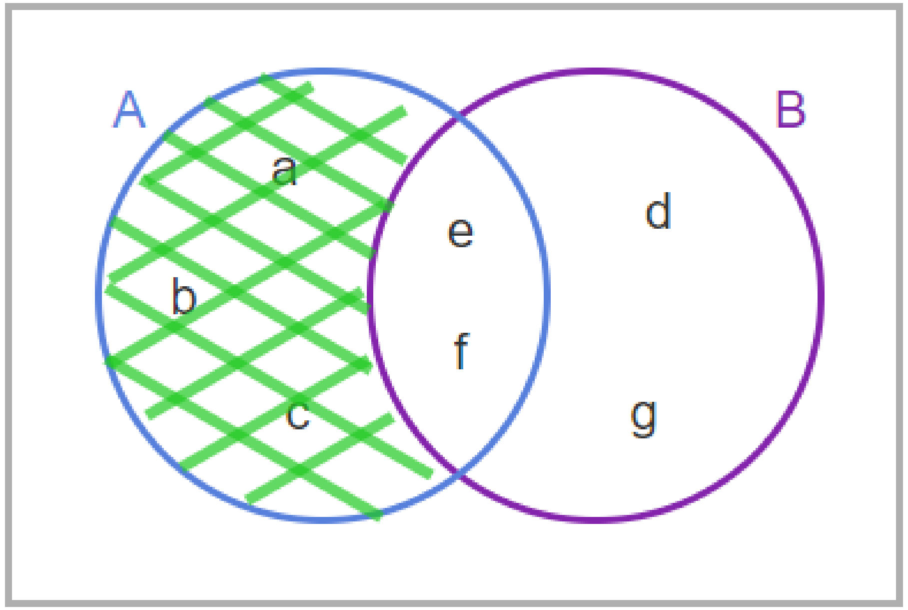
$A = \{a, b, c, e, f\}$
$B = \{d, e, f, g\}$

$A \cup B = \{a, b, c, e, f, d, g\}$

Difference Between Sets

The difference between two sets A and B, denoted **A – B**, is the set of elements that are in A but not in B.

$$\{x \mid x \in A \text{ and } x \notin B\}$$



A = { a, b, c, ~~e~~, ~~f~~}

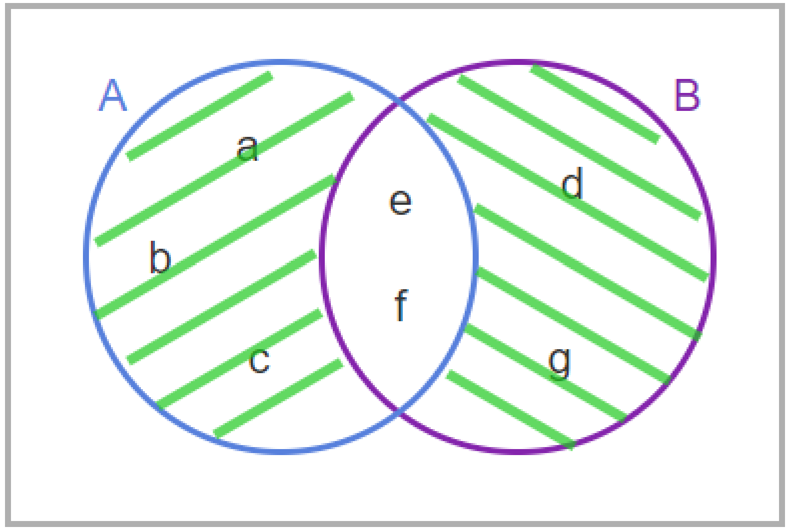
B = { d, e, f, g }

A - B ={ a , b , c }

Symmetric Difference Between Sets

The symmetric difference between two sets, A and B, denoted **A ⊕ B**, is the set of elements that are a member of exactly one of A and B, but not both.

{x | x ∈ A or x ∈ B but not both}



A = { a, b, c, ~~e~~, ~~f~~ }

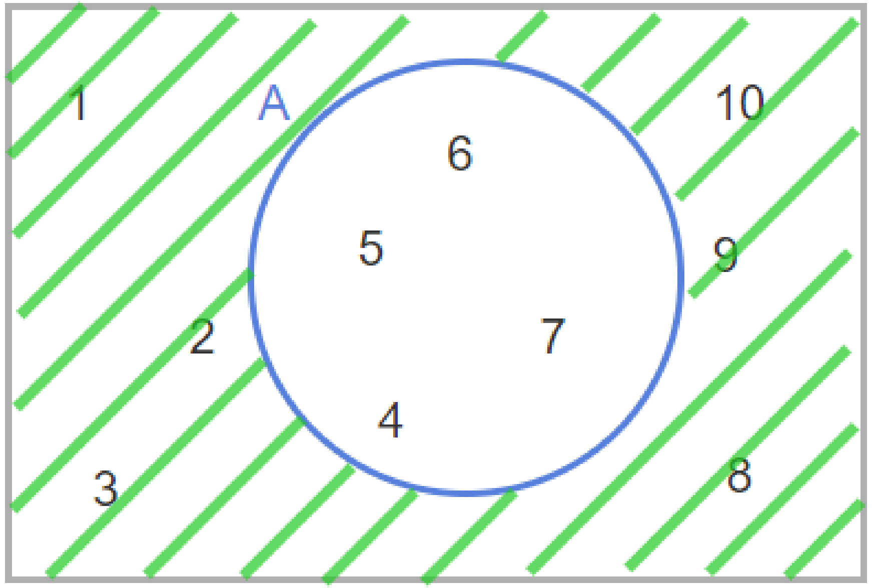
B = { d, ~~e~~, ~~f~~, g }

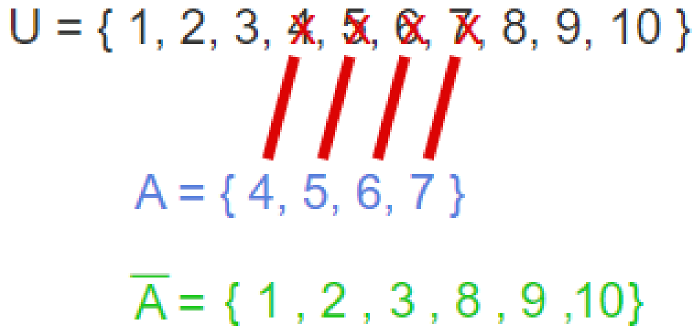
A ⊕ B = { a , b , c , d , g }

What is **A ⊕ A**? What is **A ⊕ A ⊕ A**?

Complement of a Set

Given a universe U and A ⊆ U, we write complement of A as **A' = U – A**.





Sets: Some More Proofs

Prove A = B

We show that

- $A \subseteq B$

AND

- $B \subseteq A$

Prove x ∈ (A ∩ B)

We show that

- $x \in A$

AND

- $x \in B$

Prove x ∈ (A ∪ B)

We show that

- $x \in A$

OR

- $x \in B$

Prove A ∩ B ≠ ∅

Find an element x s.t.

- $x \in A$

AND

- $x \in B$

Counting Sets: Inclusion and Difference Rules

When trying to determine the size (cardinality) of a set expression, there are a couple of helpful rules:

1. **Inclusion-Exclusion (aka Union) Rule**:

   $$|A \cup B| = |A| + |B| - |A \cap B|$$

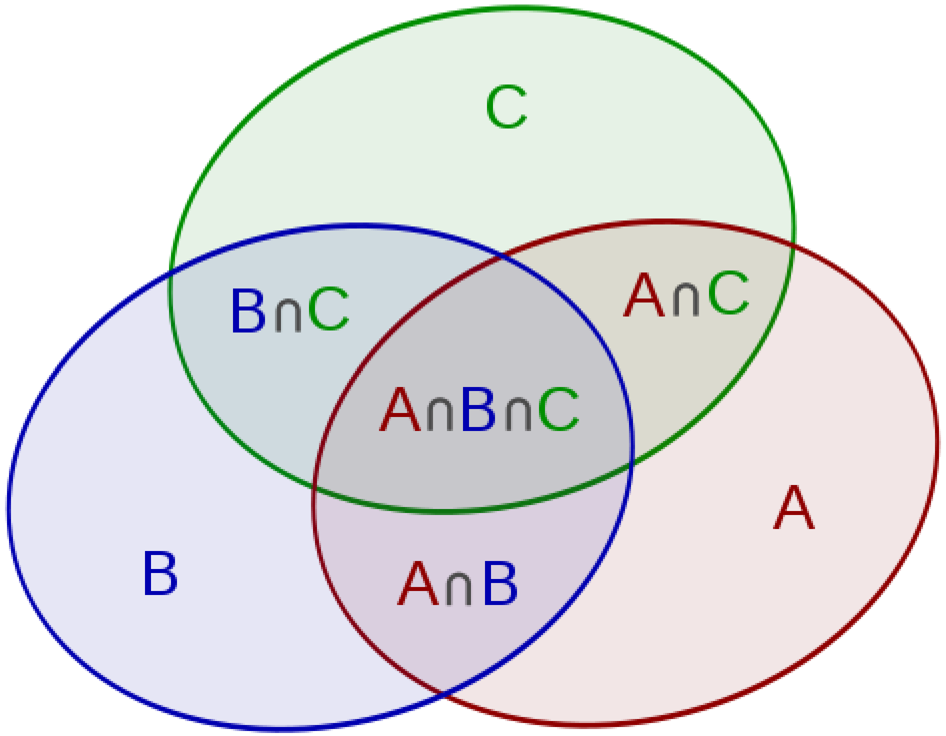
2. **Difference Rule:**

   $$|A - B| = |A| - |A \cap B|$$

Inclusion-Exclusion with 3 Sets

For finite sets A, B and C:

$$
\begin{aligned}
&|A \cup B \cup C| \\
=&|A| + |B| + |C| \\
-&|A \cap B| - |A \cap C| - |B \cap C| \\
+&|A \cap B \cap C|
\end{aligned}
$$



Inclusion-Exclusion with 3 Sets

$$|\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}|$$

$$= |A \cup (B \cup C)|$$

$$= |A| + |B \cup C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$$

Inclusion-Exclusion with 3 Sets

$$|\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}|$$

$$= |A \cup (B \cup C)|$$

$$= |A| + |B \cup C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$$

Inclusion-Exclusion with 3 Sets

$$|A \cup B \cup C|$$

$$= |A \cup (B \cup C)|$$

$$= |A| + |B \cup C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$

Inclusion-Exclusion with 3 Sets

$$|A \cup B \cup C|$$

$$= |A \cup (B \cup C)|$$

$$= |A| + |B \cup C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$

Lets, compute

$$|(A \cap B) \cup (A \cap C)| = |A \cap B| + |A \cap C| - |A \cap B \cap C|$$

Plugging it back in the above equation.

$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

Examples

- Let's say for each n ∈ ℕ let $\mathbf{D_n = \{\mathit{x} \in \mathbb{N} \mid \mathit{x} \text{ divides } n\}}$.

  (In other words, D_n *is the set of positive divisors of n*.)

- Using this definition, some examples of the previously mentioned set expressions are:

  $D_5 = \{1, 5\}$, $D_6 = \{1, 2, 3, 6\}$, and $D_9 = \{1, 3, 9\}$

  $D_5 \cup D_6 = \{1, 2, 3, 5, 6\}$

  $D_5 \cap D_6 = \{1\}$

  $D_9 - D_6 = \{9\}$

  $D_5 \oplus D_6 = \{2, 3, 5, 6\}$

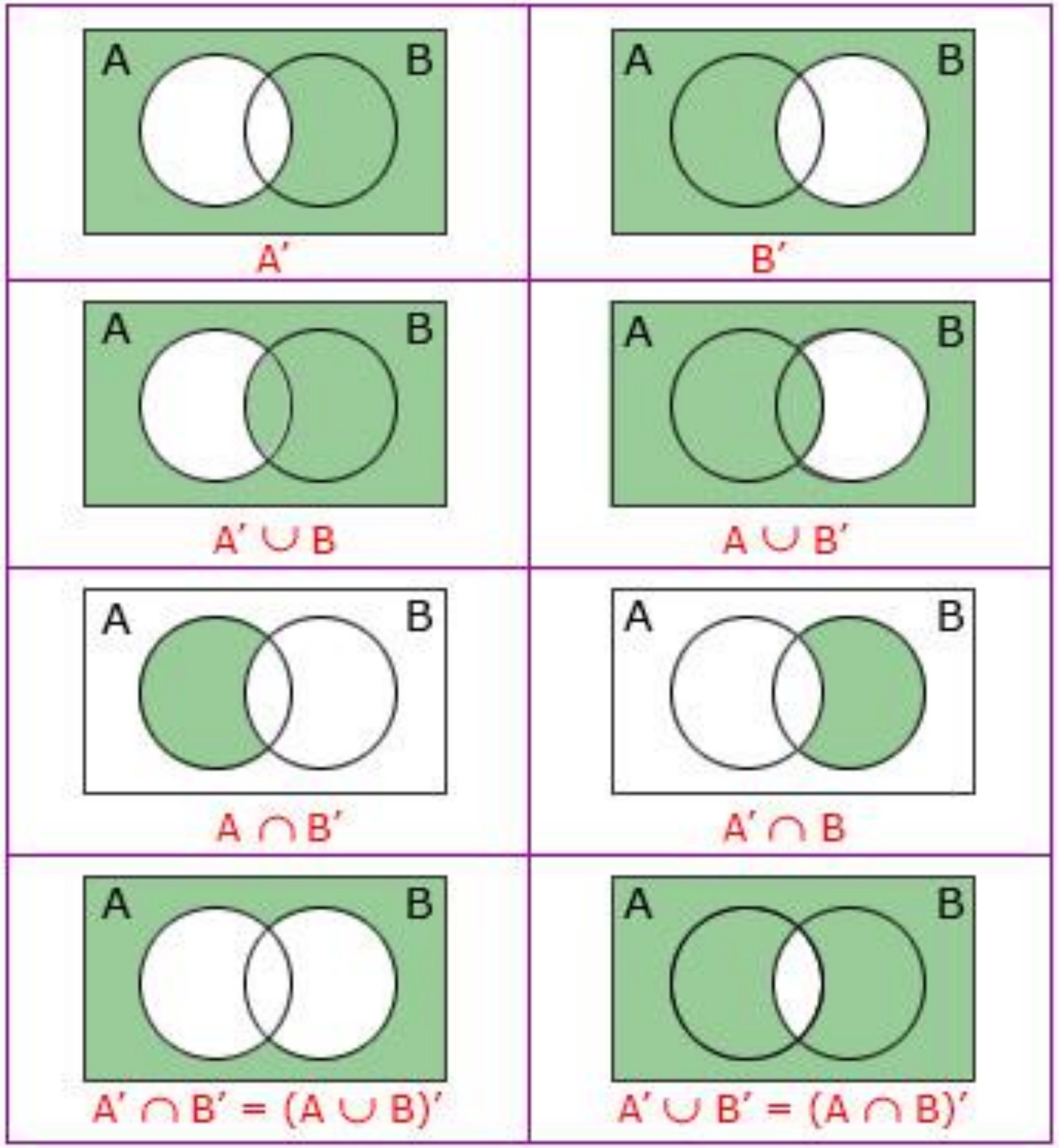
Set Identities

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Double Complement law	$A'' = A$	
Complement laws	$A \cap A' = \emptyset$ $U' = \emptyset$	$A \cup A' = U$ $\emptyset' = U$
De Morgan's laws	$(A \cup B)' = A' \cap B'$	$(A \cap B)' = A' \cup B'$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

Set Identities

$x \in \overline{A \cap B} \longleftrightarrow \neg(x \in A \cap B)$	Definition of complement
$\longleftrightarrow \neg(x \in A \wedge x \in B)$	Definition of intersection
$\equiv \neg(x \in A) \vee \neg(x \in B)$	De Morgan's law for proposition
$\longleftrightarrow x \in \overline{A} \vee x \in \overline{B}$	Definition of complement
$\longleftrightarrow x \in (\overline{A} \cup \overline{B})$	Definition of Union
$\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's set identity

DeMorgan's Law



Set Partitions

Partition: A partition of a non-empty set A is a collection of non-empty subsets of A such that:

each element of A is in exactly one of the subsets.

$\{ 1, 2, 3, a, b, \&, \$ \}$

Set Partitions

Partition: A partition of a non-empty set A is a collection of non-empty subsets of A such that:

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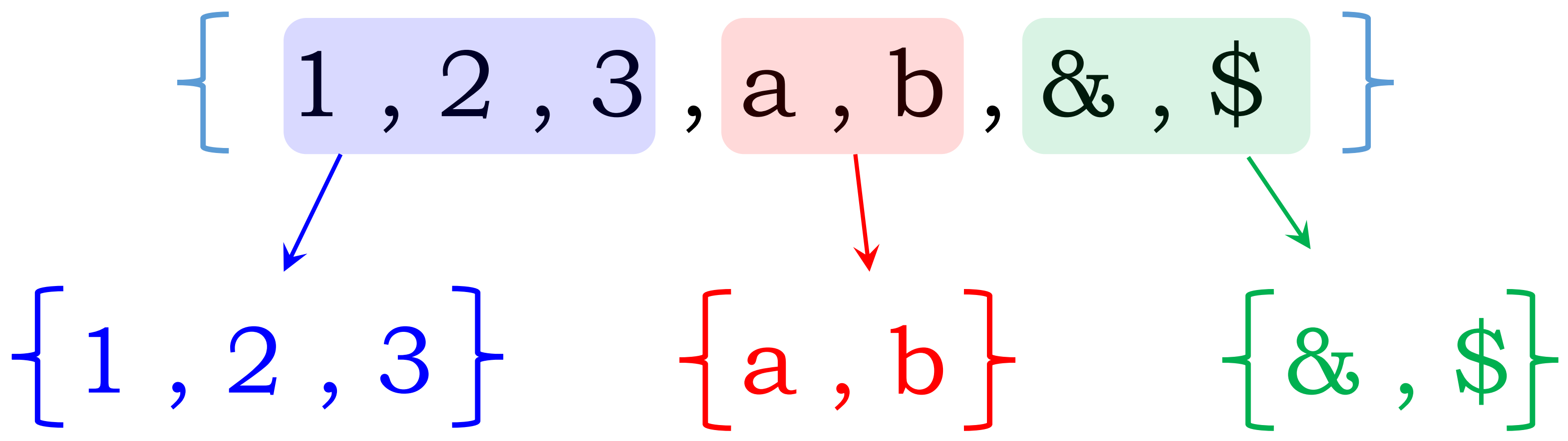


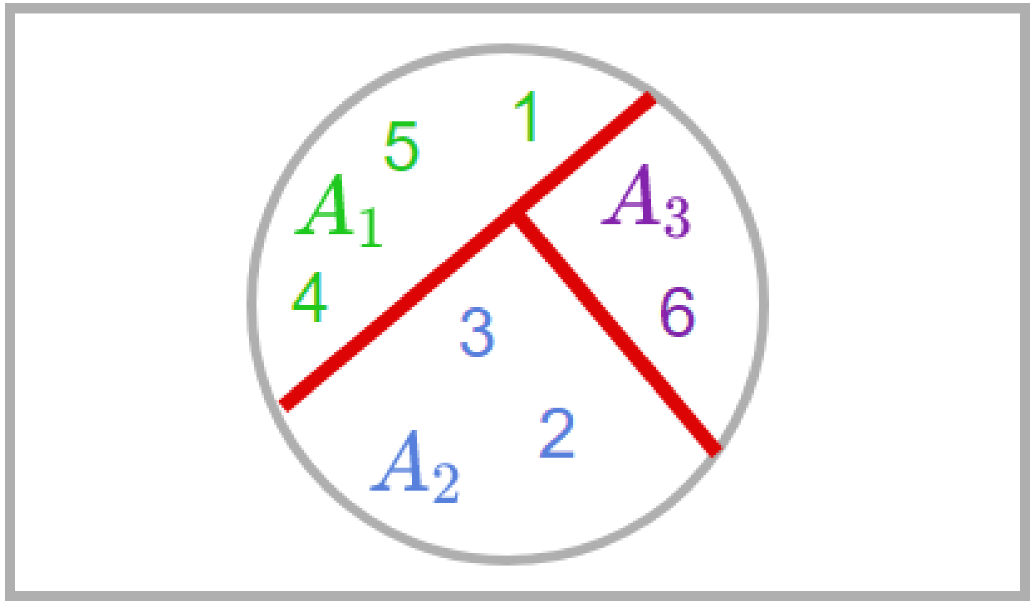
Set Partitions

In other words,

$A_1, A_2, \ldots, A_n$ is a **partition** of a non-empty subset A if

- $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots A_n = A$ (union)
- $A_i \cap A_j = \emptyset$, for all i ≠ j (disjointness)
- $A_i \neq \emptyset$, for all i. (non-emptiness)

Set Partitions (Example)



$A = \{ 1, 2, 3, 4, 5, 6 \}$

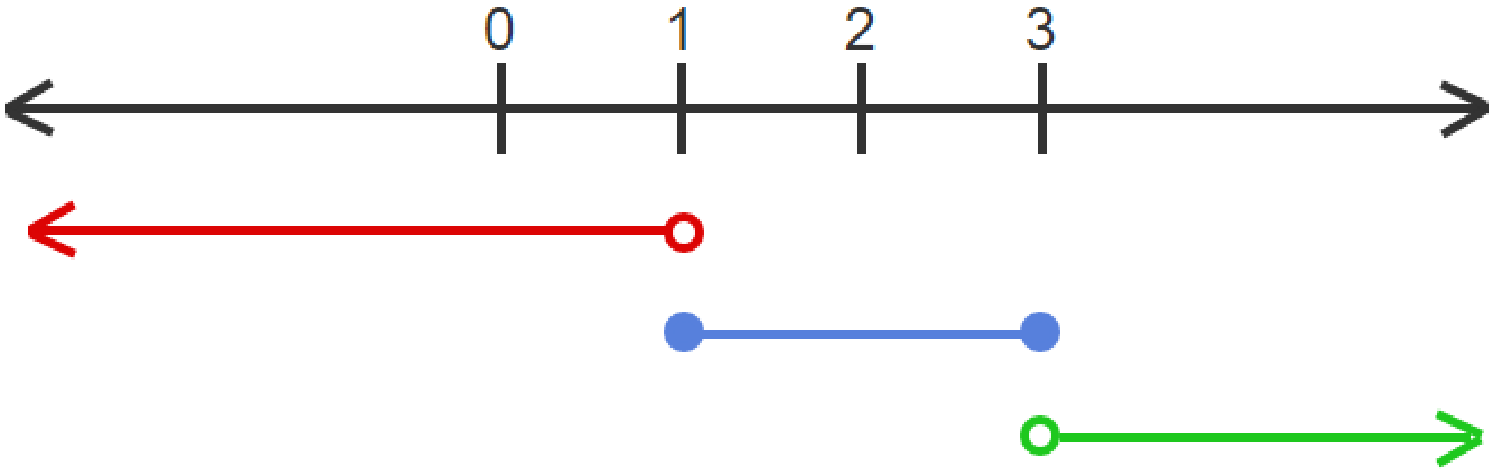
$A_1 = \{ 1, 4, 5 \}$

$A_2 = \{ 2, 3 \}$

$A_3 = \{ 6 \}$

A_1 A_2 and A_3 form a partition of A

Set Partitions (Example)



B_1 = { x ∈ R: x < 1}

B_2 = { x ∈ R: 1 ≤ x ≤ 3 }

B_3 = { x ∈ R: x > 3 }

B_1 B_2 and B_3 form a partition of R

Cartesian Product

Ordered Pair: An ordered pair of items is written (x, y).

Note that, (x, y) is not same as (y, x).

Order of terms matter here.

Cartesian Product: For two sets, A and B, the Cartesian product of A and B, denoted **$A \times B$**, is the set of all ordered pairs in which the first entry is in A and the second entry is in B.

Cartesian Product

A = {1, 2}, B = { a, b, c}

A × **B**

(1, a)	(1, b)	(1, c)
(2, a)	(2, b)	(2, c)

B × **A**

(a, 1)	(a, 2)
(b, 1)	(b, 2)
(c, 1)	(c, 2)