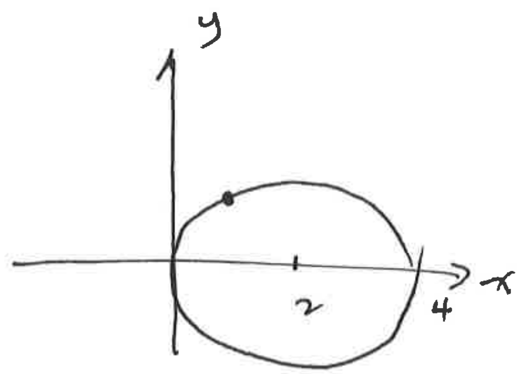
Calculus in Polar Coordinates

Derivatives

$x = r\cos\theta$ if $r = f(\theta)$ then $x = f(\theta)\cos\theta$

$y = r\sin\theta$ $y = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'\cos\theta - f\sin\theta}{f'\sin\theta + f\cos\theta}$$



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$r = 4\cos\theta \quad (2, \pi/3)$

$x = r\cos\theta = 2\cos\pi/3 = 2\frac{1}{2} = 1$

$y = r\sin\theta = 2\sin\pi/3 = 2\frac{\sqrt{3}}{2} = 1.7$

Now on curve $x = 4\cos\theta \cdot \cos\theta = 4\cos^2\theta$

$y = 4\cos\theta\sin\theta$

$$\frac{dy}{d\theta} = 4\cos^2\theta - 4\sin^2\theta$$

$$\frac{dx}{d\theta} = -8\cos\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{4(\cos^2\theta - \sin^2\theta)}{-8\cos\theta\sin\theta} = -\frac{\cos 2\theta}{\sin 2\theta}$$

$$\left.\frac{dy}{dx}\right|_p = -\frac{\cos 2\pi/3}{\sin 2\pi/3} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

Areas

Calc I

area of rectangle

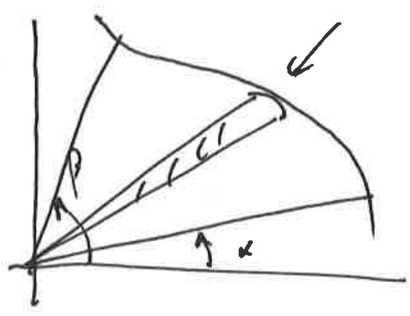
$f(x)\,dx$

height

thickness

$$A = \int_a^b f(x)\,dx$$ add up rectangles

Polar



thin slice approx. like a part of a circle

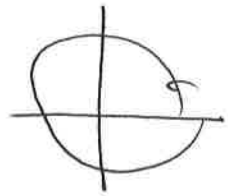
ratio $\frac{dA}{\pi r^2} = \frac{d\theta}{2\pi}$ dA - area of slice

$$\Rightarrow \quad dA = \tfrac{1}{2} r^2 d\theta$$

Now add up the slices

$$A = \int_\alpha^\beta \tfrac{1}{2} r^2 d\theta$$

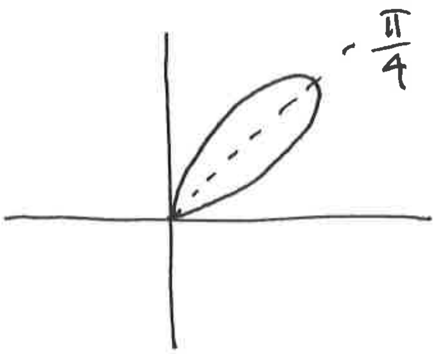
Ex 2 $r = a$ (circle)



$$A = \frac{1}{2}\int_0^{2\pi} a^2\,d\theta$$

$$= \frac{1}{2} a^2\theta \Big|_0^{2\pi} = \frac{1}{2} a^2 2\pi = \pi a^2 \quad (\text{like } \pi r^2)$$

Ex 3 $r = \sin 2\theta$ 1 leaf



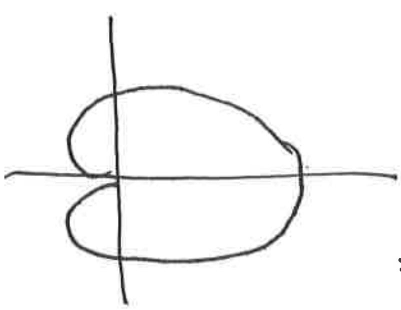
$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/4} \sin^2 2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= \frac{\theta}{2} - \frac{\sin 4\theta}{8}\Big|_0^{\pi/4} = \left(\frac{\pi}{8} - \frac{\sin \pi}{8}\right) - (0 - 0)$$

$$= \frac{\pi}{8}$$

Ex 4 $r = 1 + \cos\theta$



$$A = 2 \cdot \frac{1}{2}\int_0^{\pi} (1+\cos\theta)^2\, d\theta$$

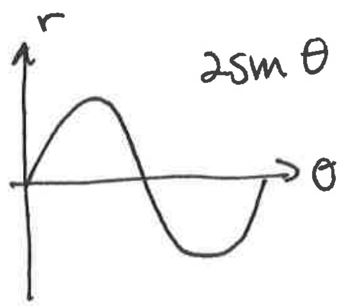
$$= \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

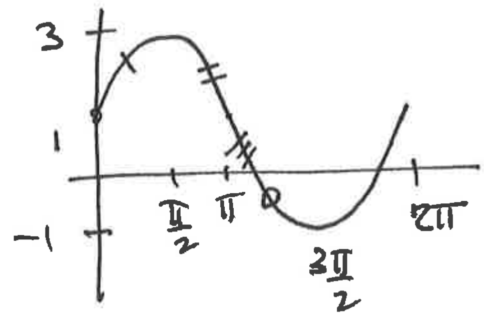
$$= \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta = \frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\Big|_0^{\pi}$$

$$= \frac{3\pi}{2}$$

ex5 Find the area of the inner loop of

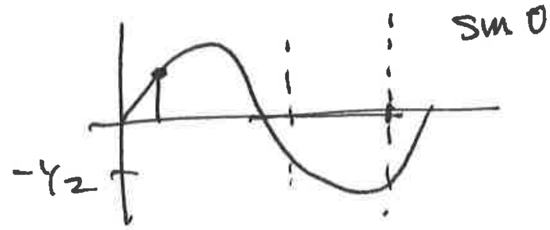
$$r = 1 + 2\sin\theta$$





Place where we cross the origin

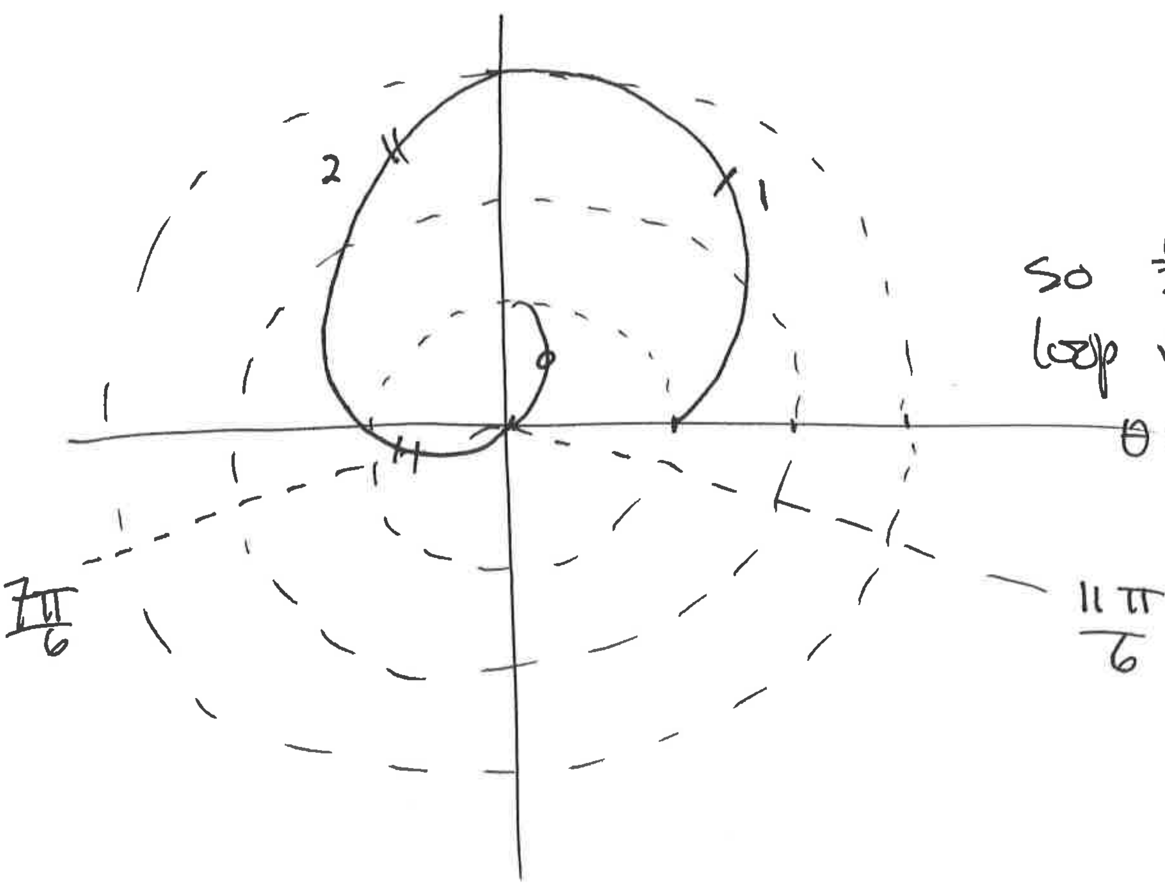
$1 + 2\sin\theta = 0 \quad \sin\theta = -1/2$



$\sin\theta = 1/2 \quad \theta = \pi/6$

$\sin\theta = -1/2 \quad \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

$= 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$



So $\frac{1}{2}$ of small loop is when

$\theta: \frac{7\pi}{6} - \frac{3\pi}{2}$

$$A = 2 \cdot \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} (1+2\sin\theta)^2 \, d\theta$$

$$= \pi - \frac{3\sqrt{3}}{2}$$

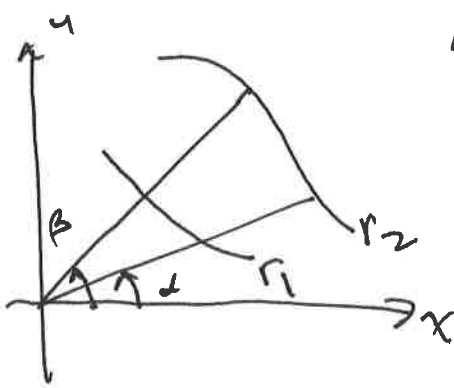
Outer Loop here $\theta : \pi/2 \to \frac{7\pi}{6}$

$$A = 2 \cdot \frac{1}{2} \int_{\pi/2}^{\frac{7\pi}{6}} (1+2\sin\theta)^2 \, d\theta$$

$$= 2\pi + \frac{3\sqrt{3}}{2}$$

Two Curve's

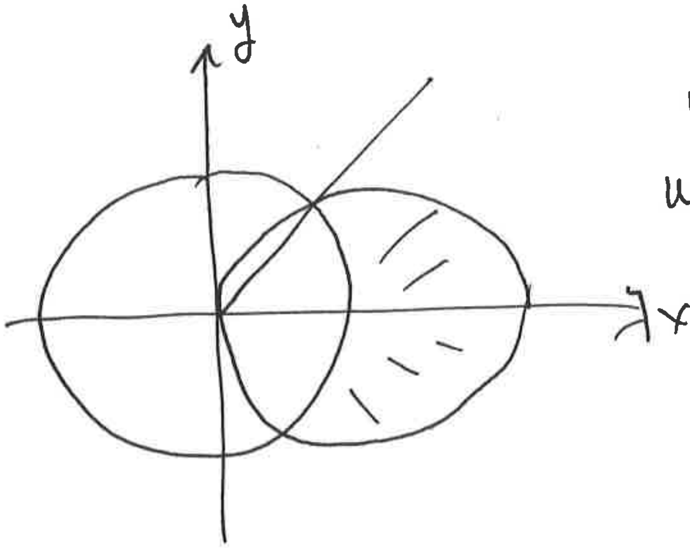
If we have 2 curves



$$A = \frac{1}{2}\int_{\alpha}^{\beta} r_2^2 \, d\theta - \frac{1}{2}\int_{\alpha}^{\beta} r_1^2 \, d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} (r_2^2 - r_1^2) \, d\theta$$

Ex Find the area outside $r=1$ and inside $r=2\cos\theta$



we will need to find where the curves intersect

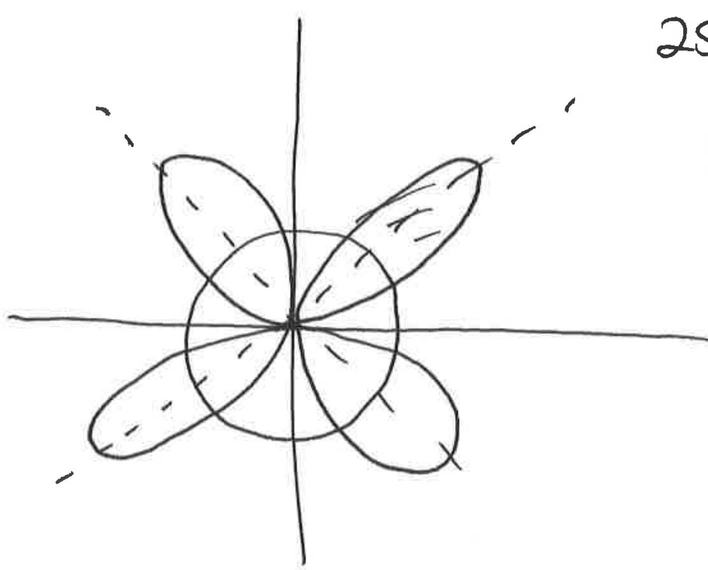
$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$ $\theta = \pi/3$

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/3} (2\cos\theta)^2 - 1 \, d\theta = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

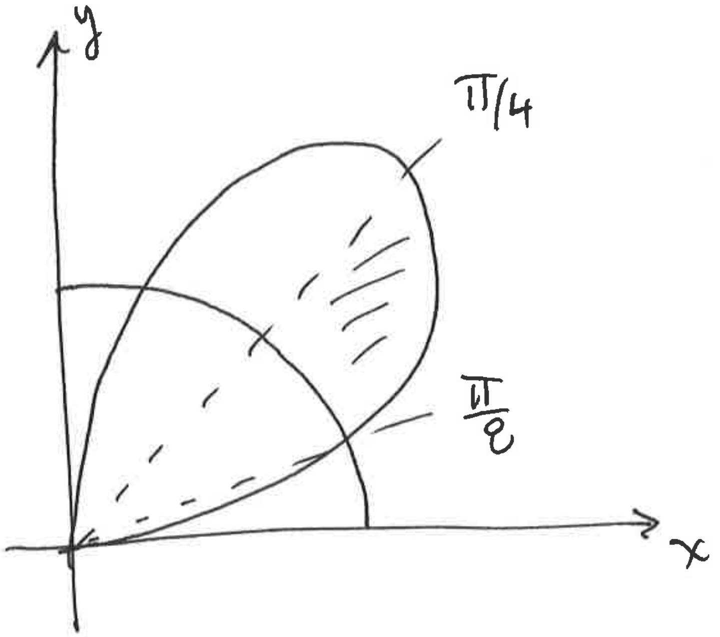
sum   outer curve   inner curve

Ex Inside $r = 2\sin 2\theta$ outside $r=1$



$2\sin 2\theta = 1$ $\sin 2\theta = 1/2$

$2\theta = \frac{\pi}{4}$ $\theta = \frac{\pi}{8}$



$$A = 8 \cdot \frac{1}{2} \int_{\pi/8}^{\pi/4} (2\sin 2\theta)^2 - 1 \, d\theta \;=$$

Finally, inside both $r = 1$, $r = 2\sin 2\theta$

$$A = \frac{1}{2} \int_0^{\pi/8} (2\sin 2\theta)^2 \, d\theta + \frac{1}{2} \int_{\pi/8}^{\pi/4} 1^2 \, d\theta$$

then multiply by 8

$$= \frac{3}{2}\pi - 2$$