Elementary ODE Review

1 First Order Equations

Ordinary differential equations of the form

$$y' = F(x, y) \tag{1}$$

are called first order ordinary differential equations. There are a variety of techniques to solve these type of equations and main methods are:

(i) separable
(ii) linear
(iii) Bernoulli
(iv) Ricatti
(v) homogeneous
(vi) linear fractional
(vii) exact
(viii) Legendre transformations

1.1 Separable Equations

A separable first order differential equation has the form:

$$\frac{dy}{dx} = f(x)g(y) \tag{2}$$

The general solution is found by separating the differential equation and integrating including a single constant of integration, *i.e.*

$$\int \frac{1}{g(y)} dy = \int f(x) dx + c.$$

For example, to solve

$$y' = xy + x + 2y + 2,$$

it is necessary to rewrite it as

$$y' = (x+2)(y+1).$$

Separating variables and writing in integral form gives

$$\int \frac{dy}{y+1} = \int x + 2\,dx,$$

and integrating yields

$$\ln|y+1| = \frac{x^2}{2} + 2x + c.$$

Letting $c = \ln(k)$ and solving for y gives

$$y = k \cdot e^{x^2/2+2x} - 1.$$

1.2 Linear Equations

Equations of this type are in the form

$$\frac{dy}{dx} + p(x)y = q(x). \tag{3}$$

To solve this, we introduce the integrating factor

$$\mu = e^{\int p(x)dx}. \tag{4}$$

This is created so that when both sides of (3) are multiplied by μ, the left side (3) is a derivative of a product, that is, it becomes

$$\mu\left(\frac{dy}{dx} + py\right) = \frac{d}{dx}(\mu y),$$

and then (3) can be integrated. For example, if

$$\frac{dy}{dx} - \frac{2y}{x} = 2x^3 - 1, \tag{5}$$

then $p(x) = -\frac{2}{x}$, so that

$$\mu = e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = \frac{1}{x^2}.$$

On multiplying (5) by μ gives

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 2x - \frac{1}{x^2},$$

which simplifies to

$$\frac{d}{dx}\left(\frac{1}{x^2}\cdot y\right) = 2x - \frac{1}{x^2}.$$

Integrating gives

$$\frac{1}{x^2}\cdot y = x^2 + \frac{1}{x} + c,$$

and solving for y gives

$$y = x^4 + x + cx^2.$$

1.3 Bernoulli

Equations of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad (n \neq 0, 1) \tag{6}$$

are called Bernoulli equations. Dividing both sides of (6) by y^n gives

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{p(x)}{y^{n-1}} = q(x) \tag{7}$$

Let $v = \frac{1}{y^{n-1}}$, then $\frac{dv}{dx} = (1-n)\frac{1}{y^n}\frac{dy}{dx}$ or $\frac{1}{(1-n)}\frac{dv}{dx} = \frac{1}{y^n}\frac{dy}{dx}$. Upon making this substitution into (7) gives

$$\frac{1}{1-n}\frac{dv}{dx} + p(x)v = q(x)$$

which is linear. So Bernoulli equations can be reduced to linear equations.

Example

Consider

$$\frac{dy}{dx} - \frac{y}{2x} = y^3.$$

This is an example of a Bernoulli equation where $n = 3$. Putting this into standard form gives

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{2x}\frac{1}{y^2} = 1 \tag{8}$$

Letting $v = \frac{1}{y^2}$, then $\frac{dv}{dx} = \frac{-2}{y^3}\frac{dy}{dx}$, and (8) is transformed to

$$-\frac{1}{2}\frac{dv}{dx} - \frac{1}{2x}v = 1,$$

or

$$\frac{dv}{dx} + \frac{v}{x} = -2.$$

As this is linear, then $p(x) = \frac{1}{x}$, and the integrating factor for this is x, so that

$$x\frac{dv}{dx} + v = \frac{d}{dx}(xv) = -2x,$$

and thus

$$xv = -x^2 + c,$$

or

$$v = -x + \frac{c}{x}.$$

So that

$$\frac{1}{y^2} = -x + \frac{c}{x},$$

or

$$y = \frac{\pm 1}{\sqrt{-x + \frac{c}{x}}}.$$

1.4 Ricatti Equations

Ricatti equations have the form:

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x). \tag{9}$$

To find a general solution to this requires having one solution first. Given this solution, it is possible to change equation (9) to a linear equation. If we let

$$y = y_0 + \frac{1}{u},$$

where y_0 is a solution to (9), then (9) is transformed to the linear equation

$$u' = -\left(2a(x)y_0 + b(x)\right)u - a(x),$$

which is linear. To illustrate, we consider the following example

$$\frac{dy}{dx} = -\frac{y^2}{x^2} + \frac{y}{x} + 1. \tag{10}$$

Since $y_0 = x$ is a solution to this equation, let $y = x + \frac{1}{u}$, and (10) becomes

$$1 - \frac{u'}{u^2} = -\frac{1}{x^2}\left(x^2 + 2\frac{x}{u} + \frac{1}{u^2}\right) + \frac{1}{x}\left(x + \frac{1}{u}\right) + 1.$$

Simplifying gives

$$-\frac{u'}{u^2} = -\frac{1}{xu} - \frac{1}{x^2u^2},$$

and multiplying by $-u^2$ and rearranging gives rise to the linear equation

$$u' - \frac{1}{x}u = \frac{1}{x^2}. \tag{11}$$

Here $p(x) = -\dfrac{1}{x}$ so this has the integrating factor $\mu = e^{\int \frac{-dx}{x}} = \frac{1}{x}$, so (11) becomes

$$\frac{u'}{x} - \frac{u}{x^2} = \frac{d}{dx}\left(\frac{u}{x}\right) = \frac{1}{x^3},$$

and upon integration gives

$$\frac{u}{x} = -\frac{1}{2x^2} + c,$$

or

$$u = cx - \frac{1}{2x}.$$

Since $y = x + \dfrac{1}{u}$, this gives y as

$$y = x + \frac{1}{cx - \dfrac{1}{2x}}.$$

1.5 Homogeneous Equations

Equations of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{12}$$

are called homogeneous equations. Substituting $y = xu$ will yield the equation

$$x\frac{du}{dx} + u = F(u).$$

which separates to

$$\frac{du}{F(u) - u} = \frac{dx}{x}.$$

Consider the previous example,

$$\frac{dy}{dx} = -\frac{y^2}{x^2} + \frac{y}{x} + 1. \tag{13}$$

If we let $y = xu$, then (13) becomes

$$\frac{d(xu)}{dx} = -u^2 + u + 1,$$

or

$$x\frac{du}{dx} + u = -u^2 + u + 1$$

and simplifying and separating gives

$$\frac{du}{1-u^2} = \frac{dx}{x}.$$

Integrating gives

$$\frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| = \ln|x| + \frac{1}{2}\ln c,$$

or

$$\frac{u+1}{u-1} = cx^2.$$

Since $y = xu$ this gives

$$\frac{\frac{y}{x}+1}{\frac{y}{x}-1} = cx^2,$$

or

$$\frac{y+x}{y-x} = cx^2,$$

solving for y leads to the solution

$$y = \frac{cx^3 + x}{cx^2 - 1}.$$

1.6 Linear Fractional

Equations that have the form

$$\frac{dy}{dx} = \frac{ax + by + e}{cx + dy + f}, \tag{14}$$

are called linear fractional. Under a change of variables,

$$x = \bar{x} + \alpha, \;\; y = \bar{y} + \beta,$$

we can change equation (14) to one that is either homogeneous (if $ad - bc \neq 0$) or to one that is separable (if $ad - bc = 0$). The following examples illustrate.

Consider

$$\frac{dy}{dx} = \frac{2x - 3y + 8}{3x - 2y + 7}. \tag{15}$$

If we let

$$x = \bar{x} + \alpha, \;\; y = \bar{y} + \beta,$$

then (15) becomes

$$\frac{d\bar{y}}{d\bar{x}} = \frac{2\bar{x} - 3\bar{y} + 2\alpha - 3\beta + 8}{3\bar{x} - 2\bar{y} + 3\alpha - 2\beta + 7}.$$

Choosing

$$2\alpha - 3\beta + 8 = 0, \;\; 3\alpha - 2\beta + 7 = 0, \tag{16}$$

leads to

$$\frac{d\bar{y}}{d\bar{x}} = \frac{2\bar{x} - 3\bar{y}}{3\bar{x} - 2\bar{y}}, \tag{17}$$

a homogeneous equation. The natural question is, "does (16) have a solution?" In this case, it does and we can find that the solution is $\alpha = -1$ and $\beta = 2$. The solution of (17) is

$$\bar{x}^2 - 3\bar{x}\bar{y} + \bar{y}^2 = c \tag{18}$$

and from our change of variables ($x = \bar{x} - 1, \;\; y = \bar{y} + 2$) we find the solution to (15) is

$$(x+1)^2 - 3(x+1)(y-2) + (y-2)^2 = c. \tag{19}$$

As a second example, consider

$$\frac{dy}{dx} = \frac{4x - 2y + 8}{2x - y + 7}. \tag{20}$$

If we let

$$x = \bar{x} + \alpha, \;\; y = \bar{y} + \beta,$$

then

$$\frac{d\bar{y}}{d\bar{x}} = \frac{4\bar{x} - 2\bar{y} + 4\alpha - 2\beta + 8}{2\bar{x} - \bar{y} + 2\alpha - \beta + 7}.$$

We choose

$$4\alpha - 2\beta + 8 = 0, \;\; 2\alpha - \beta + 7 = 0,$$

but this has no solution! However, if we let $u = 2x - y$, then (20) becomes

$$\frac{du}{dx} = \frac{6}{u+7},$$

which is separable! Its solution is given by

$$u^2 + 14u = 12x + c,$$

and upon back substitution, we obtain the solution of (20) as

$$(2x - y)^2 + 14(2x - y) = 12x + c,$$

1.7 Exact Equations

An ordinary differential equation of the form

$$\frac{dy}{dx} = F(x, y), \tag{21}$$

has the alternate form

$$M(x, y)dx + N(x, y)dy = 0. \tag{22}$$

If M and N have continuous partial derivatives of first order in some region R and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then the ODE (22) is said to be "exact" and can be integrated by setting

$$\frac{\partial \phi}{\partial x} = M, \text{ and } \frac{\partial \phi}{\partial y} = N.$$

For example, consider the differential equation

$$\frac{dy}{dx} = -\frac{2xy}{x^2 + y^2}, \tag{23}$$

which can be written as

$$2xy\,dx + (x^2 + y^2)dy = 0. \tag{24}$$

If we identify that M and N are

$$M = 2xy \text{ and } N = x^2 + y^2,$$

so

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x},$$

so (24) is exact. Setting

$$\frac{\partial \phi}{\partial x} = 2xy, \text{ and } \frac{\partial \phi}{\partial y} = x^2 + y^2,$$

and integrating the first gives

$$\phi = x^2 y + g(y),$$

taking the partial of this with respect to y gives

$$\frac{\partial \phi}{\partial y} = x^2 + g'(y).$$

Comparing this to $\frac{\partial \phi}{\partial y} = x^2 + y^2$ gives that

$$g'(y) = y^2,$$

so

$$g(y) = \frac{y^3}{3} + c,$$

so that

$$\phi = x^2 y + \frac{y^3}{3} + c.$$

From Cal III we know that

$$d\phi = \phi_x dx + \phi_y dy,$$

but in this case this is

$$d\phi = 2xydx + (x^2 + y^2)dy = 0$$

so ϕ is a constant. Thus we have as solutions to

$$2xy\,dx + (x^2 + y^2)dy = 0$$

$$\phi = k \text{ or } x^2 y + \frac{y^3}{3} + c = k,$$

and absorbing the constant k into c gives

$$x^2 y + \frac{y^3}{3} + c = 0$$

as the set of possible solutions to (23).

1.7.1 Legendre Transformations

Sometimes it is necessary to solve more general equations of the form

$$F(x, y, y') = 0, \tag{25}$$

say, for example

$$y'^2 - xy' + 3y = 0. \tag{26}$$

One possibility is to introduce a contact transformation that enables one to solve a given equation. Contact transformations, in general, are of the form

$$x = F(X, Y, Y'), \quad y = G(X, Y, Y'), \quad y' = H(X, Y, Y'), \tag{27}$$

with the contact condition that

$$\frac{G_X + G_Y y' + G_{Y'} Y''}{F_X + F_Y y' + F_{Y'} Y''} = H.$$

One such contact transformation is called a Legendre transformation and is given by

$$x = \frac{dY}{dX}, \quad y = X\frac{dY}{dX} - Y, \quad y' = X. \tag{28}$$

One can verify that

$$\frac{dy}{dx} = \frac{\frac{d}{dX}\left(X\frac{dY}{dX} - Y\right)}{\frac{d}{dX}\left(\frac{dY}{dX}\right)} = \frac{X\frac{d^2Y}{dX^2}}{\frac{d^2Y}{dX^2}} = X.$$

Substitution of (28) in (26) gives

$$2X\frac{dY}{dX} - 3Y + X^2 = 0 \tag{29}$$

a linear ODE! Solving gives

$$Y = CX^{\frac{3}{2}} - X^2. \tag{30}$$

Substituting (30) back into (28) gives

$$x = \frac{3}{2}cX^{\frac{1}{2}} - 2X, \quad y = \frac{1}{2}cX^{\frac{3}{2}} - X^2. \tag{31}$$

Solving the first of (31) for $X^{\frac{1}{2}}$ gives

$$X^{\frac{1}{2}} = \frac{3c \pm \sqrt{9c^2 + 32x}}{8}, \tag{32}$$

and from the second of (31) gives

$$y = \frac{c}{2}\left(\frac{3c \pm \sqrt{9c^2 + 32x}}{8}\right)^3 - \left(\frac{3c \pm \sqrt{9c^2 + 32x}}{8}\right)^4,$$

the exact solution of (26).

2 Linear Systems

A linear system of equations

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy, \tag{33}$$

can be can be written as a matrix ODE

$$\frac{d\bar{x}}{dt} = A\bar{x} \tag{34}$$

where $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\bar{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we consider solutions of the form

$$\bar{x} = \bar{c}e^{\lambda t},$$

then after substitution into (34) we obtain

$$\lambda \bar{c}\, e^{\lambda t} = A\, \bar{c}\, e^{\lambda t}$$

from which we deduce

$$(\lambda I - A)\, \bar{c} = 0. \tag{35}$$

In order to have nontrivial solutions $\bar{c}$, we require that

$$|\lambda I - A| = 0. \tag{36}$$

This is the eigenvalue-eigenvector problem. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then (36) becomes

$$\lambda^2 - (a+d)\lambda + a\,d - b\,c = 0,$$

from which we have three cases:

(i) two distinct eigenvalues

(ii) two repeated eigenvalues,

(iii) two complex eigenvalues.

Here we consider an example of the first, two distinct eigenvalues. If

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \bar{x} \tag{37}$$

then the characteristic equation is

$$\begin{vmatrix} \lambda - 1 & -1 \\ -2 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 2$.

Case 1: $\lambda = -1$

From (35) we have

$$\begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain after expanding $2c_1 + c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Case 2: $\lambda = 2$

From (35) we have

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain after expanding $c_1 - c_2 = 0$ and we deduce the eigenvector

$$\bar{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution to (37) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-\mathrm{t}} + \mathrm{c}_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{2\mathrm{t}}.$$