CAP 4630
Artificial Intelligence

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- http://www.ultimateaiclass.com/
- https://moodle.cis.fiu.edu/
- HW1 out 9/5 today, due today
  - Remember that you have up to 4 late days to use throughout the semester.
- HW2 out this week, due 10/17
- Midterm on 10/19
  - Review during half of class on 10/17

Kuka

Robot-soccer

Robot Soccer Goes Big Time

Adversarial search

- https://www.youtube.com/watch?v=tIIJME8-au8
- https://www.youtube.com/watch?time_continue=8&v=pDTjxlANyxs
- https://www.youtube.com/watch?v=N8bqzjZvSmc
- http://www.cs.cmu.edu/~mmv/papers/07aaai-colin.pdf
  - "Timed" zero-sum games, Markov decision processes

Adversarial search

- This 8×8 variant of draughts (checkers) was **weakly solved** on April 29, 2007 by the team of Jonathan Schaeffer, known for Chinook, the "World Man-Machine Checkers Champion." From the standard starting position, both players can guarantee a draw with perfect play. Checkers is the largest game that has been solved to date, with a search space of 5×10^{20}. The number of calculations involved was 10^{14}, which were done over a period of 18 years. The process involved from 200 desktop computers at its peak down to around 50.

Weakly vs. strongly solved

- **Weak**: Provide an algorithm that secures a win for one player, or a draw for either, against any possible moves by the opponent, from the beginning of the game. That is, produce at least one complete ideal game (all moves start to end) with proof that each move is optimal for the player making it. It does not necessarily mean a computer program using the solution will play optimally against an imperfect opponent. For example, the checkers program Chinook will never turn a drawn position into a losing position (since the weak solution of checkers proves that it is a draw), but it might possibly turn a winning position into a drawn position because Chinook does not expect the opponent to play a move that will not win but could possibly lose, and so it does not analyze such moves completely.

Weakly vs. strongly solved

- **Strong**: Provide an algorithm that can produce perfect moves from any position, even if mistakes have already been made on one or both sides.

- **Ultra-weak**: Prove whether the first player will win, lose or draw from the initial position, given perfect play on both sides. This can be a non-constructive proof (possibly involving a strategy-stealing argument) that need not actually determine any moves of the perfect play.

Connect Four

- Solved first by James D. Allen (Oct 1, 1988), and independently by Victor Allis (Oct 16, 1988). First player can force a win. Strongly solved by John Tromp's 8-ply database (Feb 4, 1995). Weakly solved for all boardsizes where width+height is at most 15 (as well as 8×8 in late 2015) (Feb 18, 2006).
- The artificial intelligence algorithms able to strongly solve Connect Four are minimax or negamax, with optimizations that include alpha-beta pruning, move ordering, and transposition tables.

Connect Four

- The solved conclusion for Connect Four is first player win. With perfect play, the first player can force a win, on or before the 41st move (ply) by starting in the middle column. The game is a theoretical draw when the first player starts in the columns adjacent to the center. For the edges of the game board, column 1 and 2 on left (or column 7 and 6 on right), the exact move-value score for first player start is loss on the 40th move, and loss on the 42nd move, respectively. In other words, by starting with the four outer columns, the first player allows the second player to force a win.

2-player limit Hold'em poker is solved (Science 2015)

Heads-up Limit Hold'em Poker is Solved

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Poker is a family of games that exhibit imperfect information, where players do not have full knowledge of past events. Whereas many perfect information games have been solved (e.g., Connect Four and checkers), no nontrivial imperfect information game played competitively by humans has previously been solved. Here, we announce that heads-up limit Texas hold'em is now essentially weakly solved. Furthermore, this computation formally proves the common wisdom that the dealer in the game holds a substantial advantage. This result was enabled by a new algorithm, CFR$^+$, which is capable of solving extensive-form games orders of magnitude larger than previously possible.

Heads-up Limit Hold 'em Poker is Solved

- Play against Cepheus here http://poker-play.srv.ualberta.ca/

Poker

- **Abstract:** Poker is a family of games that exhibit imperfect information, where players do not have full knowledge of past events. Whereas many perfect-information games have been solved (e.g., Connect Four and checkers), no nontrivial imperfect-information game played competitively by humans has previously been solved. Here, we announce that heads-up limit Texas hold’em is now **essentially weakly solved**. Furthermore, this computation formally proves the common wisdom that the dealer in the game holds a substantial advantage. This result was enabled by a new algorithm, CFR^+, which is capable of solving extensive-form games orders of magnitude larger than previously possible.

Adversarial search

- We first consider games with two players, whom we call MAX and MIN. MAX moves first, and then they take turns moving until the game is over. At the end of the game, points are awarded to the winning player, and penalties given to the loser. A game can be formally defined as a kind of search problem with the following elements:

Search problem definition

- States
- Initial state
- Actions
- Transition model
- Goal test
- Path cost

Definition for 8-queens problem

- **States**: Any arrangement of 0 to 8 queens on the board is a state.
- **Initial state**: No queens on the board.
- **Actions**: Add a queen to any empty square.
- **Transition model**: Returns the board with a queen added to the specified square
- **Goal test**: 8 queens are on the board, none attacked
- **Path cost**: (Not applicable)

Game definition

- S_0: the **initial state**, which specifies how the game starts
- PLAYER(s): defines which player has the move in a state
- ACTIONS(s): Returns the set of legal moves in a state
- RESULT(s,a): The **transition model**, which defines the result of a move.
- TERMINAL-TEST(s): A **terminal** test, which is true when the game is over and false otherwise. States where the game has ended are called **terminal states**.
- UTILITY(s,p): A **utility function** (also called an objective function or payoff function), defines the final numeric value for a game that ends in terminal state s for a player p. In chess, the outcome is a win, loss, or draw, with values +1, 0, or ½. Some games have a wider variety of possible outcomes; the payoffs in backgammon range from 0 to +192.

Zero-sum games

- A **zero-sum** game is (confusingly) defined as one where the total payoff to all players is the same for every instance of the game.
- Is chess zero-sum?
- Checkers?
- Poker?

Zero-sum games

- Chess is zero-sum because every game has payoff of either 0 +1, 1+0, or ½ + ½
- "Constant-sum" would have been a better term, but zero-sum is traditional and makes sense if you imagine that each player is charged an entry fee of ½.

Game tree

- The initial state, ACTIONS function, and RESULT function define the **game tree** for the game—a tree where the nodes are game states and the edges are moves. The figure shows part of the game tree for tic-tac-toe. From the initial state, MAX has nine possible moves. Play alternates between MAX's placing an X and MIN's placing an O until we reach leaf nodes corresponding to terminal states such that one player has three in a row or all the squares are filled. The number on each leaf node indicates the utility value of the terminal state from the point of view of MAX; high values are assumed to be good for MAX and bad for MIN (which is how the players get their names).

Game trees

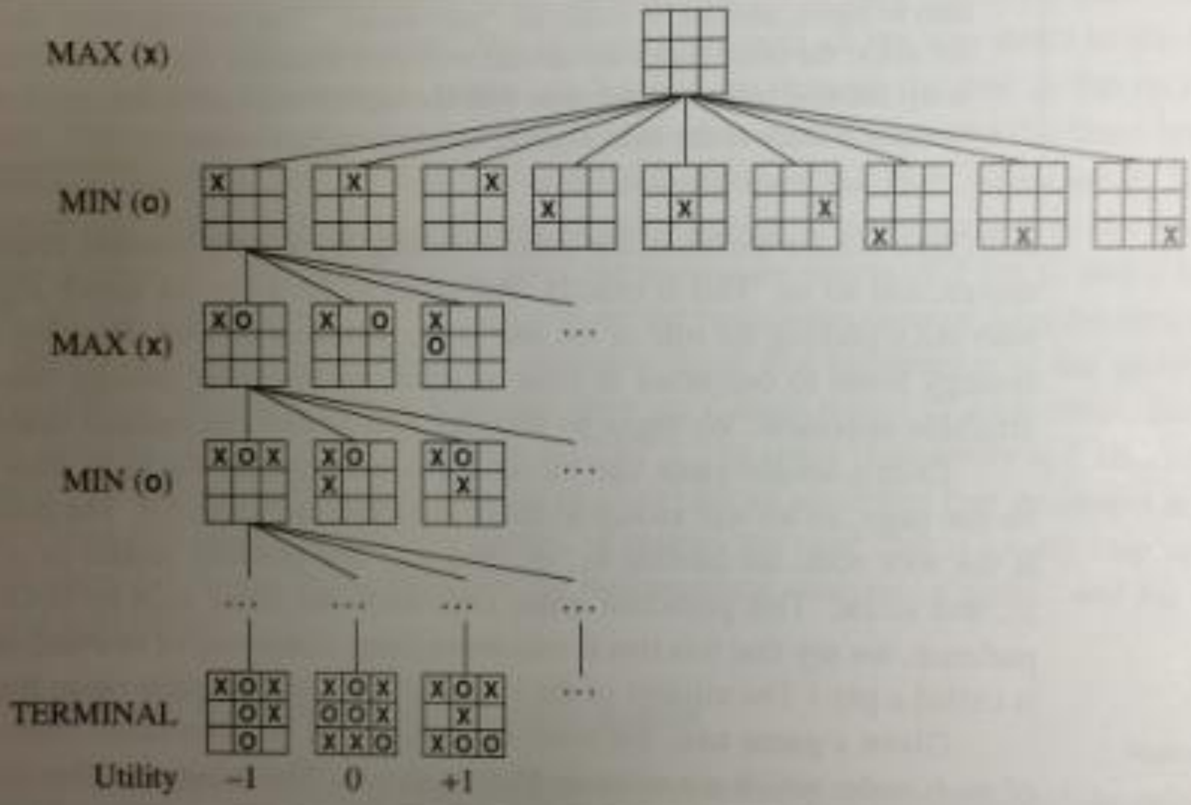


Figure 5.1 A (partial) game tree for the game of tic-tac-toe. The top node is the initial state, and MAX moves first, placing an X in an empty square. We show part of the tree, giving alternating moves by MIN (O) and MAX (X), until we eventually reach terminal states, which can be assigned utilities according to the rules of the game.

Game trees

- For tic-tac-toe the game tree is relatively small—fewer than 9! = 362,880 terminal nodes. But for chess there are over 10^40 nodes, so the game tree is best thought of as a theoretical construct that we cannot realize in the physical world. But regardless of the game tree, it is MAX’s job to search for a good move. We use the term **search tree** for a tree that is superimposed on the full game tree, and examines enough nodes to allow a player to determine what move to make.

Optimal decisions in games

- In a normal search problem, the optimal solution would be a sequence of actions leading to a goal state—a terminal state that is a win. In adversarial search, MIN has something to say about it. MAX therefore must find a contingent **strategy**, which specifies MAX's move in the initial state, then MAX's moves in the states resulting from every possible response by MIN, then MAX's moves in the states resulting by every possible response by MIN to *those* moves, and so on. Roughly speaking, an optimal strategy leads to outcomes at least as good as any other strategy when one is playing an infallible opponent.

Game tree

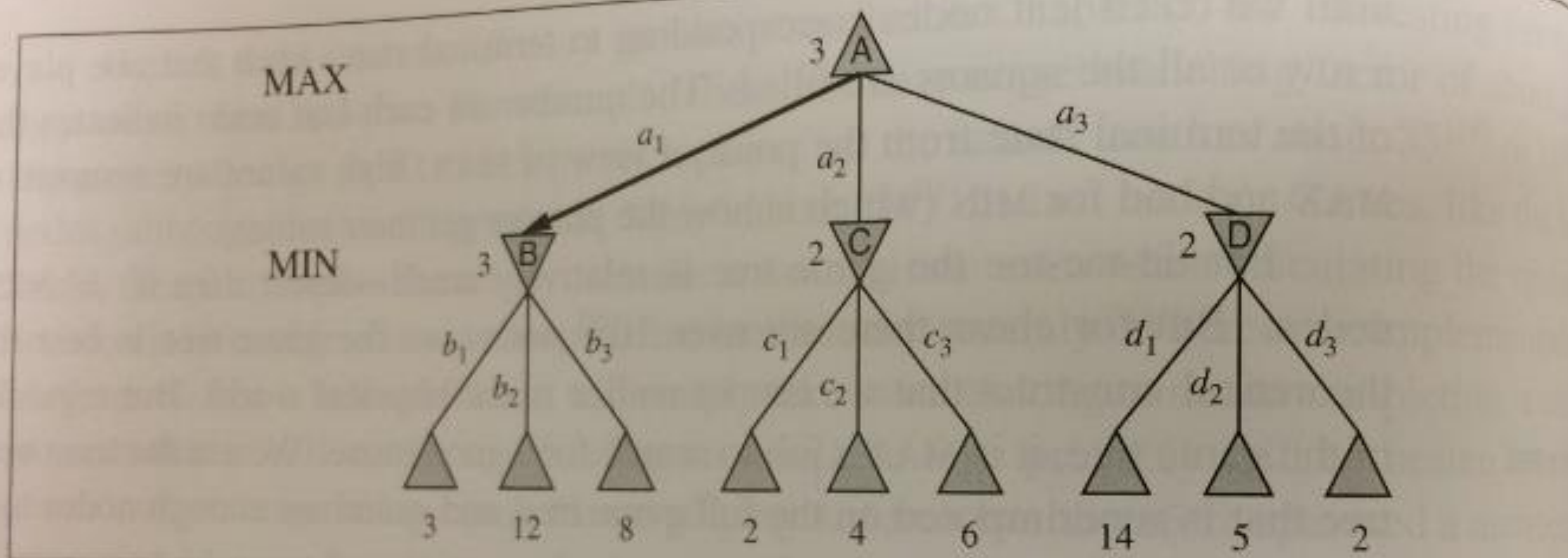


Figure 5.2 A two-ply game tree. The △ nodes are "MAX nodes," in which it is MAX's turn to move, and the ▽ nodes are "MIN nodes." The terminal nodes show the utility values for MAX; the other nodes are labeled with their minimax values. MAX's best move at the root is a_1, because it leads to the state with the highest minimax value, and MIN's best reply is b_1, because it leads to the state with the lowest minimax value.

Optimal decisions in games

- Even a simple game like tic-tac-toe is too complex for us to draw the entire game tree on one page, so we will instead examine a “trivial” game. The possible moves for MAX at the root node are labeled a1, a2, and a3. The possible replies to a1 for MIN are b1, b2, b3, and so on. This particular game ends after one move each by MAX and MIN. (We say that this tree is one move deep, consisting of two half-moves, each of which is called a **ply**.) The utilities of the terminal states in this game range from 2 to 14.

Optimal decisions in games

- Given a game tree, the optimal strategy can be determined from the **minimax value** of each node, which we write as MINIMAX(n). The minimax value of a node is the utility (for MAX) of being in the corresponding state, *assuming that both players play optimally* from there to the end of the game. Obviously, the minimax value of a terminal state is just its utility. Furthermore, given a choice, MAX prefers to move to a state of maximum value, whereas MIN prefers a state of minimum value. So we have:

$$\text{MINIMAX}(s) = \begin{cases} \text{UTILITY}(s) & \text{if TERMINAL-TEST}(s) \\ \max_{a \in Actions(s)} \text{MINIMAX}(\text{RESULT}(s, a)) & \text{if PLAYER}(s) = \text{MAX} \\ \min_{a \in Actions(s)} \text{MINIMAX}(\text{RESULT}(s, a)) & \text{if PLAYER}(s) = \text{MIN} \end{cases}$$

Optimal decisions in games

- Let us apply these definitions to the game tree considered above. The terminal nodes on the bottom level get their utility values from the game’s UTILITY function. The first MIN node, labeled B, has three successor states with values 3, 12, and 8, so its minimax value is 3. Similarly, the other two MIN nodes have minimax value 2. The root node is a MAX node; its successor states have minimax values 3, 2, and 2; so it has a minimax value of 3. We can also identify the **minimax decision** at the root: action a1 is the optimal choice for MAX because it leads to the state with the highest minimax value.

Optimal decisions in games

- This definition of optimal play for MAX assumes that MIN also plays optimally—it maximizes the *worst-case* outcome for MAX. What if MIN does not play optimally? Then it is easy to show (homework exercise) that MAX will do even better. Other strategies against suboptimal opponents may do better than the minimax strategy, but these strategies necessarily do worse against optimal opponents.

The minimax algorithm

- The **minimax algorithm** computes the minimax decision from the current state. It uses a simple recursive computation of the minimax values of each successor state, directly implementing the defining equations. The recursion proceeds all the way down to the leaves of the tree, and then the minimax values are **backed up** through the tree as the recursion unwinds. For example, in the figure the algorithm first recurses down to the three bottom-left nodes and uses the UTILITY function on them to discover that their values are 3, 12, and 8, respectively. Then it takes the minimum of these values, 3, and returns it as the backed-up value of node B. A similar process gives the backed-up values of 2 for C and 2 for D. Finally, we take the maximum of 3, 2, and 2 to get the backed-up value of 3 for the root node.

Minimax algorithm

```
function MINIMAX-DECISION(state) returns an action
  return arg max_{a ∈ ACTIONS(s)} MIN-VALUE(RESULT(state, a))

function MAX-VALUE(state) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← −∞
  for each a in ACTIONS(state) do
    v ← MAX(v, MIN-VALUE(RESULT(s, a)))
  return v

function MIN-VALUE(state) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← ∞
  for each a in ACTIONS(state) do
    v ← MIN(v, MAX-VALUE(RESULT(s, a)))
  return v
```

Figure 5.3 An algorithm for calculating minimax decisions. It returns the action corresponding to the best possible move, that is, the move that leads to the outcome with the best utility, under the assumption that the opponent plays to minimize utility. The functions MAX-VALUE and MIN-VALUE go through the whole game tree, all the way to the leaves, to determine the backed-up value of a state. The notation $\text{argmax}_{a \in S} f(a)$ computes the element a of set S that has the maximum value of $f(a)$.

Minimax algorithm

- Does the minimax algorithm resemble any algorithms we have seen previously?

- How does it rate on the “big 4”?
  - Recall that game-tree search is still a form of search.

Minimax algorithm

- The minimax algorithm performs a complete depth-first exploration of the game tree. If the maximum depth of the tree is m and there are b legal moves at each point, then the time complexity of the minimax algorithm is $O(b^m)$. The space complexity is O(bm) for an algorithm that generates all actions at once, or O(m) for an algorithm that generates actions one at a time. For real games, of course, the time cost is totally impractical, but this algorithm serves as the basis for the mathematical analysis of games and for more practical algorithms.

Game-tree search pruning

- The problem with minimax search is that the number of game states it has to examine is exponential in the depth of the tree. Unfortunately, we can't eliminate the exponent, but it turns out that we can effectively cut it in half. The trick is that it is possible to compute the correct minimax decision without looking at every node in the game tree. That is, we can borrow the idea of **pruning** from the search section (recall that A* pruned the subtree following below Timisoara) to eliminate large parts of the tree from consideration. The particular technique we consider is **alpha-beta pruning**. When applied to a standard minimax tree, it returns the same move as minimax would, but prunes away branches that cannot possibly influence the final decision.

Game tree

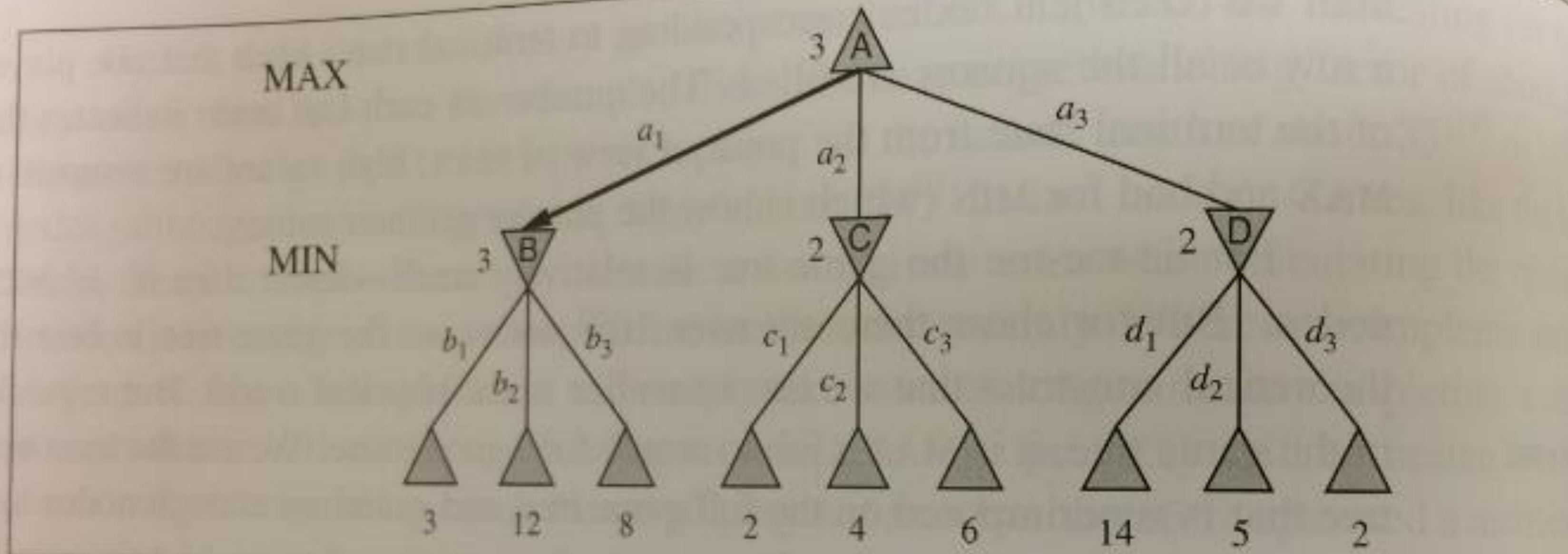
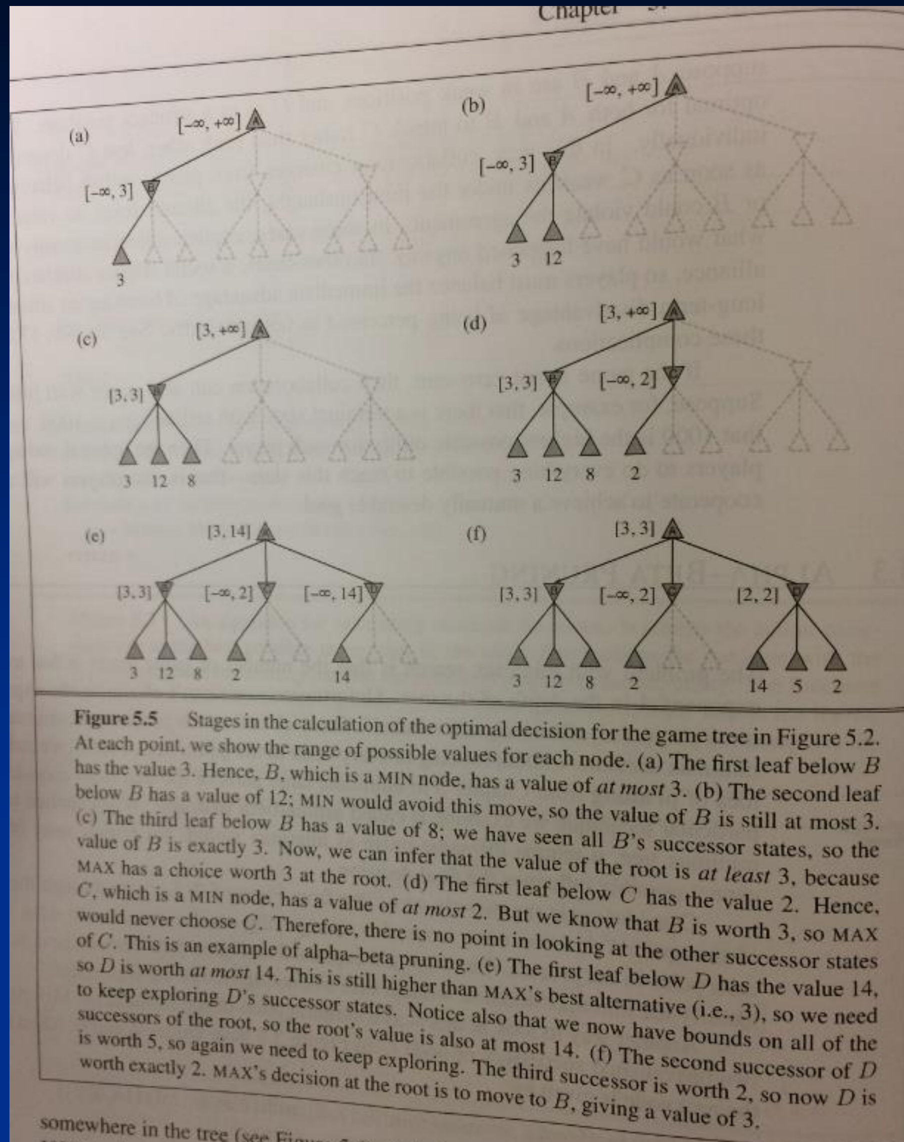


Figure 5.2 A two-ply game tree. The △ nodes are "MAX nodes," in which it is MAX's turn to move, and the ▽ nodes are "MIN nodes." The terminal nodes show the utility values for MAX; the other nodes are labeled with their minimax values. MAX's best move at the root is a_1, because it leads to the state with the highest minimax value, and MIN's best reply is b_1, because it leads to the state with the lowest minimax value.

Alpha-beta pruning

- Consider again the two-play game tree. Let’s go through the calculation of the optimal decision once more, this time paying careful attention to what we know at each point in the process. The steps are explained in the figure on the next page. The outcome is that we can identify the minimax decision without ever evaluating two of the leaf nodes.

Alpha-beta pruning



(a) $[-\infty, +\infty]$ A $[-\infty, 3]$ B 3

(b) $[-\infty, +\infty]$ A $[-\infty, 3]$ B 3 12

(c) $[3, +\infty]$ A $[3, 3]$ B 3 12 8

(d) $[3, +\infty]$ A $[3, 3]$ B $[-\infty, 2]$ C 3 12 8 2

(e) $[3, 14]$ A $[3, 3]$ B $[-\infty, 2]$ C $[-\infty, 14]$ D 3 12 8 2 14

(f) $[3, 3]$ A $[3, 3]$ B $[-\infty, 2]$ C $[2, 2]$ D 3 12 8 2 14 5 2

Figure 5.5 Stages in the calculation of the optimal decision for the game tree in Figure 5.2. At each point, we show the range of possible values for each node. (a) The first leaf below B has the value 3. Hence, B, which is a MIN node, has a value of *at most* 3. (b) The second leaf below B has a value of 12; MIN would avoid this move, so the value of B is still at most 3. (c) The third leaf below B has a value of 8; we have seen all B's successor states, so the value of B is exactly 3. Now, we can infer that the value of the root is *at least* 3, because MAX has a choice worth 3 at the root. (d) The first leaf below C has the value 2. Hence, C, which is a MIN node, has a value of *at most* 2. But we know that B is worth 3, so MAX would never choose C. Therefore, there is no point in looking at the other successor states of C. This is an example of alpha–beta pruning. (e) The first leaf below D has the value 14, so D is worth *at most* 14. This is still higher than MAX's best alternative (i.e., 3), so we need to keep exploring D's successor states. Notice also that we now have bounds on all of the successors of the root, so the root's value is also at most 14. (f) The second successor of D is worth 5, so again we need to keep exploring. The third successor is worth 2, so now D is worth exactly 2. MAX's decision at the root is to move to B, giving a value of 3.

Alpha-beta pruning

- Another way to look at this is as a simplification of the formula for MINIMAX. Let the two unevaluated successors of node C in the figure have values x and y. Then the value of the root node is given by:

  MIMIMAX(root)

  = max(min(3,12,8),min(2,x,y),min(14,5,2))

  = max(3,min(2,x,y),2)

  = max(3,z,2) where z = min(2,x,y) <= 2

  = 3.
- In other words, the value of the root and hence the minimax decision are *independent* of the values of the pruned leaves x and y.

Alpha-beta pruning

- Alpha-beta pruning can be applied to trees of any depth, and it is often possible to prune entire subtrees rather than just leaves. The general principle is this: consider a node n somewhere in the tree (see next figure) such that Player has a choice of moving to that node. If Player has a better choice m either at the parent node of n or at any choice point further up, then *n will never be reached in actual play*. So once we have found out enough about n (by examining some of its descendants) to reach this conclusion, we can prune it.

General alpha-beta pruning

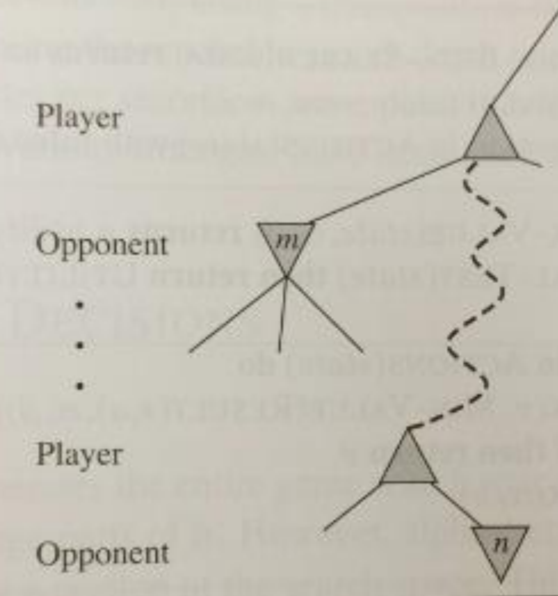


Figure 5.6 The general case for alpha–beta pruning. If m is better than n for Player, we will never get to n in play.

Alpha-beta search

- Remember that minimax search is depth-first, so at any one time we just have to consider the nodes along a single path in the tree. Alpha-beta pruning gets its name from the following two parameters that describe bounds on the backed-up values that appear anywhere along the path:
  - α = the value of the best (i.e., highest-value) choice we have found so far at any choice point along the path for MAX.
  - β = the value of the best (i.e., lowest-value) choice we have found so far at any choice point along the path for MIN.

Alpha-beta search algorithm

- Alpha-beta search updates the values of α and β as it goes along and prunes the remaining branches at a node (i.e., terminates the recursive call) as soon as the values of the current node is known to be worse than the current α or β value for MAX or MIN, respectively. The complete algorithm is given on the next slide. We can trace its behavior when applied to the example.

Alpha-beta search algorithm

```
function ALPHA-BETA-SEARCH(state) returns an action
   v ← MAX-VALUE(state, −∞, +∞)
   return the action in ACTIONS(state) with value v

function MAX-VALUE(state, α, β) returns a utility value
   if TERMINAL-TEST(state) then return UTILITY(state)
   v ← −∞
   for each a in ACTIONS(state) do
      v ← MAX(v, MIN-VALUE(RESULT(s,a), α, β))
      if v ≥ β then return v
      α ← MAX(α, v)
   return v

function MIN-VALUE(state, α, β) returns a utility value
   if TERMINAL-TEST(state) then return UTILITY(state)
   v ← +∞
   for each a in ACTIONS(state) do
      v ← MIN(v, MAX-VALUE(RESULT(s,a) , α, β))
      if v ≤ α then return v
      β ← MIN(β, v)
   return v
```

Figure 5.7 The alpha–beta search algorithm. Notice that these routines are the same as the MINIMAX functions in Figure 5.3, except for the two lines in each of MIN-VALUE and MAX-VALUE that maintain α and β (and the bookkeeping to pass these parameters along).

Adding dynamic move-ordering

Move ordering

- The effectiveness of alpha-beta pruning is highly dependent on the order in which the states are examined. For example, in the figure we could not prune any successors of D at all because the worst successors (from the point of view of MIN) were generated first. If the third successor of D had been generated first, we would have been able to prune the other two. This suggests that it might be worthwhile to try to examine first the successors that are likely to be best.

Alpha-beta move ordering

- If this can be done, then it turns out that alpha-beta needs to examine only O(b^(m/2)) nodes to pick the best move, instead of O(b^m) for minimax. This means that the effective branching factor becomes sqrt(b) instead of b – for chess, about 6 instead of 35. Put another way, alpha-beta can solve a tree roughly twice as deep as minimax in the same amount of time. If successors are examined in random order rather than best-first, the total number of nodes examined will be roughly O(b^(3m/4)) for moderate b. For chess, a fairly simple ordering function (such as trying captures first, then threats, then forward moves, and then backward moves) gets to within about a factor of 2 of the best-case O(b^(m/2)) result.

Alpha-beta move ordering

- Adding dynamic move-ordering schemes, such as trying the moves that were found to be best in the past, brings us quite close to the theoretical limit. The past could be the previous move—often the same threats remain– or it could come from previous exploration of the current move. One way to gain information from the current move is with iterative deepening search. First, search 1 ply deep and record the best path of moves. Then search 1 ply deeper, but use the recorded path to inform move ordering. As we saw in the search module, iterative deepening on an exponential game three adds only a constant fraction to the total search time, which can be more than made up from better move ordering. The best moves are often called **killer moves** and to try them first is called the **killer move heuristic**.

Alpha-beta move ordering

- In the search module, we noted that repeated states in the search tree can cause an exponential increase in search cost. In many games, repeated states occur frequently because of **transpositions**—different permutations of the move sequence that end up in the same position. For example, if White has one move, a1, that can be answered by Black with b1 and an unrelated move a2 on the other side of the board that can be answered by b2, then the sequences [a1,b1,a2,b2] and [a2,b2,a1,b1] both end up in the same position. It is worthwhile to store the evaluation of the resulting position in a hash table the first time it is encountered so that we don't have to recompute it on subsequent occurrences. The hash table of previously seen positions is called a **transposition table**; it is analogous to the *explored* list in GRAPH-SEARCH.

Transposition table

- Using a transposition table can have a dramatic effect, sometimes as much as doubling the reachable search depth in chess. On the other hand, if we are evaluating a million nodes per second, at some point it is not practical to keep *all* of them in the transposition table. Various strategies have been used to choose which nodes to keep and which to discard.

Evaluation function

- The minimax algorithm generates the entire game search space, whereas the alpha-beta algorithm allows us to prune large parts of it. However, alpha-beta still has to search all the way to terminal states for at least a portion of the search space. This depth is usually not practical, because moves must be made in a reasonable amount of time—typically a few minutes at most. Claude Shannon's paper *Programming a Computer for Playing Chess* (1950) proposed instead that programs should cut off the search earlier and apply a heuristic **evaluation function** to states in the search, effectively turning nonterminal nodes into terminal leaves.

Evaluation function

- In other words, the suggestion is to alter minimax or alpha-beta in two ways:
  - Replace the utility function by a heuristic evaluation function EVAL, which estimates the position's utility
  - Replace the terminal test by a **cutoff test** that decides when to apply EVAL.
- This gives the following for heuristic minimax for state s and maximum depth d:

$$\textsc{H-Minimax}(s,d) = \begin{cases} \textsc{Eval}(s) & \text{if } \textsc{Cutoff-Test}(s,d) \\ \max_{a \in Actions(s)} \textsc{H-Minimax}(\textsc{Result}(s,a), d+1) & \text{if } \textsc{Player}(s) = \textsc{max} \\ \min_{a \in Actions(s)} \textsc{H-Minimax}(\textsc{Result}(s,a), d+1) & \text{if } \textsc{Player}(s) = \textsc{min}. \end{cases}$$

Adversarial search summary

- A game can be defined by the **initial state**, legal actions at each state, the result of each **action**, a **terminal test**, and a **utility function** that applies to terminal states.
- In two-player zero-sum games with **perfect information**, the **minimax** algorithm can select optimal moves by a depth-first enumeration of the game tree.
- The **alpha-beta** search algorithm computes the same optimal move as minimax, but achieves much greater efficiency by eliminating subtrees that are provably irrelevant.
- Usually it is not feasible to consider the whole game tree (even with alpha-beta), so we need to cut the search off at some point and apply a heuristic **evaluation function** that estimates the utility of a state.

Adversarial search extensions

- Many game programs precompute tables of best opening and endgame moves so they can look up a move rather than search.
- Games of chance can be handled by an extension to the minimax algorithm that evaluates a **chance node** by taking the average utility of all children, weighted by the probability of each child.
- Optimal play in games of **imperfect information**, such as Kriegspiel and bridge, requires reasoning about the current and future belief states of each player. A simple approximation can be obtained by averaging the value of an action over each possible configuration of missing information.
- Programs have bested even champion human players at games such as chess, checkers, and Othello. Humans retain the edge in several games of imperfect information, such as poker, bridge, and Kriegspiel, and in games with very large branching factors and little good heuristic knowledge, such as Go (outdated).

Constraint satisfaction

- In the first portion of the search module, we explored the idea that problems can be solved by searching in a space of **states**. These states can be evaluated by domain-specific heuristics and tested to see whether they are goal states. From the point of view of the search algorithm, however, each state is atomic, or divisible—a black box with no internal structure.
- We now describe a way to solve a wide variety of problems more efficiently. We use a **factored representation** for each state: a set of variables, each of which has a value. A problem is solved when each variable has a value that satisfies all the constraints on the variable. A problem describe this way is called a **constraint satisfaction problem**, or CSP.

Constraint satisfaction

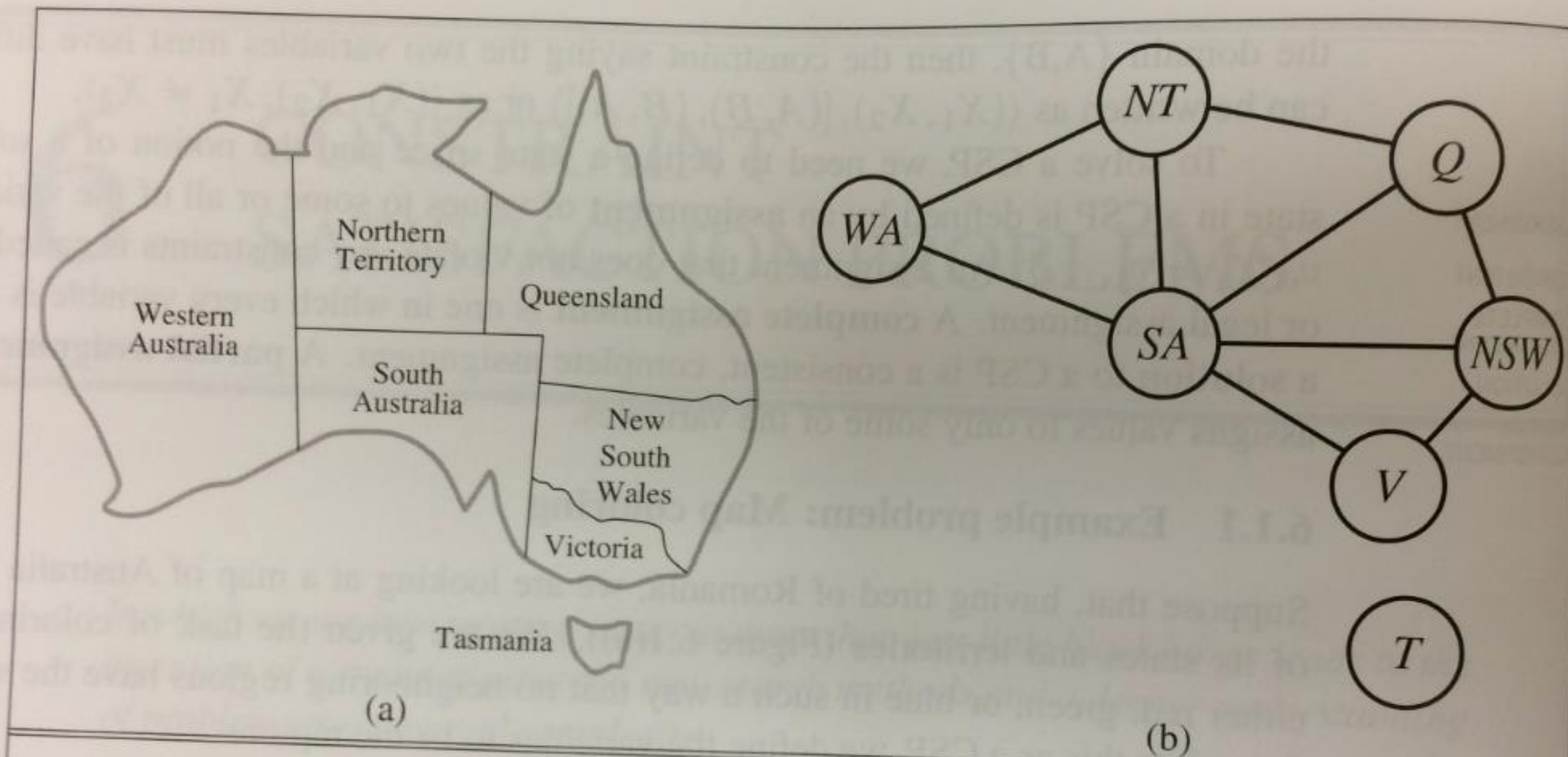


Figure 6.1 (a) The principal states and territories of Australia. Coloring this map can be viewed as a constraint satisfaction problem (CSP). The goal is to assign colors to each region so that no neighboring regions have the same color. (b) The map-coloring problem represented as a constraint graph.

Constraint satisfaction problems

- A constraint satisfaction problem consists of three components, X, D, and C:
  - X is a set of variables, $\{X_1,\ldots,X_n\}$.
  - D is a set of domains, $\{D_1,\ldots,D_n\}$, one for each variable.
  - C is a set of constraints that specify allowable combinations of values.

Example problem: Map coloring

- Suppose that, having tired of Romania, we are looking at a map of Australia showing each of its states and territories. We are given the task of coloring each region either red, green, or blue in such a way that no neighboring regions have the same color.
- To formulate this as a CSP, we define the variables to be the regions: X = {WA, NT, Q, NSW, V, SA, T}
- The domain of each variable is the set D_i = {red, green, blue}.
- The constraints require neighboring regions to have distinct colors. Since there are nine places where regions border, there are nine constraints: C = {SA!=WA, SA!=NT,SA!=Q, etc.}
- SA!=WA is shortcut for ((SA,WA),SA!=WA), where SA!=WA can be fully enumerated in turn as {(red,green),(red,blue),…}

Example problem: Map coloring

- There are many possible solutions to this problem, such as …

Example problem: Map coloring

- There are many possible solutions to this problem, such as ...

{WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=red}

- It can be helpful to visualize a CSP as a **constraint graph**. The nodes of the graph correspond to variables of the problem, and a link connects any two variables that participate in a constraint.

Constraint satisfaction problem

- Each domain D_i consists of a set of allowable values, $\{v_1,\ldots,v_n\}$ for variable X_i. Each constraint consists of a pair (*scope*, *rel*), where *scope* is a tuple of variables that participate in the constraint and *rel* is a relation that defines the values that those variables can take on. A relation can be represented as an explicit list of all tuples of values that satisfy the constraint, or as an abstract relation that supports two operations: testing if a tuple is a member of the relation and enumerating the members of the relation. For example, if X1 and X2 both have the domain {A,B}, then the constraint saying the two variables must have different values can be written as ((X1,X2),[(A,B),(B,A)]) or as ((X1,X2),X1 != X2).

CSP

- To solve a CSP, we need to define a state space and the notion of a solution. Each state in a CSP is defined by an **assignment** of values to some or all of the variables, {X1=v1,X2=v2,…} An assignment that does not violate any constraints is called a **consistent** or legal assignment. A **complete assignment** is one in which every variable is assigned, and a **solution** to a CSP is a consistent, complete assignment. A **partial assignment** is one that assigns values to only some of the variables.

Why formulate a problem as a CSP?

- One reason is that the CSPs yield a natural representation for a wide variety of problems; if you already have a CSP-solving system, it is often easier to solve a problem using it than to design a custom solution using another search technique. In addition, CSP solvers can be faster than state-space searchers because the CSP solver can quickly eliminate large swatches of the search space. For example, once we have chosen {SA=blue} in the Australia problem, we can conclude that none of the five neighboring variables can take on the value *blue*. Without taking advantage of constraint propagation, a search procedure would have to consider 3^5=243 assignments for the five neighboring variables; with constraint propagation we never have to consider *blue* as a value, so we have only 2^5=32 assignments to look at, a reduction of 87%.

Why formulate a problem as a CSP?

- In regular state-space search we can only ask: is this specific state a goal? No? What about this one? With CSPs, once we find out that a partial assignment is not a solution, we can immediately discard further refinements of the partial assignment. Furthermore, we can see *why* the assignment is not a solution—we see which variables violate a constraint—so we can focus attention on the variables that matter. As a result, many problems that are intractable for regular state-space search can be solved quickly when formulated as a CSP.

Example problem: Job-shop scheduling

- Factories have the problem of scheduling a day's worth of jobs, subject to various constraints. In practice, many of these problems are solved with CSP techniques. Consider the problem of scheduling the assembly of a car. The whole job is composed of tasks, and we can model each task as a variable, where the value of each variable is the time that the task starts, expressed as an integer number of minutes. Constraints can assert that one task must occur before another—for example, a wheel must be installed before the hubcap is put on—and that only so many tasks can go on at once. Constraints can also specify that a task takes a certain amount of time to complete.

- We consider a small part of the car assembly, consisting of 15 tasks: install axles (front and back), affix all four wheels (right and left, front and back), tighten nuts for each wheel, affix hubcaps, and inspect the final assembly. We can represent the tasks with 15 variables:

- X = {AxleF, AxleB, WheelRF,…,NutsRF,…, CapRF,…,Inspect}
- The value of each variable is the time that the task starts. Next we represent **precedence constraints** between individual tasks. Whenever a task T1 must occur before task T2, and task T1 takes duration d1 to complete, we add an arithmetic constraint of the form…

- T1 + d1 <= T2
- In our example, the axles have to be in place before the wheels are put on, and it takes 10 minutes to install an axle, so we write:
- AxleF + 10 <= WheelRF;
- AxleF + 10 <= WheelLF;
- AxleB + 10 <= WheelRB;
- AxleB + 10 <= WheelLB.

- Next we say that, for each wheel, we must affix the wheel (which takes 1 minute), then tighten the nuts (2 minutes), and finally attach the hubcab (1 minute, but not represented yet):

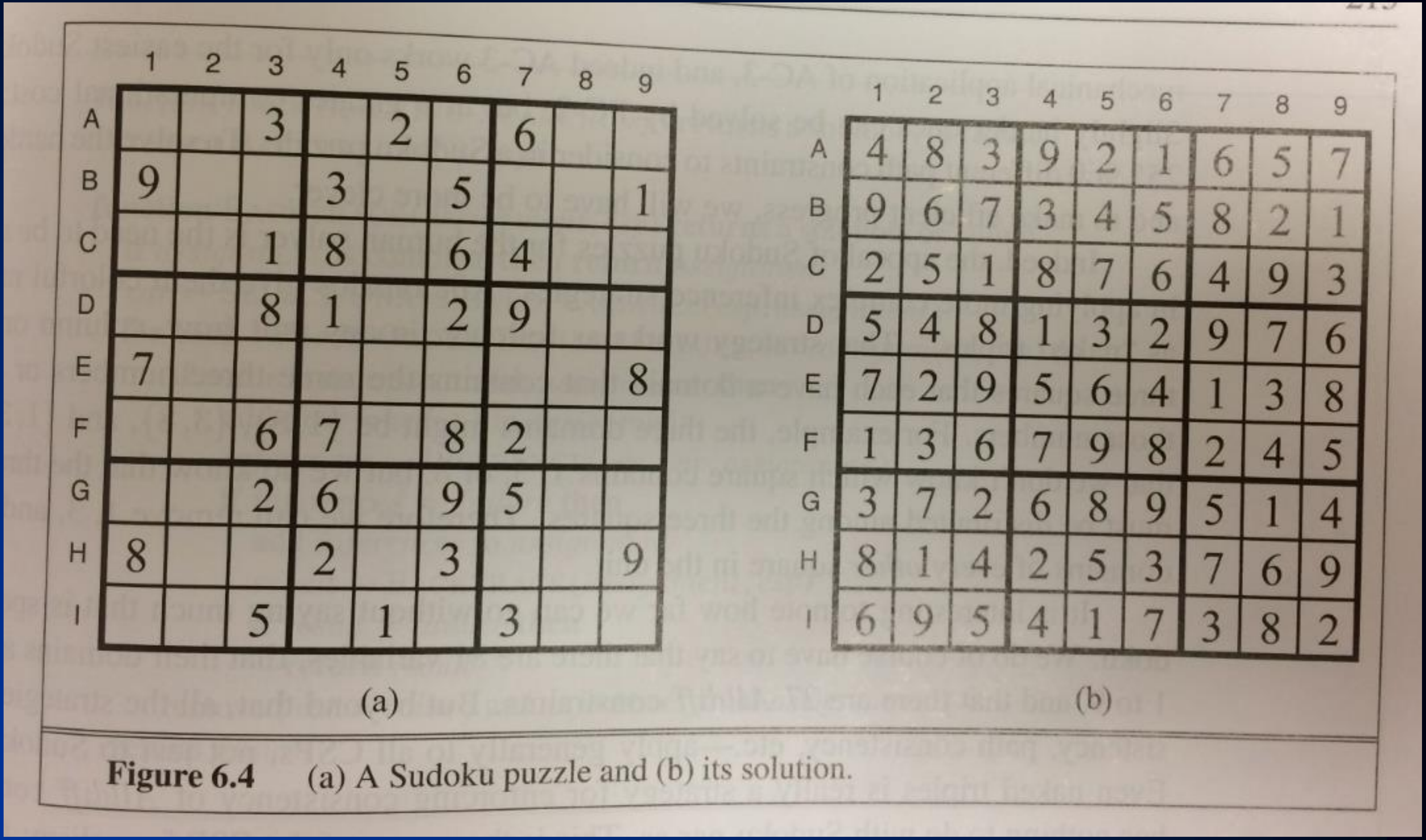
- WheelRF + 1 <= NutsRF;
- WheelLF + 1 <= NutsLF;
- WheelRB + 1 <= NutsRB;
- WheelLB + 1 <= NutsLB;
- NutsRF + 2 <= CapRF;
- NutsLF + 2 <= CapLF;
- NutsRB + 2 <= CapRB;
- NutsLB + 2 <= CapLB.

- Suppose we have four workers to install wheels, but they have to share one tool that helps put the axle in place. We need a **disjunctive constraint** to say that AxleF and AxleB must not overlap in time; either one comes first or the other does:
- (AxleF + 10 <= AxleB) OR (AxleB + 10 <= AxleF)
- This looks like amore complicated constraint, combining arithmetic and logic. But it still reduces to a set of pairs of values that AxleF and AxleB can take on.

- We also need to assert that the inspection comes last and takes 3 minutes. For every variable except *Inspect* we add a constraint of the form X + dX <= Inspect.
- Finally, suppose there is a requirement to get the whole assembly done in 30 minutes. We can achieve that by limiting the domain of all variables: Di = {1,2,3,…,27}

- This particular problem is trivial to solve, but CSPs have been applied to job-shop scheduling problems like this with thousands of variables. In some cases, there are complicated constraints that are difficult to specify in the CSP formalism, and more advanced planning techniques are used, which will discuss in the Planning Module of the course.

Example problem: Sudoku



	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

(a)

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

(b)

Figure 6.4 (a) A Sudoku puzzle and (b) its solution.

CSP applications

- Examples of simple problems that can be modeled as a constraint satisfaction problem include:
  - Eight queens puzzle
  - Map coloring problem
  - Sudoku, Crosswords, Futoshiki, Kakuro (Cross Sums), Numbrix, Hidato and many other logic puzzles
- These are often provided with tutorials of ASP, Boolean SAT and SMT solvers. In the general case, constraint problems can be much harder, and may not be expressible in some of these simpler systems.
- "Real life" examples include automated planning and resource allocation. An example for puzzle solution is using a constraint model as a Sudoku solving algorithm.

Variations on the CSP formalism

- Standard variant: **discrete, finite domains**
  - E.g., map coloring and job-shop coloring
  - The 8-queens problem can also be viewed as a finite-domain CSP, where the variables Q1-Q8 are the positions of each queen in columns 1-8 and each variable has the domain Di={1,…,8}
- Discrete domain can be **infinite**, such as the set of integers or strings
  - Can no longer enumerate all combinations of values.
  - Instead use **constraint language** that can understand constraints such as T1 + d1 <= T2 directly, without enumerating the set of pairs of allowable values for (T1,T2)

Variations on the CSP formulation

- Special solution algorithms (which we will see shortly) exist for **linear constraints** on integer variables—that is, constraints such as the one just given, in which each variable appears only in linear form. It can be shown that no algorithm exists for solving general **nonlinear constraints** on integer variables.

Variations on the CSP formulation

- Constraint satisfaction problems with **continuous domains** are common in the real world and are widely studied in the field of operations research. For example, the scheduling of experiments on the Hubble Space Telescope requires very precise timing of observations; the start and finish of each observation and maneuver are continuous-valued variables that must obey a variety of astronomical, precedence, and power constraints. The best-known category of continuous-domain CSPs is that of **linear programming** problems, where constraints must be linear equalities or inequalities. Linear programming problems can be solved in time polynomial in the number of variables. Problems with different types of constraints and objective functions have also been studied—quadratic programming, second-order conic programming, and so on.

CSP variations

- The simplest type of constraint is a **unary constraint**, which restricts the value of a single variable. For example, in the map-coloring problem it could be the case that South Australians won't tolerate the color green; we can express that with the unary constraint <(SA), SA != green>
- A **binary constraint** relates two variables. For example, SA != NSW is a binary constraint. A binary CSP is one with only binary constraints; it can be represented as a constraint graph
- We can also describe higher-order constraints, such as asserting that the value of Y is between X and Z, with the ternary constraint *Between*(X,Y,Z)

Constraint graph

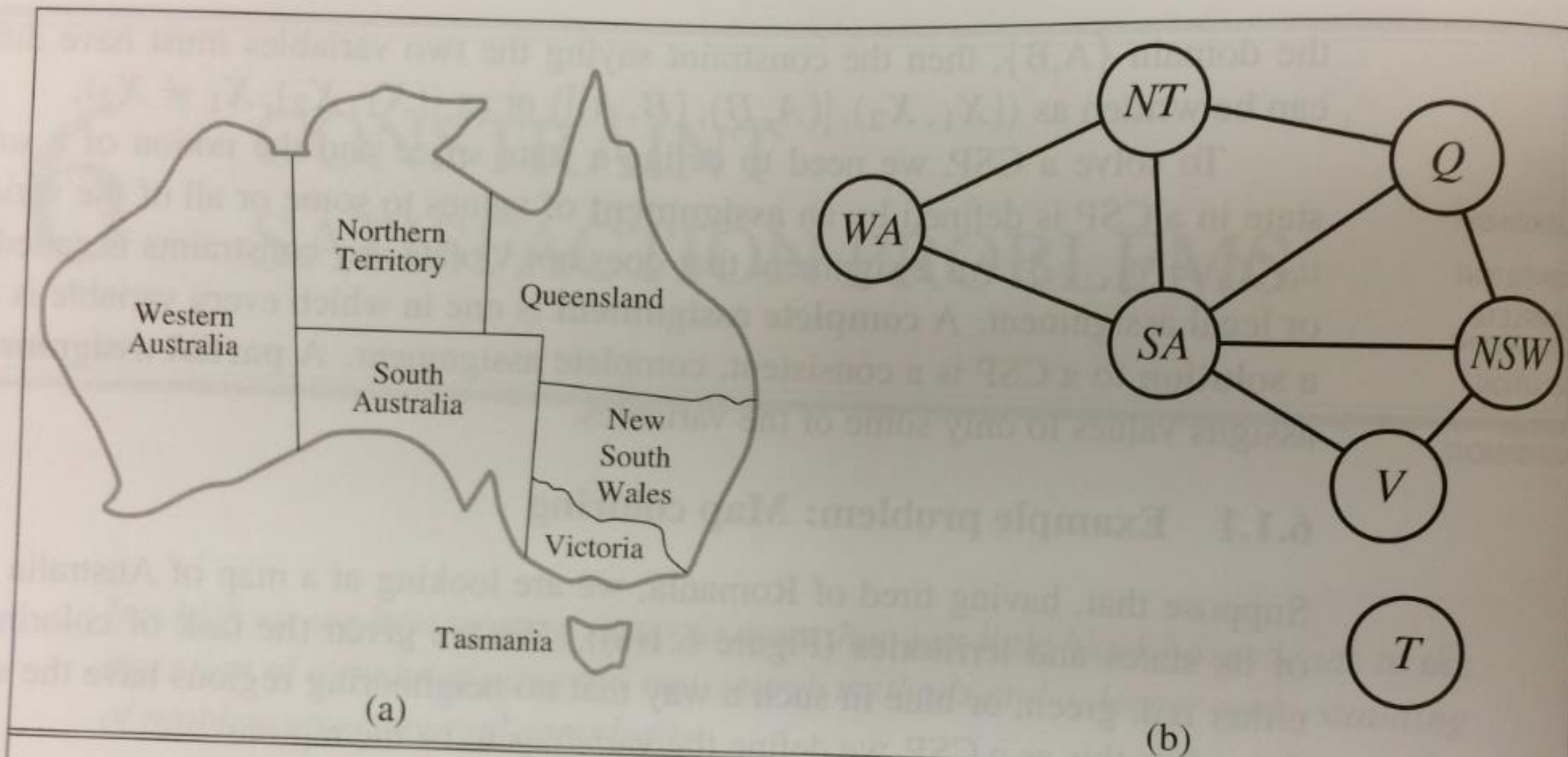


Figure 6.1 (a) The principal states and territories of Australia. Coloring this map can be viewed as a constraint satisfaction problem (CSP). The goal is to assign colors to each region so that no neighboring regions have the same color. (b) The map-coloring problem represented as a constraint graph.

CSP variations

- A constraint involving an arbitrary number of variables is called a **global constraint** (The name is traditional but confusing because it need not involve *all* the variables in a problem). One of the most common global constraints is *Alldiff*, which says that all of the variables involved in the constraint must have different values. In Sudoku problems, all variables in a row or column must satisfy an *Alldiff* constraint. Another example is provided by **cryptarithmetic puzzles**. Each letter represents a different digit. For the example problem this would be represented as the global constraint *Alldiff*(F,T,U,W,R,O).

Cryptarithmetic problem

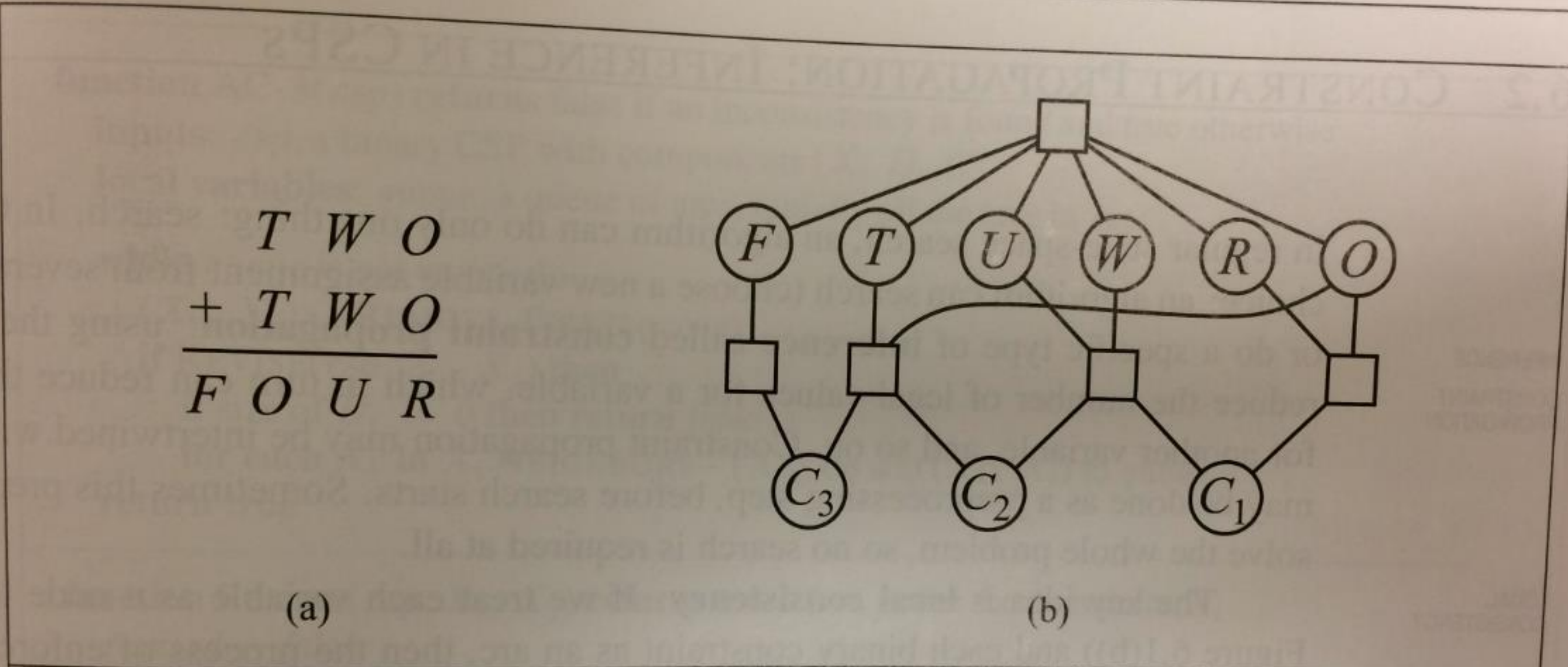


Figure 6.2 (a) A cryptarithmetic problem. Each letter stands for a distinct digit; the aim is to find a substitution of digits for letters such that the resulting sum is arithmetically correct, with the added restriction that no leading zeroes are allowed. (b) The constraint hypergraph for the cryptarithmetic problem, showing the *Alldiff* constraint (square box at the top) as well as the column addition constraints (four square boxes in the middle). The variables C_1, C_2, and C_3 represent the carry digits for the three columns.

Cryptarithmetic problem

- The addition constraints on the four columns of the puzzle can be written as the following n-ary constraints:
  - …

Cryptarithmetic problem

- The addition constraints on the four columns of the puzzle can be written as the following n-ary constraints:
  - O + O = R + 10 * C10
  - C10 + W + W = U + 10*C100
  - C100 + T + T = O + 10 * C1000
  - C1000 = F
- Where C10, C100, and C1000 are auxiliary variables representing the digit carried over into the tens, hundreds, or thousands column. These constraints can be represented in a **constraint hypergraph**. A hypergraph consists of ordinary nodes (the circles in the figure) and hypernodes (the squares) which represent n-ary constraints.

CSP variations

- Alternatively, homework exercise asks you to prove every finite-domain constraint can be reduced to a set of binary constraints if enough auxiliary variables are introduced, so we could transform any CSP into one with only binary constraints; this makes the algorithms simpler.
- There are however two reasons why we might prefer a global constraint such as *Alldiff* rather than a set of binary constraints. First, it is easier and less error-prone to write the problem description using *Alldiff*. Second, it is possible to design special-purpose inference algorithms for global constraints that are not available for a set of more primitive constraints, which we will describe.

CSP variations

- The constraints we have described so far have all been absolute constraints, violation of which rules out a potential solution. Many real-world CSPs include **preference constraints** indicating which solutions are preferred. For example, in a university class-scheduling problem there are absolute constraints that no professor can teach two classes at the same time. But we also may allow preference constraints: Prof. R might prefer teaching in the morning, whereas Prof. N prefers teaching in the afternoon. A schedule that has Prof. R teaching at 2 p.m. would still be an allowable solution (unless Prof. R happens to be the department chair) but would not be an optimal one.

CSP variations

- Preference constraints can often be encoded as costs on individual variable assignments—for example, assigning an afternoon slot for Prof. R costs 2 points against the overall objective function, whereas a morning slot costs 1. With this formulation, CSPs with preferences can be solved with optimization search methods, either path-based or local. We call such a problem a **constraint optimization problem**, or COP. Linear/integer/nonlinear programming problems do this kind of optimization.

Inference in CSPs

- In regular state-space search, an algorithm can only do one thing: search. In CSPs there is a choice: an algorithm can search (choose a new variable assignment from several possibilities) or do a specific type of **inference** called **constraint propagation**: using the constraints to reduce the number of legal values for a variable, which in turn can reduce the legal values for another variable, and so on. Constraint propagation may be intertwined with search, or it may be done as a preprocessing step, before search starts. Sometimes this preprocessing can solve the whole problem, so no search is required at all.

CSP inference

- The key idea is **local consistency**. If we treat each variable as a node in a graph (like for the Australia constraint graph) and each binary constraint as an arc (edge), then the process of enforcing local consistency in each part of the graph causes inconsistent values to be eliminated throughout the graph. There are several different types of local consistency: node consistency, arc consistency, path consistency, *k*-consistency.

Arc consistency

- A variable in a CSP is **arc-consistent** if every value in its domain satisfies the variable’s binary constraints. For formally, Xi is arc-consistent with respect to another variable Xj if for every value in the current domain Di there is some value in the domain Dj that satisfies the binary constraint on the arc (Xi,Xj). A network is arc-consistent if every variable is arc-consistent with every other variable.
- For example, consider the constraint Y = X^2 where the domain of both X and Y is the set of digits. We can write this explicitly as …

Arc consistency

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- For example, consider the constraint Y = X^2 where the domain of both X and Y is the set of digits. We can write this explicitly as … <(X,Y),{(0,0),(1,1),(2,4),(3,9)}>. To make X arc-consistent with respect to Y, we reduce X's domain to {0,1,2,3}. If we also make Y arc-consistent with respect to X, then Y's domain becomes {0,1,4,9} and the whole CSP is arc-consistent.

Arc consistency

- How about for the Australia map-coloring problem?

Arc consistency

- On the other hand, arc consistency can do nothing for the Australia map-coloring problem. Consider the following inequality constraint on (SA,WA):
  - {(red,green),(red,blue),(green,red),(green,blue),(blue,red),(blue,green)}
- No matter what value you choose for SA (or for WA), there is a value value for the other variable. So applying arc consistency has no effect on the domains of either variable.

Arc consistency

- The most popular algorithm for arc consistency is called AC-3. To make every variable arc-consistent, the AC-3 algorithm maintains a queue of arcs to consider (Actually the order of consideration is not important, so the data structure is really a set, but tradition calls it a queue). Initially the queue contains all the arcs in the CSP. (Each binary constraint becomes two arcs, one in each direction.) AC-3 then pops off an arbitrary arc (Xi,Xj) from the queue and makes Xi arc-consistent with respect to Xj. If this leaves Di unchanged, the algorithm just moves to the next arc. But if this revises Di (makes the domain smaller), then we add to the queue all arcs (Xk,Xi) where Xk is a neighbor of Xi. We need to do that because the change in Di might enable further reductions in the domains of Dk, even if we have previously considered Xk.

Arc consistency

- If Di is revised down to nothing, then we know the whole CSP has no consistent solution, and AC-3 can immediately return failure. Otherwise, we keep checking, trying to remove values from the domains of variables until no more arcs are in the queue. At that point, we are left with a CSP that is equivalent to the original CSP—they both have the same solutions—but the arc-consistent CSP will in most cases be faster to search because its variables have smaller domains.

Arc consistency

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
  inputs: *csp*, a binary CSP with components (X, D, C)
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

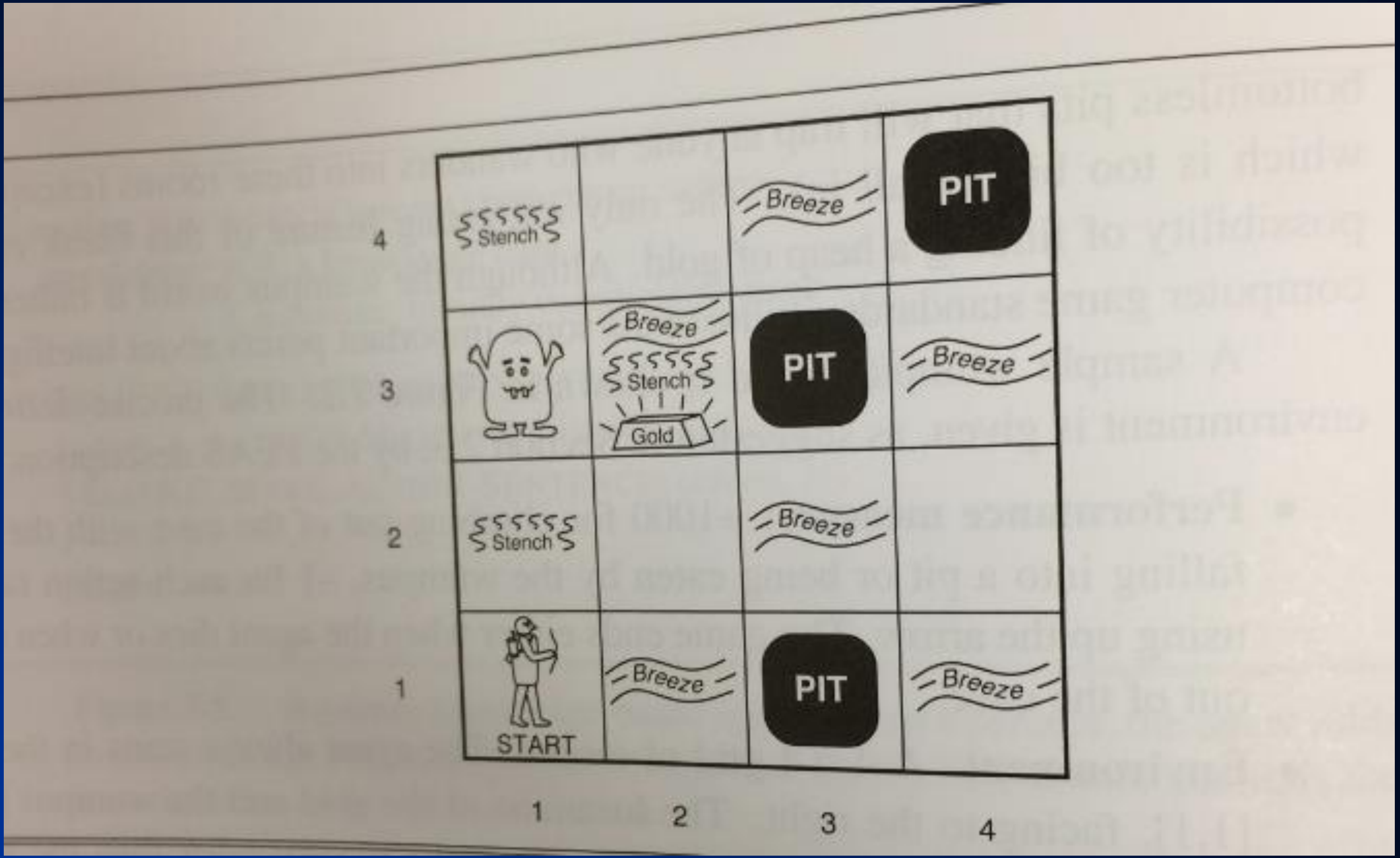
  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REVISE(*csp*, X_i, X_j) **then**
      if size of $D_i = 0$ **then return** *false*
      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do** add (X_k, X_i) to *queue*
  return *true*

---

function REVISE(*csp*, X_i, X_j) **returns** true iff we revise the domain of X_i
  revised $\leftarrow$ *false*
  for each x **in** D_i **do**
    if no value y in D_j allows (x, y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      revised $\leftarrow$ *true*
  return *revised*

Figure 6.3 The arc-consistency algorithm AC-3. After applying AC-3, either every arc is arc-consistent, or some variable has an empty domain, indicating that the CSP cannot be solved. The name "AC-3" was used by the algorithm's inventor (Mackworth, 1977) because it's the third version developed in the paper.

Logical agents



Wumpus world

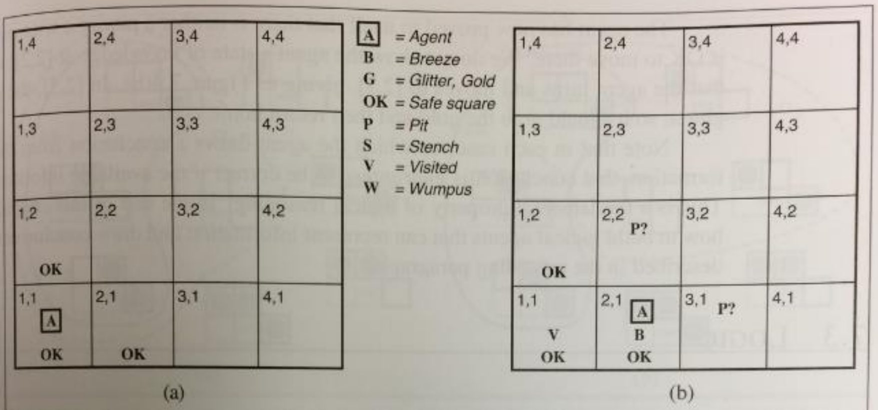


Figure 7.3 The first step taken by the agent in the wumpus world. (a) The initial situation, after percept [*None*, *None*, *None*, *None*, *None*]. (b) After one move, with percept [*None*, *Breeze*, *None*, *None*, *None*].

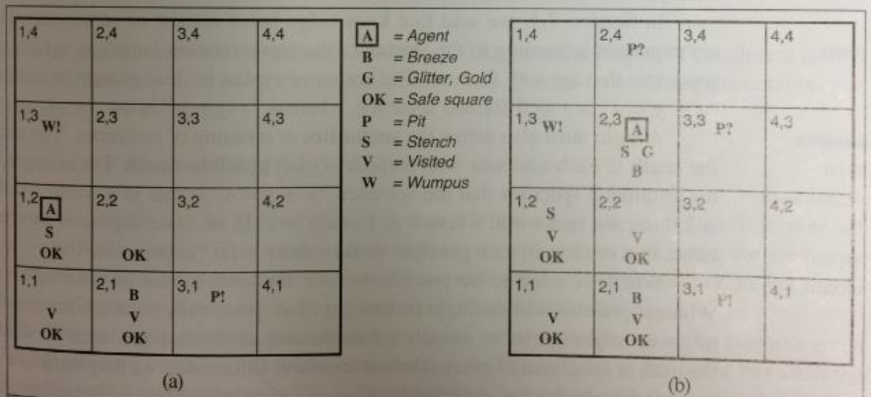


Figure 7.4 Two later stages in the progress of the agent. (a) After the third move, with percept [*Stench*, *None*, *None*, *None*, *None*]. (b) After the fifth move, with percept [*Stench*, *Breeze*, *Glitter*, *None*, *None*].

Wumpus world

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	R_1	R_2	R_3	R_4	R_5	KB
false	*false*	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*false*	*false*
false	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*false*	*true*	*false*	*false*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
false	*true*	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*false*	*true*	*true*	*false*
false	*true*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*false*	*false*	*true*	*false*	*true*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*false*	*true*	*false*	*false*	*true*	*false*	*false*	*true*	*true*	*false*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
true	*true*	*true*	*true*	*true*	*true*	*true*	*false*	*true*	*true*	*false*	*true*	*false*

Figure 7.9 A truth table constructed for the knowledge base given in the text. KB is true if R_1 through R_5 are true, which occurs in just 3 of the 128 rows (the ones underlined in the right-hand column). In all 3 rows, $P_{1,2}$ is false, so there is no pit in [1,2]. On the other hand, there might (or might not) be a pit in [2,2].

Optimization

- An **integer programming** problem is a mathematical optimization or feasibility program in which some or all of the variables are restricted to be integers. In many settings the term refers to **integer linear programming** (ILP), in which the objective function and the constraints (other than the integer constraints) are linear.
- Integer programming is NP-hard. A special case, 0-1 integer linear programming, in which unknowns are binary, and only the restrictions must be satisfied, is one of Karp's 21 NP-complete problems.
  - By contrast, polynomial-time algorithms exist for **linear programming**, which we will see next.

Integer linear programs

Canonical and standard form for ILPs [edit]

An integer linear program in canonical form is expressed as:[1]

$$\begin{aligned} &\text{maximize} && \mathbf{c}^\mathrm{T}\mathbf{x} \\ &\text{subject to} && A\mathbf{x} \le \mathbf{b}, \\ & && \mathbf{x} \ge \mathbf{0}, \\ &\text{and} && \mathbf{x} \in \mathbb{Z}^n, \end{aligned}$$

and an ILP in standard form is expressed as

$$\begin{aligned} &\text{maximize} && \mathbf{c}^\mathrm{T}\mathbf{x} \\ &\text{subject to} && A\mathbf{x} + \mathbf{s} = \mathbf{b}, \\ & && \mathbf{s} \ge \mathbf{0}, \\ &\text{and} && \mathbf{x} \in \mathbb{Z}^n, \end{aligned}$$

where $\mathbf{c}, \mathbf{b}$ are vectors and A is a matrix. Note that similar to linear programs, ILPs not in standard form can be converted to standard form by eliminating inequalities by introducing slack variables ($\mathbf{s}$) and replacing variables that are not sign-constrained with the difference of two sign-constrained variables

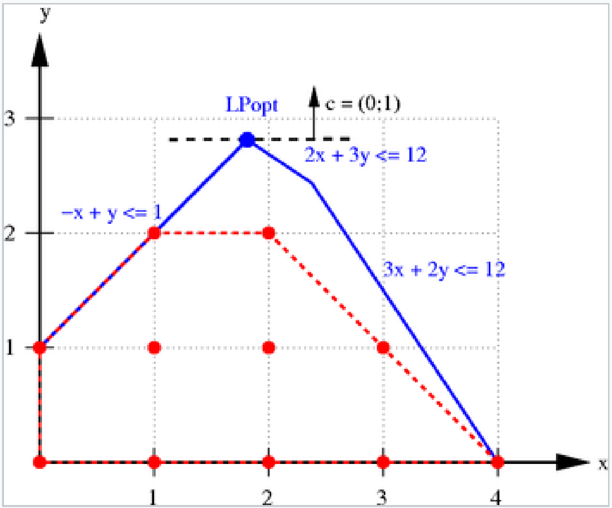
ILP example

Example [edit]

The graph on the right shows the following problem.

$$\begin{aligned} \max y \\ -x + y \leq 1 \\ 3x + 2y \leq 12 \\ 2x + 3y \leq 12 \\ x, y \geq 0 \\ x, y \in \mathbb{Z} \end{aligned}$$

The feasible integer points are shown in red, and the red dashed lines indicate their convex hull, which is the smallest polyhedron that contains all of these points. The blue lines together with the coordinate axes define the polyhedron of the LP relaxation, which is given by the inequalities without the integrality constraint. The goal of the optimization is to move the black dotted line as far upward while still touching the polyhedron. The optimal solutions of the integer problem are the points $(1, 2)$ and $(2, 2)$ which both have an objective value of 2. The unique optimum of the relaxation is $(1.8, 2.8)$ with objective value of 2.8. Note that if the solution of the relaxation is rounded to the nearest integers, it is not feasible for the ILP.



IP polytope with LP relaxation

ILP applications

Applications [edit]

There are two main reasons for using integer variables when modeling problems as a linear program:

1. The integer variables represent quantities that can only be integer. For example, it is not possible to build 3.7 cars.
2. The integer variables represent decisions and so should only take on the value 0 or 1 .

These considerations occur frequently in practice and so integer linear programming can be used in many applications areas, some of which are briefly described below.

Production planning [edit]

Mixed integer programming has many applications in industrial production, including job-shop modelling. One important example happens In agricultural production planning involves determining production yield for several crops that can share resources (e.g. Land, labor, capital, seeds, fertilizer, etc.). A possibl objective is to maximize the total production, without exceeding the available resources. In some cases, this can be expressed in terms of a linear program, but variables must be constrained to be integer.

Scheduling [edit]

These problems involve service and vehicle scheduling in transportation networks. For example, a problem may involve assigning buses or subways to individual routes so that a timetable can be met, and also to equip them with drivers. Here binary decision variables indicate whether a bus or subway is assigned to a route and whether a driver is assigned to a particular train or subway.

Telecommunications networks [edit]

The goal of these problems is to design a network of lines to install so that a predefined set of communication requirements are met and the total cost of the network is minimal.[4] This requires optimizing both the topology of the network along with the setting the capacities of the various lines. In many cases, the capacities are constrained to be integer quantities. Usually there are, depending on the technology used, additional restrictions that can be modeled as a linea inequalities with integer or binary variables.

Cellular networks [edit]

The task of frequency planning in GSM mobile networks involves distributing available frequencies across the antennas so that users can be served and interference is minimized between the antennas.[5] This problem can be formulated as an integer linear program in which binary variables indicate whether a frequency is assigned to an antenna.

NP-hardness of ILP

Proof of NP-hardness [edit]

The following is a reduction from minimum vertex cover to integer programming that will serve as the proof of NP-hardness.

Let $G = (V, E)$ be an undirected graph. Define a linear program as follows:

$$\min \sum_{v \in V} y_v$$

$$\begin{aligned} y_v + y_u &\geq 1 && \forall uv \in E \\ y_v &\geq 0 && \forall v \in V \\ y_v &\in \mathbb{Z} && \forall v \in V \end{aligned}$$

Given that the constraints limit y_v to either 0 or 1, any feasible solution to the integer program is a subset of vertices. The first constraint implies that at least one end point of every edge is included in this subset. Therefore, the solution describes a vertex cover. Additionally given some vertex cover C, y_v can be set to 1 for any $v \in C$ and to 0 for any $v \notin C$ thus giving us a feasible solution to the integer program. Thus we can conclude that if we minimize the sum of y_v we have also found the minimum vertex cover.[2]

Homework for next class

- Chapters 2-3 from Jensen textbook.
- HW1: out 9/5 due today
- HW2: out this week due 10/17