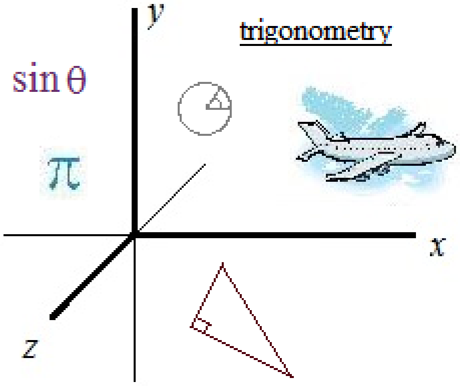
Trigonometry Identities III – Solving and Graphing

Strategies, examples, formulas, and practice questions (with solutions)



Trig identities, functions, and strategies

Various strategies and trig identities are utilized in the following equations.
Try them yourself. (What approach will you use in each question?)
Then, check out steps and solutions on the following pages...

1) $\dfrac{\sin(90^\circ - x)}{\sin x} = -\sqrt{3}$

2) $\tan 2(x + 41^\circ) = 1$

for $0^\circ < x < 360^\circ$

3) $\sin 2x = \cos x$

$0 \leq x < 2\pi$

4) $\cos^2 x = \dfrac{1}{4}$

$x \in [-180^\circ, 180^\circ]$

5) $2\sin(x + 63^\circ) = 1$

where x is in the interval $[0, 360^\circ)$

6) $2\cos^2\left(\dfrac{1}{2}x\right) - 2 = 2\cos x$

7) $4\sin x\cos x = 1$

for x in $[0, 360^\circ)$

8) $3 - 3\sin x - 2\cos^2 x = 0$

9) $\tan x - 1 = 2\tan x$

Trig identities, functions, and strategies

1) $\frac{\sin(90^\circ - x)}{\sin x} = -\sqrt{3}$

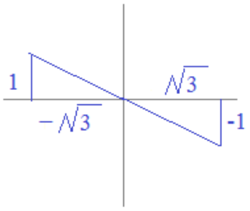
$\frac{\cos x}{\sin x} = -\sqrt{3}$ cofunction identity

$\cot x = -\sqrt{3}$ quotient identity

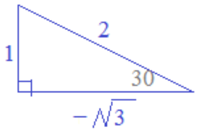
$\boxed{x = 150^\circ + n(180)^\circ}$

test $x = 150^\circ$

$\frac{\sin(90 - (150))}{\sin(150)} = \frac{\sin(-60)}{(1/2)} = \frac{\frac{-\sqrt{3}}{2}}{(1/2)} = -\sqrt{3}$ ✓



30-60-90



$\text{cotangent} = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{-\sqrt{3}}{1}$

x = -30, 150, 330, 510, ...

2) $\tan 2(x + 41^\circ) = 1$

for $0^\circ < x < 360^\circ$

First, find the inverse tangent of 1...

Let $U = 2(x + 41)$ substitution

$\tan U = 1$

U = -135, 45, 225, 405, 585...

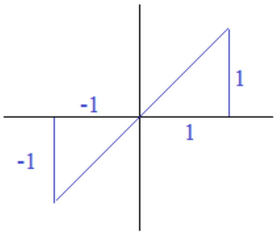
Now, substitute to find *x values between 0 and 360*

$2(x + 41) = 45$
$x + 41 = 22.5$
$x = \cancel{-18.5}$

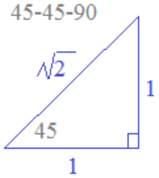
-18.5 is not in specified domain

$2(x + 41) = 225$
$x + 41 = 112.5$

$$\boxed{\begin{array}{c} x = 71.5 \\ \text{and, } x = 161.5, 251.5, 341.5 \end{array}}$$



45-45-90



$\text{tangent} = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{1}{1}$

U = 45, 225, 405, ...

3) $\sin 2x = \cos x$

$0 \leq x < 2\pi$

$2\sin x\cos x = \cos x$ double angle identity

$2\sin x\cos x - \cos x = 0$ (note: we do not divide both sides by cosx)

$\cos x(2\sin x - 1) = 0$ factor and solve

$\cos x = 0$ $\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}}$

$2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$ $\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}}$

Quick check:

let $x = \frac{\pi}{2}$

$\sin 2(\frac{\pi}{2}) = \cos(\frac{\pi}{2})$

$\sin \pi = \cos(\frac{\pi}{2})$

$0 = 0$ ✓

let $x = \frac{\pi}{6}$

$\sin 2(\frac{\pi}{6}) = \cos(\frac{\pi}{6})$

$\sin \frac{\pi}{3} = \cos(\frac{\pi}{6})$

$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$ ✓

Trig identities, functions, and strategies

4) $\cos^2 x = \frac{1}{4}$

$x \in [-180^\circ, 180^\circ]$

note:

$\cos^2 x = (\cos x)(\cos x)$

$\cos^2 x \neq (\cos x^2)$

$$\sqrt{\cos^2 x} = \sqrt{\frac{1}{4}}$$

$$\cos x = \pm\frac{1}{2}$$

(since we "square rooted a squared term", the solutions are +/–)

$$\boxed{x = 60^\circ,\ -60^\circ,\ 120^\circ,\ -120^\circ}$$

5) $2\sin(x + 63^\circ) = 1$

where x is in the interval $[0, 360^\circ)$

$\sin(x + 63) = \frac{1}{2}$

let $U = (x + 63)$

find U, where $\sin(U) = \frac{1}{2}$

$U = \arcsin\frac{1}{2} = 30, 150, 390, 510, \ldots$
and, $-210, -330, \ldots$

Now, we need to <u>find x</u>:

Since $U = (x + 63)$,

$U = 30 \dashrightarrow x = -33$ not inside [0, 360)

$U = 150 \dashrightarrow \boxed{x = 87}$

$U = 390 \dashrightarrow \boxed{x = 327}$

$U = 510 \dashrightarrow x = 447$ not inside [0, 360)

quick check:

$x = 87$

$2\sin((87) + 63) = 1$

$2\sin 150 = 1$

$2(1/2) = 1$ ✓

$x = 327$

$2\sin((327 + 63) = 1$

$2\sin 390 = 1$

$2(1/2) = 1$ ✓

6) $2\cos^2\left(\frac{1}{2}x\right) - 2 = 2\cos x$

divide both sides of equation by 2

$\cos^2\left(\frac{1}{2}x\right) - 1 = \cos x$

recognize (and isolate) the half angle..

$\cos^2\left(\frac{1}{2}x\right) = \cos x + 1$

$\frac{1 + \cos x}{2} = \cos x + 1$

multiply both sides by 2

$1 + \cos x = 2\cos x + 2$

$-1 = \cos x$

$$\boxed{x = -\pi + 2\pi k \text{ where k is any integer..}}$$

half angle identity

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$$

$$\left(\cos\frac{x}{2}\right)^2 = \frac{1 + \cos x}{2}$$

quick check: let $x = \pi$

$2\cos^2(\frac{1}{2}\pi) - 2 = 2\cos(\pi)$

$2(0)^2 - 2 = 2(-1)$

$-2 = -2$ ✓

Trig identities, functions, and strategies

7) $4\sin x\cos x = 1$

for x in $[0, 360^\circ)$

recognize the double angle within the left side of the equation! Then, re-write...

$2(2\sin x\cos x) = 1$ double angle identity

$2(\sin 2x) = 1$

$\sin 2x = \frac{1}{2}$ take inverse sine of both sides

$2x = 30, 150, 390, 510, 750, 870...$

$\boxed{x = 15^\circ, 75^\circ, 195^\circ, 255^\circ}$, ~~375~~

check w/calculator:

$x = 15^\circ$

$4\sin 15\cos 15 \cong 4(.26)(.97) \cong 1$ ✓

$x = 255^\circ$

$4\sin 255\cos 255 \cong 4(-.97)(-.26) \cong 1$ ✓

8) $3 - 3\sin x - 2\cos^2 x = 0$

using trig identity, write $\cos^2 x$ in terms of sine

pythagorean trig identity:

$\sin^2 x + \cos^2 x = 1$

so, $\cos^2 x = 1 - \sin^2 x$

$3 - 3\sin x - 2(1 - \sin^2 x) = 0$ collect like terms; set equal to zero..

$2\sin^2 x - 3\sin x + 1 = 0$

$(2\sin x - 1)(\sin x - 1) = 0$ factor and solve..

$2\sin x - 1 = 0$

$\sin x = 1/2$

$\sin x - 1 = 0$

$\sin x = 1$

$x = 30, 150, 390, 510...$

$x = 90, 450, ...$

or

$\frac{\pi}{6} + 2\pi k$

$\frac{5\pi}{6} + 2\pi k$

$\frac{\pi}{2} + 2\pi k$

check a solution:

let $x = 30^\circ$

$3 - 3\sin(30) - 2\cos^2(30) =$

$3 - 3 \cdot \frac{1}{2} - 2\left(\frac{\sqrt{3}}{2}\right)^2 =$

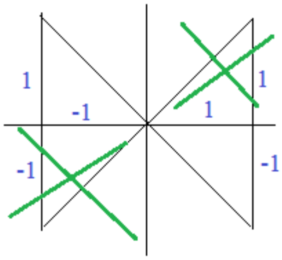
$3 - 3/2 - 2 \cdot (3/4) = 0$ ✓

9) $\tan x - 1 = 2\tan x$

$-1 = 2\tan x - \tan x$

$-1 = \tan x$

$x = 135^\circ$ or 315°



reference angle 45°

quadrants II and IV

Example: *Let* $x = 2\sin\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Simplify $\frac{x}{\sqrt{4-x^2}}$

Use substitution: $\frac{2\sin\theta}{\sqrt{4-(2\sin\theta)^2}} = \frac{2\sin\theta}{\sqrt{4-4\sin^2\theta}} = \frac{2\sin\theta}{\sqrt{4(1-\sin^2\theta)}} = \frac{2\sin\theta}{\sqrt{4(\cos^2\theta)}} = \frac{2\sin\theta}{2\cos\theta}$ $\boxed{=\tan\theta}$

Example: Find $\cos(\theta - \phi)$ where $\cos\theta = \frac{3}{5}$ in Quadrant IV

$\tan\phi = -\sqrt{3}$ in Quadrant II

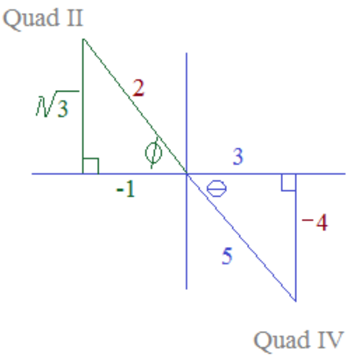
Use Trig Difference Identity

$$\cos\theta\cos\phi + \sin\theta\sin\phi$$

$$\frac{3}{5}\cdot\frac{-1}{2} + \frac{-4}{5}\cdot\frac{\sqrt{3}}{2}$$

$$\frac{-3}{10} + \frac{-4\sqrt{3}}{10}$$

$$\boxed{\frac{-1}{10}(3+4\sqrt{3})}$$



Example: Verify: $\sin(\frac{\pi}{2} - x) = \sin(\frac{\pi}{2} + x)$

$\sin(\frac{\pi}{2} - x)$ Use subtraction/difference trig identity

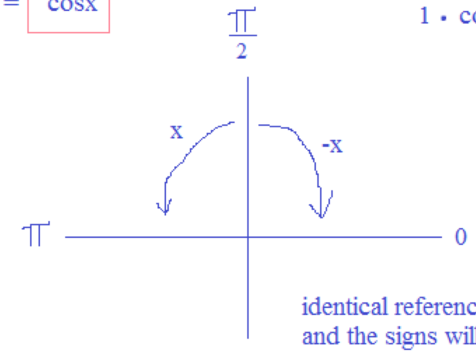
$\sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x =$

$1 \cdot \cos x - 0 \cdot \sin x = \boxed{\cos x}$

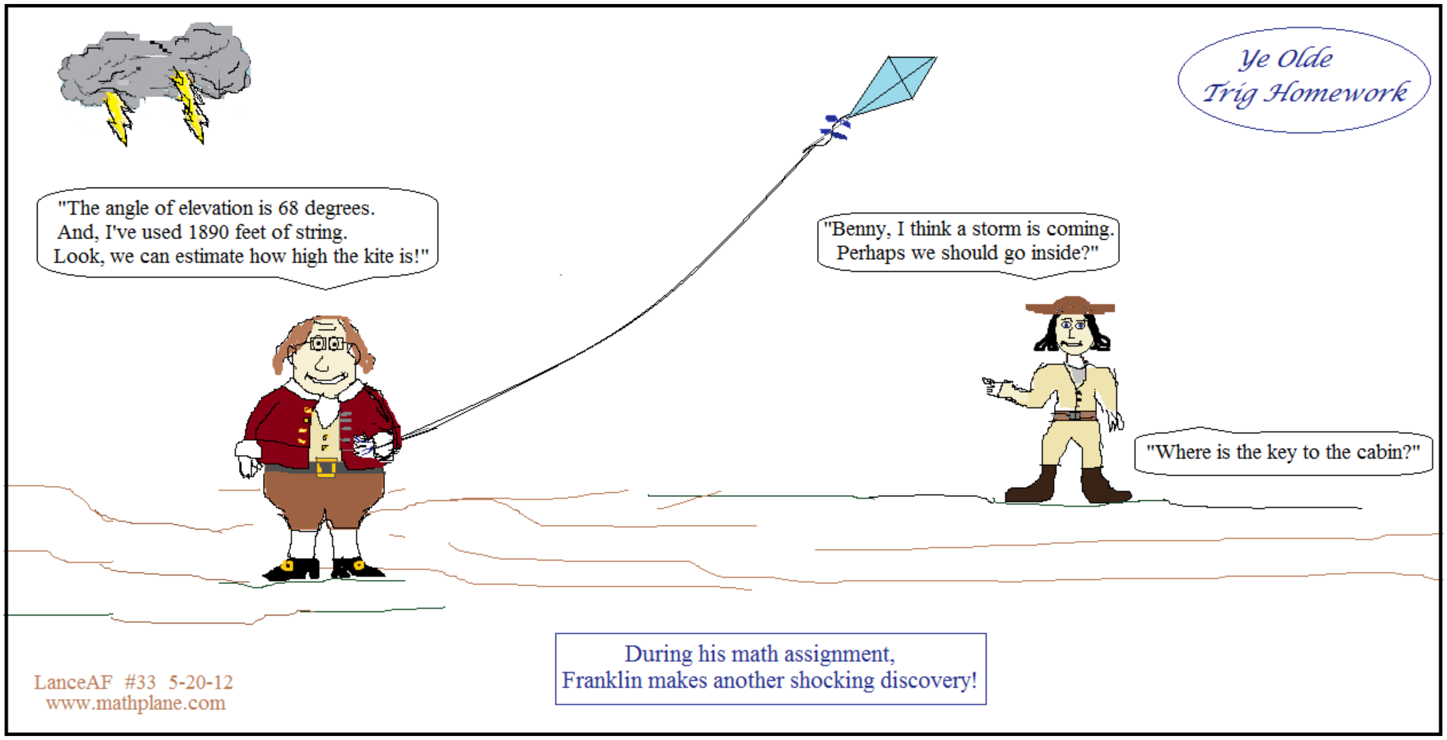
$\sin(\frac{\pi}{2} + x)$ Use addititon/sum trig identity

$\sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x$

$1 \cdot \cos x + 0 \cdot \sin x = \boxed{\cos x}$



identical reference angles
and the signs will be the same



More Questions -→

Solve the following trigonometric equations:

1) $\sin x\cos x = \frac{1}{4}$

$0 < x < 2\pi$

2) $\cos(\frac{\pi}{6} + x) + \cos(\frac{\pi}{6} - x) = \sqrt{3}$

$0 < x < 2\pi$

3) $\cos\theta\cos3\theta - \sin\theta\sin3\theta = 0$

$0 \leq \theta < 360$

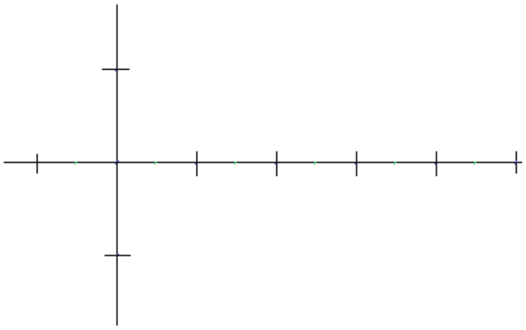
4) $3\tan^3 x = \tan x$

$0 \leq \theta < 360$

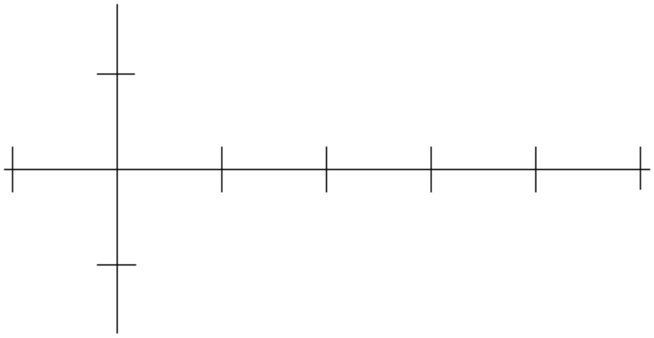
Trigonometry Equations

Solve algebraically. Then, graph to verify your solutions...

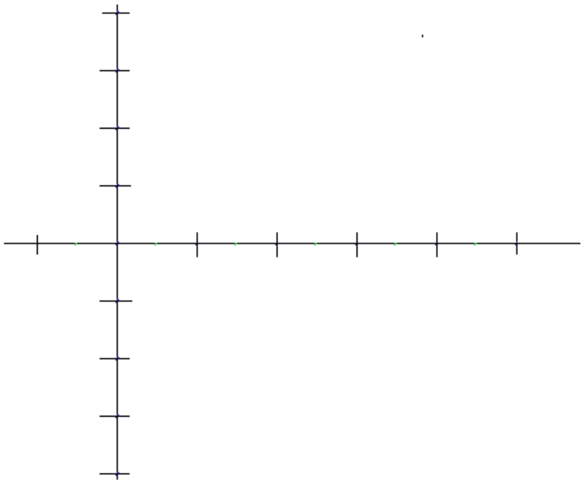
1) $\cos 2\theta = -\cos\theta \qquad 0^\circ \leq \theta < 360^\circ$



2) $-\cos 3x = \sin 3x \qquad 0 \leq x < 2\pi$



3) $3\sin\theta = \sin\theta + 2 \qquad 0^\circ \leq \theta < 360^\circ$

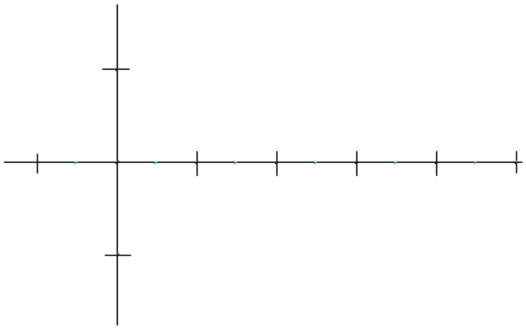


Trigonometry Equations

Solve algebraically. Then, graph to verify your solutions...

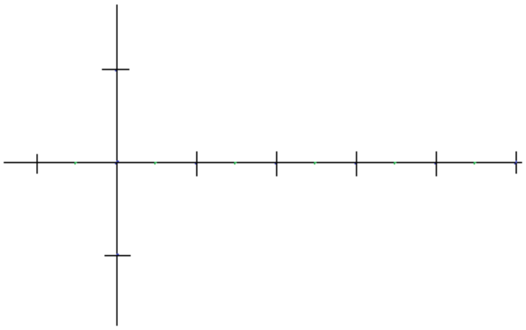
4) $\sec\theta\csc\theta + 2\csc\theta = 0$ $\quad 0^\circ \leq \theta < 360^\circ$

5) $\sec x + \tan x = 1$ $\quad 0 \leq x < 2\pi$



6) $4\sin\theta = \cos\theta - 2$ $\quad 0^\circ \leq \theta < 360^\circ$

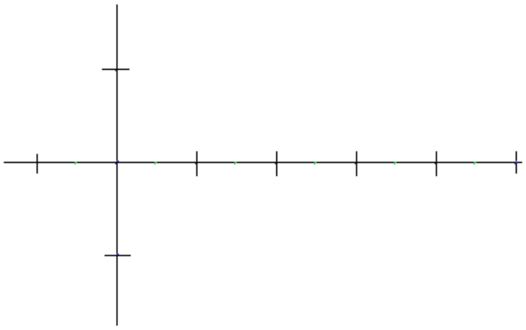
(with calculator)



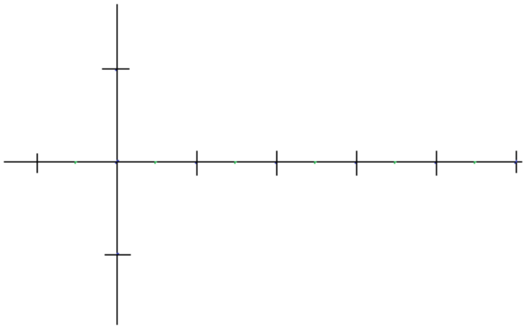
Trigonometry Equations

Solve algebraically. Then, graph to verify your solutions...

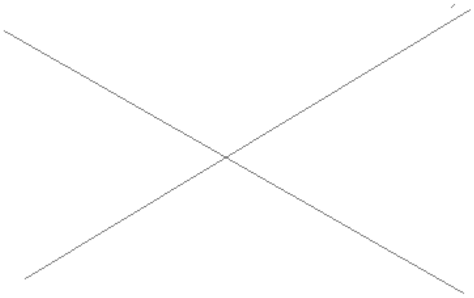
7) $\sin^2 x + \cos x = -1 \qquad 0 \leq x < 2\pi$



8) $2\sin\ominus + \csc\ominus = 0 \qquad 0^\circ \leq \ominus < 360^\circ$



9) $2\sec^2 x + \tan^2 x - 3 = 0 \qquad 0 \leq x < 2\pi$



Solve the following trigonometric equations:

1) $\sin x\cos x = \frac{1}{4}$

$0 < x < 2\pi$

$\sin 2x = 2\sin x\cos x$

$2\sin x\cos x = \frac{2}{4}$

$\sin 2x = \frac{1}{2}$

let U = 2x:

$\sin U = \frac{1}{2}$ $\quad U = \frac{\pi}{6}\quad \frac{5\pi}{6}\quad \frac{13\pi}{6}\quad \frac{17\pi}{6}\quad \frac{25\pi}{6}\quad \frac{29\pi}{6}\ \ldots \quad = 2x$

Therefore, $\boxed{x = \frac{\pi}{12}\quad \frac{5\pi}{12}\quad \frac{13\pi}{12}\quad \frac{17\pi}{12}}\quad \frac{25\pi}{12}\ \ldots.$

four solutions between 0 and 2π

2) $\cos(\frac{\pi}{6} + x) + \cos(\frac{\pi}{6} - x) = \sqrt{3}$

$0 < x < 2\pi$

$\cos x\cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$

$\frac{1}{2}[\cos(\frac{\pi}{6} + x) + \cos(\frac{\pi}{6} - x)] = \frac{1}{2} \cdot \sqrt{3}$

$\cos\frac{\pi}{6}\cos x = \frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{3}}{2}$

$\cos x = 1$ $\quad$ therefore, $\boxed{x = 0}$

3) $\cos\theta\cos3\theta - \sin\theta\sin3\theta = 0$

$0 \leq \theta < 360$

$\cos(x + y) = \cos x\cos y + \sin x\sin y$

$\cos(\theta + 3\theta) = 0$

$\cos(4\theta) = 0$

$4\theta = \arccos(0)$

$4\theta = 90^\circ, 270^\circ, 450^\circ, 630^\circ, 810^\circ, 990^\circ, 1170^\circ, 1350^\circ\ \ldots$

$\boxed{\theta = 22.5^\circ, 67.5^\circ, 112.5^\circ, 157.5^\circ, 202.5^\circ, 292.5^\circ, 337.5^\circ,}$

*plug in a value to check:

112.5:

cos(112.5)cos3(112.5) - sin(112.5)sin3(112.5) =

cos(112.5)cos(337.5) - sin(112.5)sin(337.5) =

$(-.38) \cdot (.92) - (.92) \cdot (-.38) = 0$ ✓

4) $3\tan^3 x = \tan x$

$0 \leq \theta < 360$

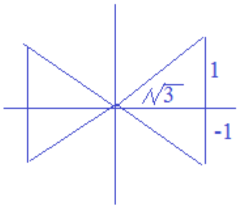
$3\tan^3 x - \tan x = 0$

$\tan x(3\tan^2 x - 1) = 0$

$\tan x = 0$ $\qquad 3\tan^2 x - 1 = 0$

$\boxed{x = 0^\circ, 180^\circ}$ $\qquad \tan^2 x = \frac{1}{3}$

*do not divide both sides by tanx!!
(that may eliminate a solution)



$\tan x = \begin{matrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{matrix}$ $\qquad \boxed{x = 30^\circ, 150^\circ, 210^\circ, 330^\circ}$

Trigonometry equations

Solve algebraically. Then, graph to verify your solutions...

1) $\cos 2\theta = -\cos\theta$ $\qquad 0^\circ \leq \theta < 360^\circ$

$2\cos^2\theta - 1 = -\cos\theta$

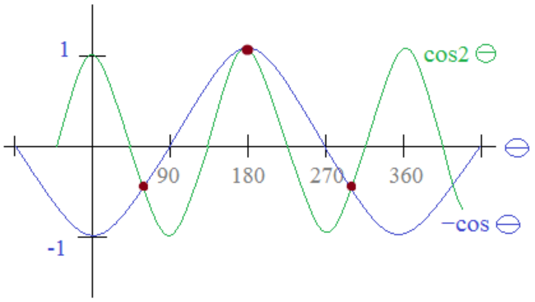
$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$2\cos\theta - 1 = 0 \qquad\qquad \cos\theta + 1 = 0$

$\cos\theta = \frac{1}{2} \qquad\qquad \cos\theta = -1$

$\boxed{\theta = 60^\circ, 300^\circ} \qquad \boxed{\theta = 180^\circ}$



The graphs intersect at
60°, 180°, 300°

2) $-\cos 3x = \sin 3x$ $\qquad 0 \leq x < 2\pi$

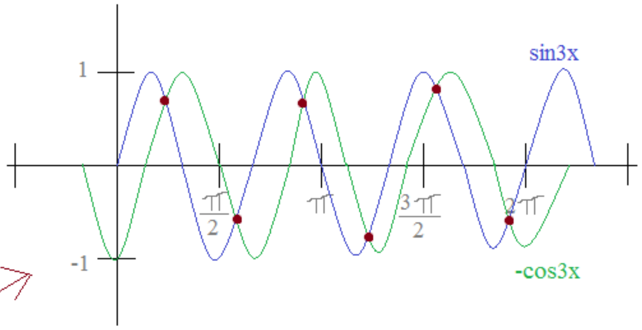
$-1 = \frac{\sin 3x}{\cos 3x}$

$-1 = \tan 3x$

If $3x = U$, then $\tan U = -1$

$U = \frac{3\pi}{4} \quad \frac{7\pi}{4} \quad \frac{11\pi}{4} \quad \frac{15\pi}{4} \quad \frac{19\pi}{4} \quad \frac{23\pi}{4} = 3x$

therefore, $\boxed{x = \frac{\pi}{4} \quad \frac{7\pi}{12} \quad \frac{11\pi}{12} \quad \frac{5\pi}{4} \quad \frac{19\pi}{12} \quad \frac{23\pi}{12}}$



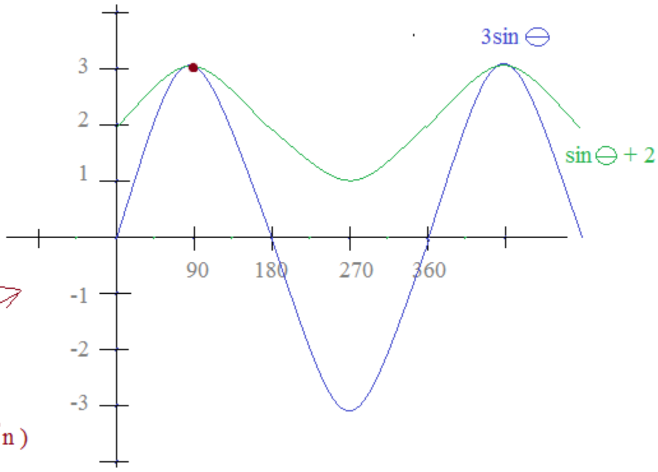
3) $3\sin\theta = \sin\theta + 2$ $\qquad 0^\circ \leq \theta < 360^\circ$

$3\sin\theta - \sin\theta = 2$

$2\sin\theta = 2$

$\sin\theta = 1$

$\boxed{\theta = 90^\circ}$



The graphs intersect at 90° (and 90° + 360°n)

Trigonometry Equations

Solve algebraically. Then, graph to verify your solutions...

4) $\sec\theta\csc\theta + 2\csc\theta = 0 \qquad 0^\circ \leq \theta < 360^\circ$

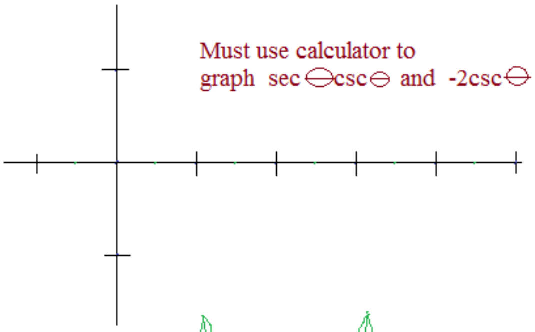
$\csc\theta\,(\sec\theta + 2) = 0$

$\sec\theta + 2 = 0$ $\qquad$ $\csc\theta = 0$

$\sec\theta = -2$

no solution
(cosecant must be greater
than 1 or less than -1)

$\boxed{\theta = 120^\circ,\ 240^\circ}$



Must use calculator to
graph $\sec\theta\csc\theta$ and $-2\csc\theta$

5) $\sec x + \tan x = 1 \qquad 0 \leq x < 2\pi$

$\tan x = \sec x - 1$

$\tan^2 x = (\sec x - 1)^2$

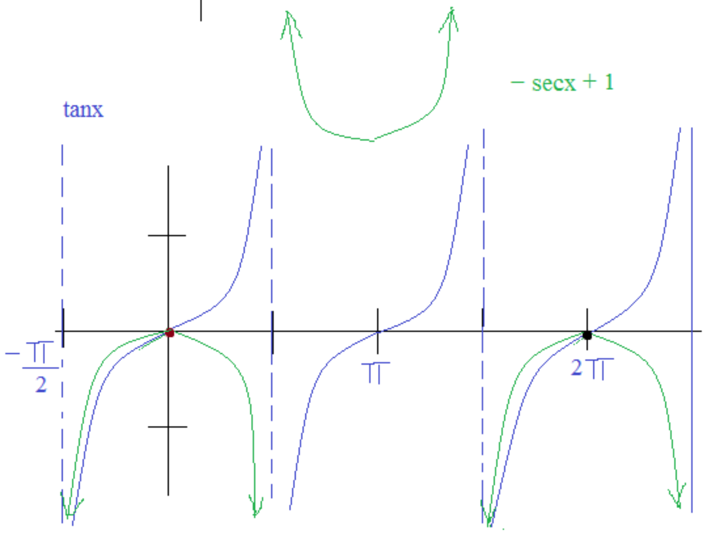
$\tan^2 x = \sec^2 x - 2\sec x + 1$

$\sec^2 x - 1 = \sec^2 x - 2\sec x + 1$

$-2 = -2\sec x$

$\sec x = 1$

$\boxed{x = 0}$



6) $4\sin\theta = \cos\theta - 2 \qquad 0^\circ \leq \theta < 360^\circ$

square both sides

$16\sin^2\theta = \cos^2\theta - 4\cos\theta + 4$

Pythagorean Identity

$16 - 16\cos^2\theta = \cos^2\theta - 4\cos\theta + 4$

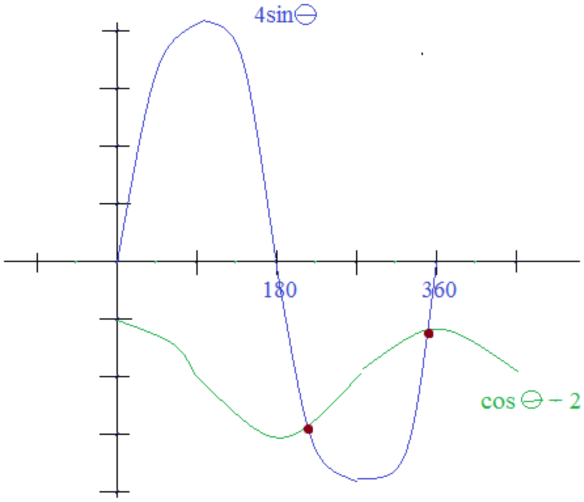
$17\cos^2\theta - 4\cos\theta - 12 = 0$

Use Quadradic formula

$\cos\theta = -.731\ ,\ .966$

$\theta = \cancel{137^\circ},\ 223^\circ\ \ \cancel{15^\circ},\ 345^\circ$

$\boxed{\theta = 223^\circ,\ 345^\circ}$



Trigonometry Equations

Solve algebraically. Then, graph to verify your solutions...

7) $\sin^2 x + \cos x = -1 \qquad 0 \leq x < 2\pi$

$1 - \cos^2 x + \cos x + 1 = 0$

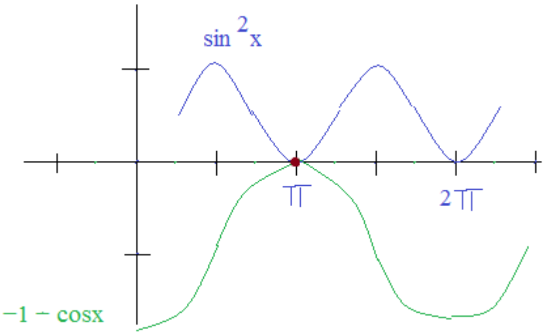
$\cos^2 x - \cos x - 2 = 0$

$(\cos x - 2)(\cos x + 1) = 0$

$\cos x - 2 = 0 \qquad \cos x + 1 = 0$

$\cos x = 2 \qquad \cos x = -1$

no solution $\qquad \boxed{x = \pi}$



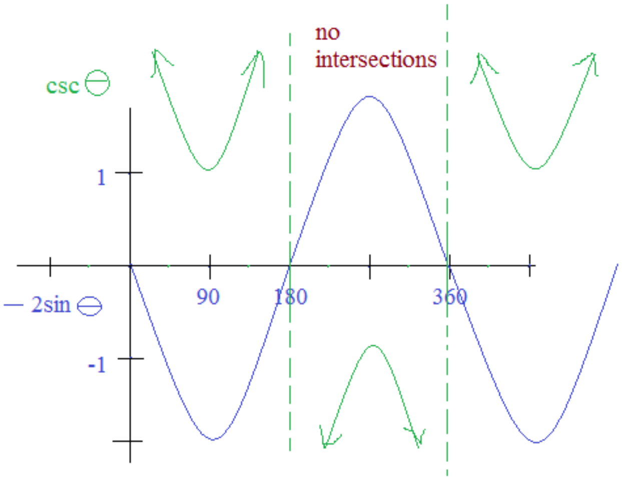
8) $2\sin\theta + \csc\theta = 0 \qquad 0^\circ \leq \theta < 360^\circ$

$$2\sin\theta + \frac{1}{\sin\theta} = 0$$

$$2\sin^2\theta + 1 = 0$$

$$\sin^2\theta = \frac{-1}{2}$$

$\boxed{\text{NO SOLUTION!!}}$



9) $2\sec^2 x + \tan^2 x - 3 = 0 \qquad 0 \leq x < 2\pi$

approach 1:

$2(1 + \tan^2 x) + \tan^2 x - 3 = 0$

$2 + 2\tan^2 x + \tan^2 x - 3 = 0$

$3\tan^2 x = 1$

$$\tan x = \pm\sqrt{\frac{1}{3}}$$

$$\boxed{x = \frac{\pi}{6} \quad \frac{5\pi}{6} \quad \frac{7\pi}{6} \quad \frac{11\pi}{6}}$$

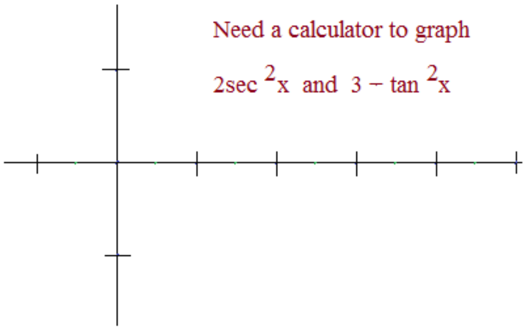
approach 2:

$2\sec^2 x + (\sec^2 x - 1) - 3 = 0$

$3\sec^2 x - 4 = 0$

$$\sec^2 x = \frac{4}{3}$$

$$\sec x = \pm\frac{2}{\sqrt{3}}$$



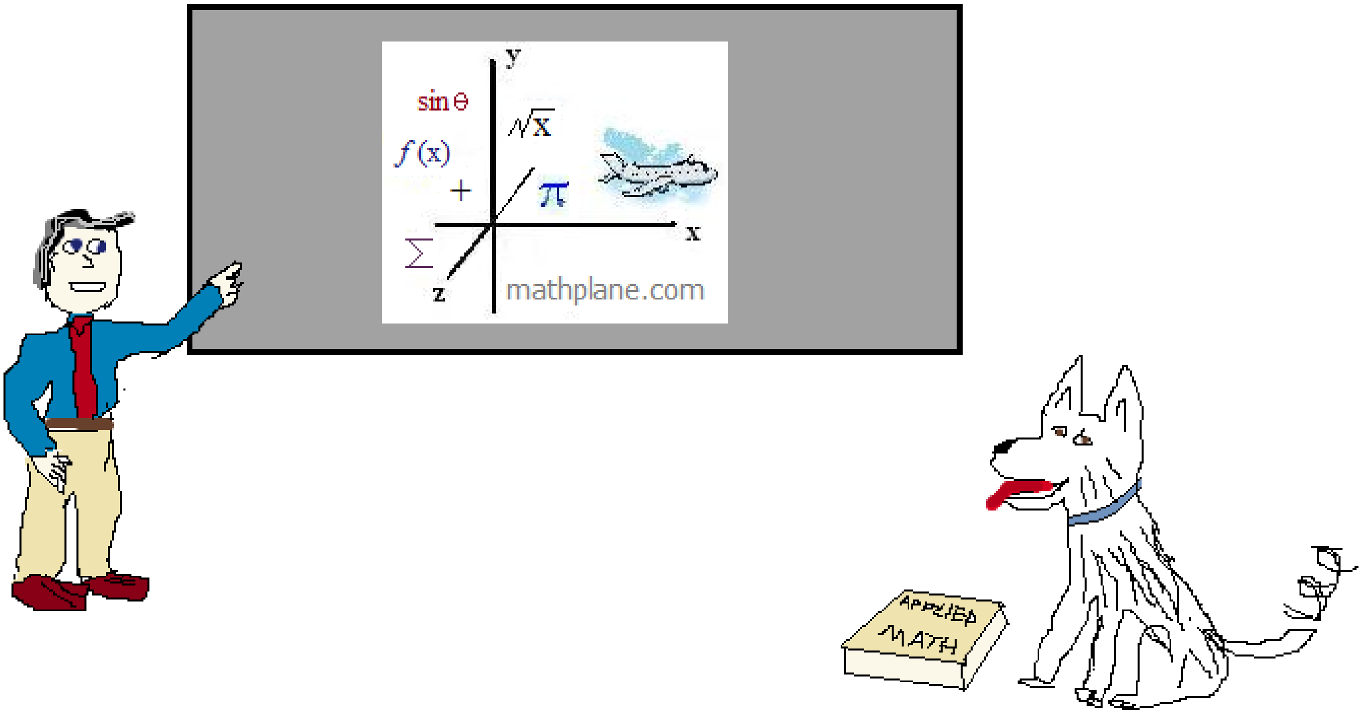
Need a calculator to graph

$2\sec^2 x$ and $3 - \tan^2 x$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Good luck!



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