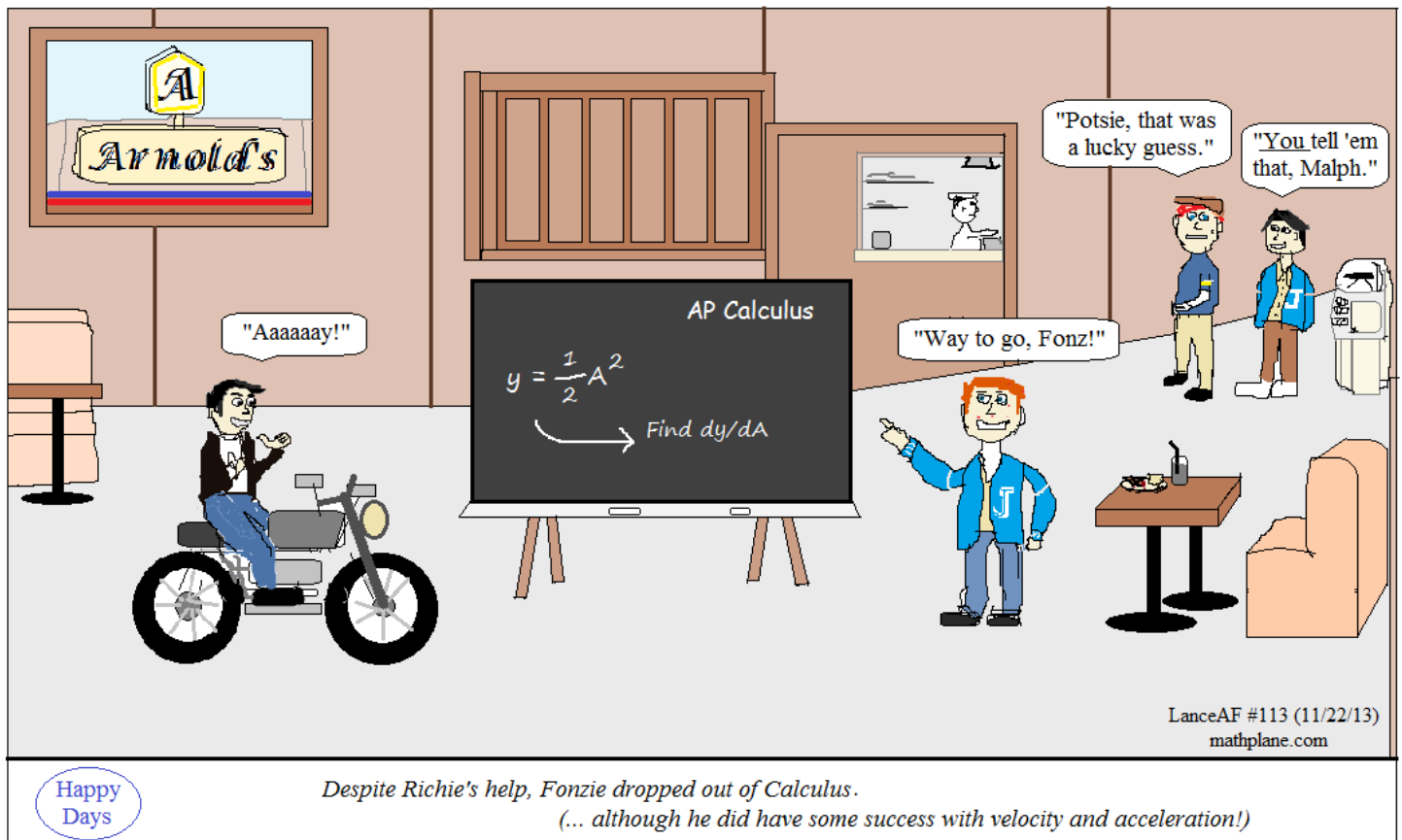


Calculus: Applications of Derivatives

Examples and explanations



Topics include linearization, velocity, sketching rational expressions, volume, slope, and more.

Projectile Motion: Calculus and Algebra Applications

A kid launches a water balloon from a balcony.
The trajectory is described by the function

$$h(t) = -16t^2 + 64t + 80 \quad \text{where}$$

t is time (in seconds)

$h(t)$ is the height of the balloon (in feet)

Using properties of quadratics and Algebra:

- 1) What is the maximum height the water balloon reaches?
- 2) When does the balloon hit the ground?
- 3) What is the height of the balcony?

Answers and Graph:

Since the degree of the polynomial is 2, it's a quadratic (parabola facing down).

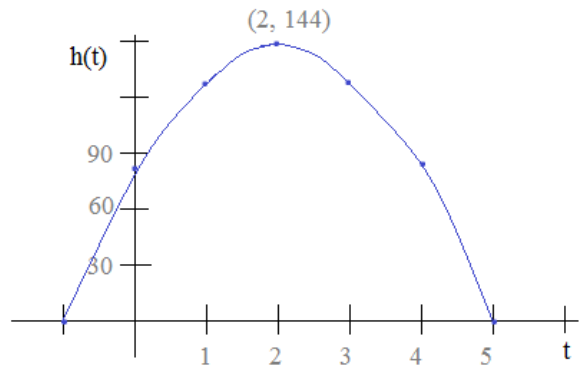
- 1) The maximum height will be at the vertex!

The vertex is $(-b/2a, h(-b/2a))$

$$\begin{aligned} a &= -16 \\ b &= 64 \\ c &= 80 \end{aligned} \quad \frac{-b}{2a} = \frac{-64}{2(-16)} = 2$$

$$h(2) = -16(2)^2 + 64(2) + 80 = 144$$

The maximum height is 144 feet,
and it occurs at 2 seconds...



- 2) Balloon hits the ground when the height is 0.

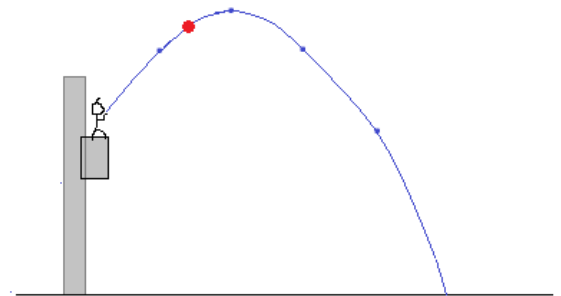
$$h(t) = 0$$

$$-16t^2 + 64t + 80 = 0$$

$$-16(t^2 - 4t - 5) = 0$$

$$-16(t - 5)(t + 1) = 0$$

The balloon hits the ground
at -1 and 5 seconds...
(since time cannot be negative,
the answer is 5 seconds after launch.



**Note: the domain is $0 \leq t \leq 5$

- 3) The height of the balcony would be the $h(t)$ when $t = 0$. (assuming we omit the height of the kid!)

$$h(0) = -16(0) + 64(0) + 80 = 80 \text{ feet}$$

Projectile Motion: Calculus and Algebra Applications

A kid launches a water balloon from a balcony.
The trajectory is described by the function $h(t) = -16t^2 + 64t + 80$

where t is time (in seconds)
 $h(t)$ is the height of the balloon (in feet)

Using Calculus applications

- 1) What is the initial velocity of the balloon?
- 2) What is the maximum height reached?
- 3) What is the acceleration of the balloon at 3 seconds?
- 4) What is the height of the balloon at 4 seconds?
- 5) What is the speed of the balloon at 4 seconds?
- 6) When is the balloon 100 feet high?
- 7) What is the AROC during the entire flight of the balloon?

$h(t)$ describes the position of the balloon

$$h(t) = -16t^2 + 64t + 80 \quad (\text{feet})$$

$h'(t)$ describes the velocity of the balloon ('instantaneous rate of change')

$$h'(t) = -32t + 64 \quad (\text{feet/second})$$

$h''(t)$ describes the acceleration (rate the velocity is changing)

$$h''(t) = -32 \quad (\text{feet/second}^2)$$

$|h'(t)|$ absolute value of velocity is the speed of the balloon

Answers:

- 1) The initial velocity is the rate of change at $t = 0$.

$$h'(0) = -32(0) + 64 = 64 \text{ feet/second}$$

- 2) The maximum height occurs when the balloon changes direction.
(in other words, when the rate of change is zero, the balloon is at a max)

$$h'(t) = -32t + 64 \quad \text{When } h'(t) = 0, \quad t = 2$$

Since max height occurs at $t = 2$ seconds, the balloon is at $h(2) = 144$ feet...

- 3) The acceleration at 3 seconds can be found using the 2nd derivative.

$$h''(t) = -32, \quad h''(3) = -32 \text{ feet/second}^2$$

- 4) and 5) The height is "location", so use the function: $h(4) = -16(4)^2 + 64(4) + 80 = 80$

The velocity is "rate of change", so use the first derivative: $h'(4) = -32(4) + 64 = -64$ feet/second

The speed is the absolute value of the velocity: 64 feet/second

- 6) The "position/location" of the balloon is found using the function:

$$100 = -16t^2 + 64t + 80$$

$$16t^2 - 64t + 20 = 0$$

$$4t^2 - 16t + 5 = 0$$

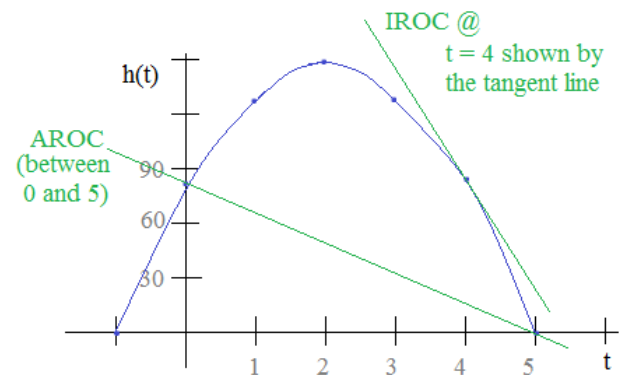
(quadratic formula)

$$t = 0.34 \text{ seconds} \quad \text{and} \quad 3.66 \text{ seconds}$$

- 7) AROC (average rate of change) is the slope between 2 points.

The points in this case are $(0, 80)$ and $(5, 0)$

$$\frac{h(5) - h(0)}{5 - 0} = \frac{0 - 80}{5 - 0} = -16 \text{ feet/second}$$



Example: Two particles that move along a horizontal axis have the following models:

$$x(t) = 3\cos\left(\frac{\pi}{4}t\right) \quad s(t) = t^3 - 6t^2 + 9t + 4$$

On the interval $0 \leq t \leq 6$, when do the particles move in the same direction?

Find the intervals where each particle increases and decreases...

First derivative....

$$x'(t) = -3\sin\left(\frac{\pi}{4}t\right) \cdot \frac{\pi}{4}$$

$$s'(t) = 3t^2 - 12t + 9 + 0$$

Then, set equal to zero (to find where particle changes direction)

$$-3 \frac{\pi}{4} \sin\left(\frac{\pi}{4}t\right) = 0$$

$$3t^2 - 12t + 9 = 0$$

$$\sin\left(\frac{\pi}{4}t\right) = 0$$

$$3(t^2 - 4t + 3) = 0$$

$$t = 4k \text{ (where } k \text{ is any integer)}$$

$$(t-3)(t-1) = 0$$

So, in interval $[0, 6]$, 0 and 4

$$t = 1 \text{ and } 3$$

Then, test each sub-interval to determine whether increasing or decreasing...

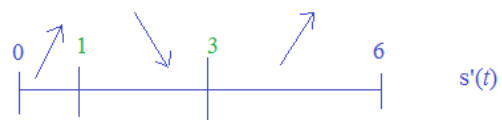
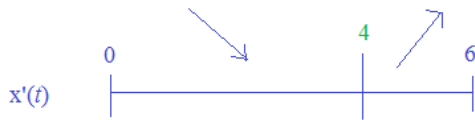
$$x'(1) = -3 \frac{\pi}{4} \sin\left(\frac{\pi}{4} \cdot 1\right) < 0$$

$$s'(1/2) = 3(1/2 - 3)(1/2 - 1) > 0$$

$$x'(5) = -3 \frac{\pi}{4} \sin\left(\frac{\pi}{4} \cdot 5\right) > 0$$

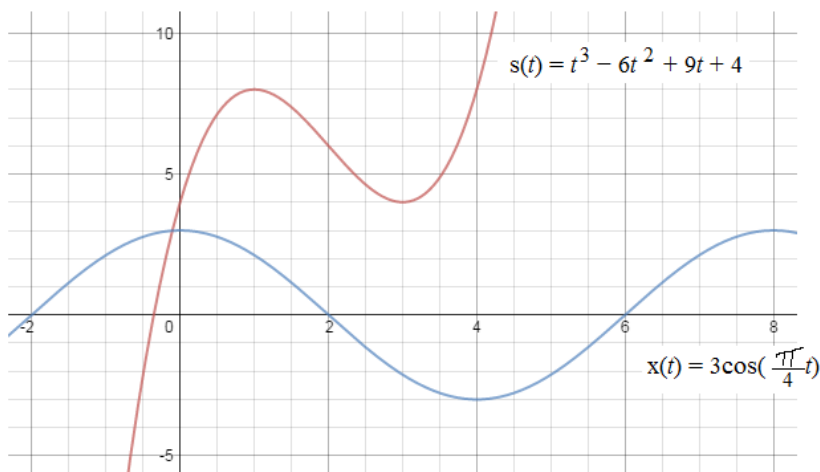
$$s'(2) = 2(2-3)(2-1) < 0$$

$$s'(4) = 2(4-3)(4-1) > 0$$



Finally, determine the sub-intervals where $x(t)$ and $s(t)$ move in the same direction....

Interval $(1, 3)$ where both are decreasing (i.e. moving to the left)
and, Interval $(4, 6]$ where both are increasing (i.e. moving to the right)



Derivatives and Linear Equations

Example: Find equation of a normal line to $f(x) = 3x^3 - 2x^2 + 5x - 3$ at $x = 2$

To find the equation of a line, we need a point and the slope.

To determine the slope, find the first derivative of $f(x)$.

$$f'(x) = 9x^2 - 4x + 5$$

Then, to find the slope of the tangent at $x = 2$: $f'(2) = 9(2)^2 - 4(2) + 5 = 33$

Since the slope of the tangent is 33, the slope of the normal (perpendicular) is $-1/33$

And, a point on the normal will be where $x = 2$:

$$f(2) = 3(2)^3 - 2(2)^2 + 5(2) - 3 = 23$$

Therefore, an equation with slope $-1/33$ going through $(2, 23)$ is

$$y - 23 = \frac{-1}{33}(x - 2)$$

Example: Write the equation of a line tangent to $x^2 + 5x + 6$ at $x = 1$

Then, graph the equation and the tangent line.

If $y = x^2 + 5x + 6$, then

$$y' = 2x + 5$$

Therefore, the instantaneous rate of change at $x = 1$ is

$$f'(1) = 2(1) + 5 = 7$$

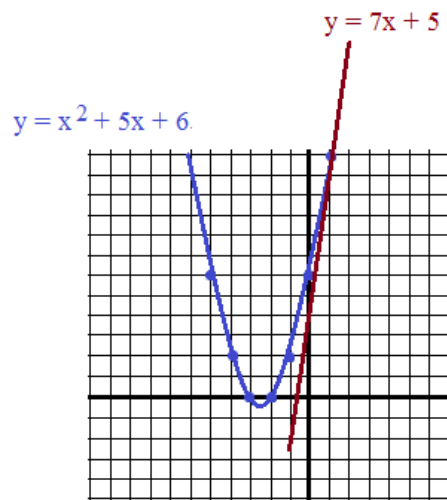
And, to find a point on the tangent line, we use $x = 1$

$$f(1) = (1)^2 + 5(1) + 6 = 12$$

So, the equation of the tangent line is

$$y - 12 = 7(x - 1)$$

$$y = 7x + 5$$



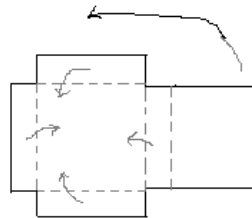
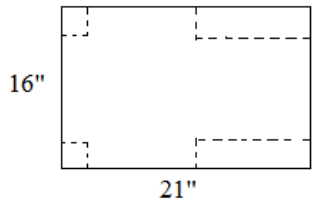
You are given a 16" x 21" cardboard sheet.

After cutting out the corners, you can fold up 3 of the sides.

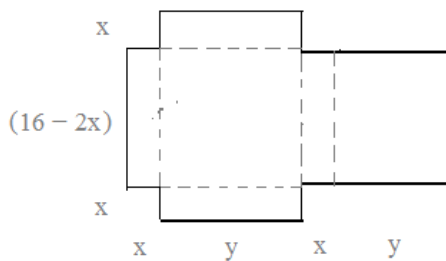
Then, the fourth side will be folded up and extended over the other 3 to form a lid.

What are the dimensions of the enclosed box with the largest volume?

Step 1: Draw a diagram to visualize the question



Step 2: Label diagram, establish variables, and write equations



since $2x + 2y = 21$, $y = \frac{21 - 2x}{2}$

Volume = (length)(width)(height)

length = $(16 - 2x)$

height = x

width = $\frac{(21 - 2x)}{2}$

$V = (16 - 2x)\left(\frac{21 - 2x}{2}\right)(x)$

Step 3: Solve.

To find the maximum (or minimum) volume, find dV/dx and set it equal to 0...

$V = (16x - 2x^2)\left(\frac{21 - 2x}{2}\right)$

$\frac{dV}{dx} = 6x^2 - 74x + 168$

$V = (8x - x^2)(21 - 2x)$

then, set derivative equal to zero...

$V = 168x - 16x^2 - 21x^2 + 2x^3$

$6x^2 - 74x + 168 = 0$

$V = 2x^3 - 37x^2 + 168x$

$3x^2 - 37x + 84 = 0$

$x = 3$ or $28/3$

Step 4: Answer question and check solutions

If $x = 28/3$, height = 9.33
length = $(16 - 2(9.33)) = -2.66$

Impossible... $x \neq 28/3$

If $x = 3$, height = 3
length = $(16 + 2(3)) = 10$
width = $\frac{(21 + 2(3))}{2} = 7.5$

The dimensions of the box (with lid) are

10" x 7.5" x 3"

Check: If $x = 2$, then dimensions are 12" x 8.5" x 2" 204 cubic inches

If $x = 2.5$, then dimensions are 11" x 8" x 2.5" 220 cubic inches

If $x = 3$, then dimensions are 10" x 7.5" x 3" 225 cubic inches

If $x = 3.5$, then dimensions are 9" x 7" x 3.5" 220.5 cubic inches

If $x = 4$, then dimensions are 8" x 6.5" x 4" 208 cubic inches



Linearization formula is similar to point slope form of line.

$$(y - y_1) = m(x - x_1) \quad \text{point slope form where } m \text{ is slope}$$

and (x_1, y_1) is a point

$$L(x) = f(a) + f'(x)(x - a)$$

$$L(x) - f(a) = f'(x)(x - a)$$

$$y - y_1 = m(x - x_1)$$

Linearization Application:

Example: Approximate the value of $\sqrt{145}$

The general equation for square root is $f(x) = \sqrt{x}$

To approximate, we'll use the *nearby* point $(144, 12)$

To find the linear approximation, we need the slope (derivative)...

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(144) = \frac{1}{2\sqrt{144}} = \frac{1}{24}$$

$$(y - y_1) = m(x - x_1) \quad L(x) - 12 = \frac{1}{24}(x - 144)$$

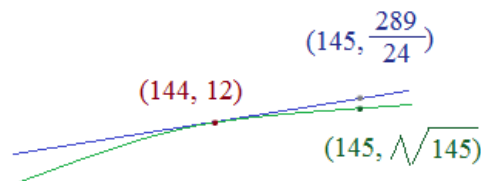
$$y = mx + b \quad L(x) = \frac{1}{24}x + 6$$

Approximate $\sqrt{145}$, using the linear equation:

$$L(145) = \frac{145}{24} + 6 = \frac{289}{24} \approx 12.04167$$

$$\text{True value: } \sqrt{145} \approx 12.04159$$

Very close!!



Finding derivatives of inverses and slope of inverses

Method 1: (If possible), find the inverse and take the derivative...

Example: $f(x) = x^3 + 7$

If $f^{-1}(x)$ is the inverse, find the slope of the curve at $f^{-1}(15)$

Find the inverse...

$y = x^3 + 7$ switch the x's and y's...

$x = y^3 + 7$ solve for y...

$x - 7 = y^3$
 $\sqrt[3]{x - 7} = y$

Take the derivative...

$f^{-1}(x) = \sqrt[3]{x - 7}$

The derivative is $\frac{1}{3}(x - 7)^{-2/3}$

So, the slope at 15 is

$\frac{1}{3}(15 - 7)^{-2/3} = \frac{1}{12}$

Unfortunately, sometimes it's difficult to find the inverse.

Method 2: $g'(x) = \frac{1}{f'(g(x))}$

Example: $f(x) = x^3 + 3x + 6$

If $g(x)$ is the inverse, find $g'(2)$

The relevant coordinate is $(-1, 2)$ $f(x)$
 $(2, -1)$ $g(x)$

The slope of tangent at $(-1, 2)$ is

$3(-1)^2 + 3(-1) + 6 = 6 \leftarrow f'(g(x))$

***Therefore the slope of tangent at $(2, -1)$ must be $\frac{1}{6}$

$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(-1)} = \frac{1}{6}$

NOTE: Since $f(x)$ and $g(x)$ are inverses, their coordinates are flipped...

$f(x)$ has a coordinate $(?, 2)$
 and
 $g(x)$ has a coordinate $(2, ?)$

What is $g(2)$?

$2 = x^3 + 3x + 6$

$x^3 + 3x + 4 = 0$

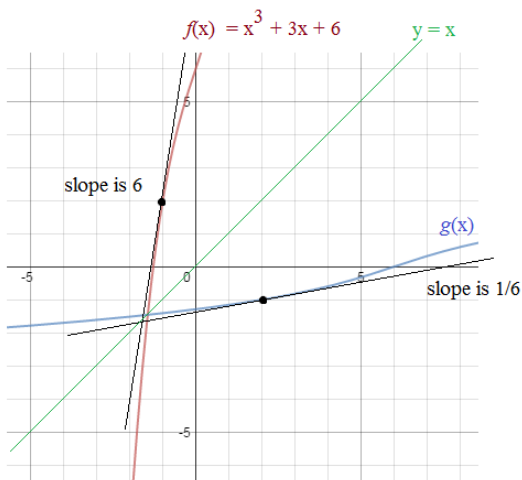
(Using a calculator or Rational Root/Factor Theorem, we can find that (-1) is a solution!)

Possible rational roots: 1, 2, 4, -1, -2, -4

$(-1)^3 + 3(-1) + 4 = 0$ ✓

NOTE: inverse functions reflect over the line $y = x$... Therefore, the slopes of mirror points are reciprocals!

Graph of $f(x)$ and $g(x)$...
 Note the symmetry/reflection over $y = x$



Method 3: Flip the x and y; Use implicit differentiation

$x = y^3 + 3y + 6$

$1 = 3y^2 \frac{dy}{dx} + 3 \frac{dy}{dx} + 0$

$1 = \frac{dy}{dx} (3y^2 + 3)$

$\frac{dy}{dx} = \frac{1}{3y^2 + 3}$

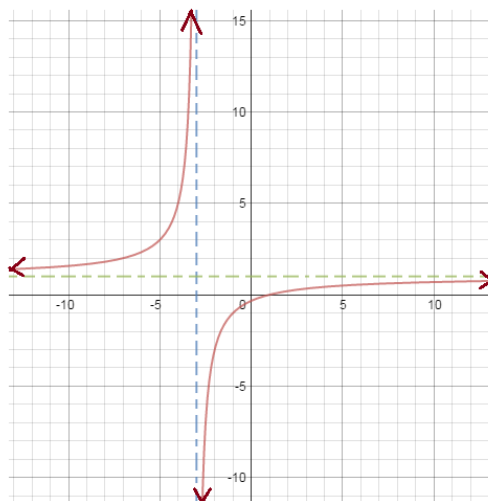
at $(2, -1)$, the slope is $\frac{1}{6}$

Derivative Application: Sketching Rational Expressions

Example: $f(x) = \frac{x-1}{x+3}$

Use algebra concepts to graph the function.
Then, use calculus concepts to verify the shape!

Vertical Asymptotes: $x = -3$	Where the function is undefined. (denominator = 0 and numerator $\neq 0$)
Horizontal Asymptote: $y = 1$	End behavior (since degree of numerator is same as degree of denominator, use the lead coefficients)
"Holes": None	
x-intercept: (1, 0)	x-intercept: when $y = 0$
y-intercept: $(0, -\frac{1}{3})$	y-intercept: when $x = 0$



Now, let's use derivatives to verify the shape:

Use quotient rule

$$f'(x) = \frac{(1)(x+3) - (1)(x-1)}{(x+3)^2} = \frac{2}{(x+3)^2}$$

Critical values: Since $f'(x)$ never equals 0, there are no extrema (no max or min)

The derivative $f'(x)$ is undefined at $x = -3$:
an asymptote

Test -4: $\frac{2}{(-4+3)^2} > 0$ increasing in the interval $(-\infty, -3)$

Test -2: $\frac{2}{(-2+3)^2} > 0$ increasing in the interval $(-3, \infty)$

$$f''(x) = \frac{(0)(x+3)^2 - 2(x+3)^1 \cdot (2)}{(x+3)^2)^2} = \frac{-4(x+3)}{(x+3)^4} = \frac{-4}{(x+3)^3}$$

Concavity: Since $f''(x)$ never equals 0, there are no points of inflection

The second derivative has a critical value at $x = -3$
(where it is undefined)

Test -4: $\frac{-4}{(-4+3)^3} > 0$ concave up between $(-\infty, -3)$

Test -2: $\frac{-4}{(-2+3)^3} < 0$ concave down between $(-3, \infty)$

Example: $g(x) = \frac{x^2-2}{x+1}$ Find the relative extrema.
Determine the concavity and inflection points (if any).

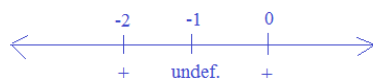
Sketch the graph.

$$g'(x) = \frac{2x(x+1) - 1(x^2-2)}{(x+1)^2} = \frac{x^2+2x+2}{(x+1)^2}$$

$g'(x)$ will never equal zero,
so no max or min...

Note: the discriminant of x^2+2x+2 is -4 ...
So, the two solutions are imaginary!

There is a critical value at -1, where the derivative is undefined...

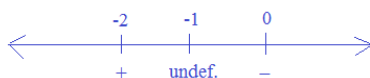


Then, test a point on left and right of -1... increasing for all x, except -1

$$g''(x) = \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x+2)}{(x+1)^4}$$

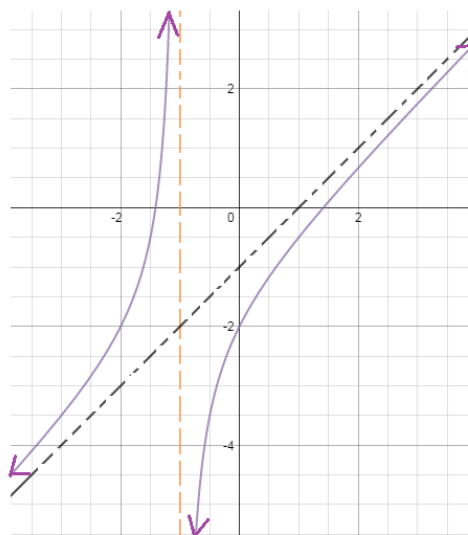
$$= \frac{(2x+2)(x+1) - 2(x^2+2x+2)}{(x+1)^3} = \frac{-2}{(x+1)^3}$$

$g''(x)$ never equals zero, so there are no points of inflection.
But, the 2nd derivative is undefined at $x = -1$



After testing points on the left and right of -1, we find its concavity...

concave up: $(-\infty, -1)$ concave down: $(-1, \infty)$

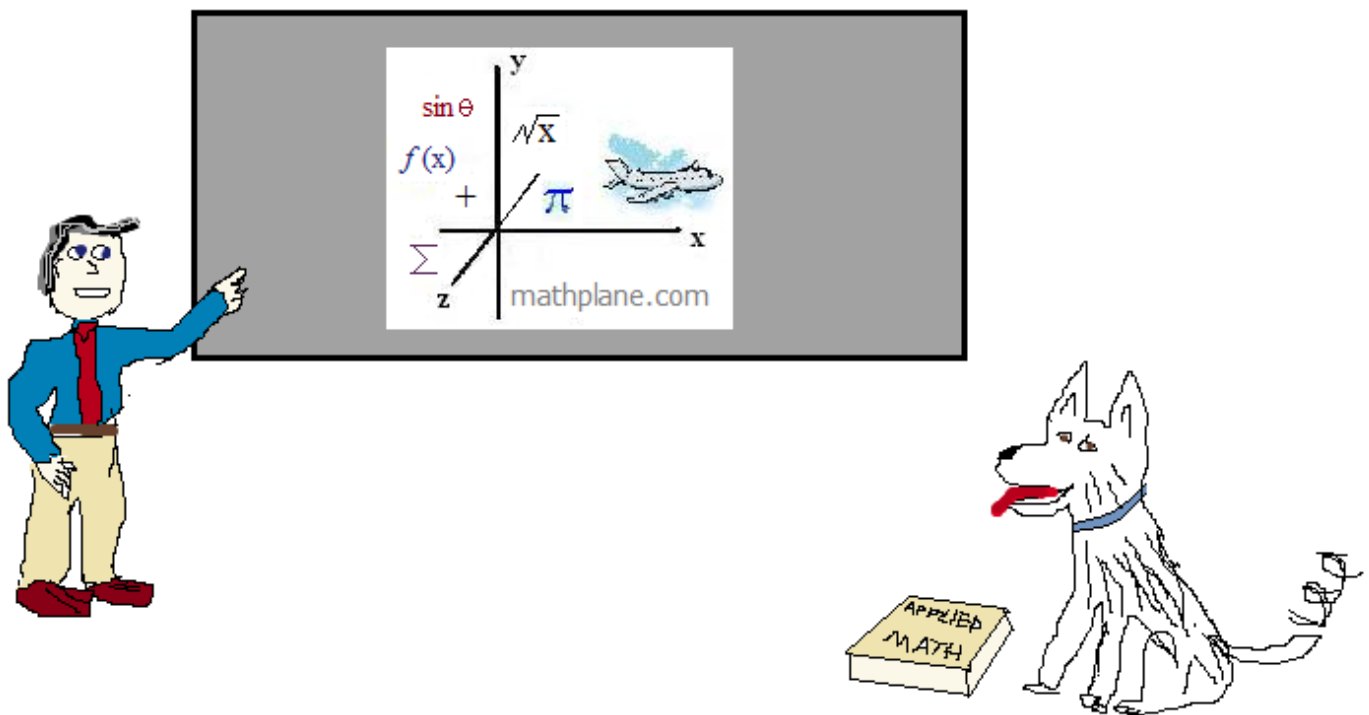


Vertical Asymptote: $x = -1$
Horizontal Asymptote: None
"Slant" (Oblique) Asymptote: $y = x - 1$
x-intercepts: $(\sqrt{2}, 0)$ $(-\sqrt{2}, 0)$
y-intercept: $(0, -2)$

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

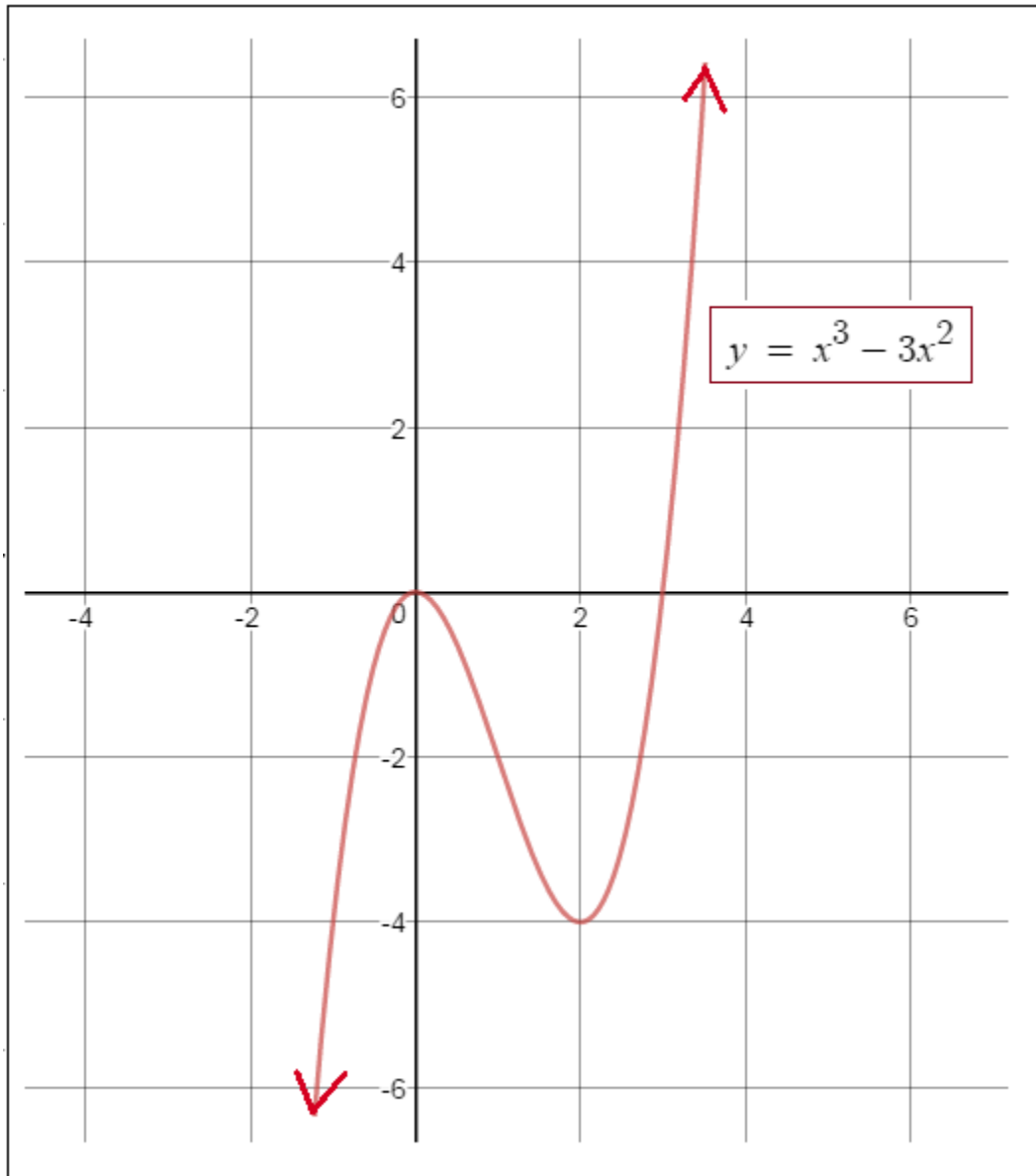
Cheers



Also, at Facebook, Google+, Pinterest, TES, and TeachersPayTeachers

One more question →

What is the equation of the line tangent to the curve *at the point of inflection*?



ANSWER-→

What is the equation of the line tangent to the curve *at the point of inflection*?

$$y = x^3 - 3x^2$$

ANSWER

First, where is the point of inflection?

Where 2nd derivative equals zero.

$$y' = 3x^2 - 6x$$

$$y'' = 6x - 6$$

$$y'' = 0 \text{ when } x = 1$$

Therefore, point of inflection is (1, -2)

$$-2 = (1)^3 - 3(1)^2$$

Now, find the slope at $x = 1$

$$y' = 3x^2 - 6x$$

$$y' = 3(1)^2 - 6(1) = -3$$

Equation of the line:

slope: -3 point: (1, -2)

$$y + 2 = -3(x - 1)$$

or

$$y = -3x + 1$$