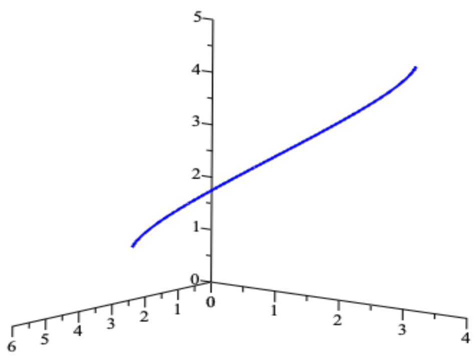
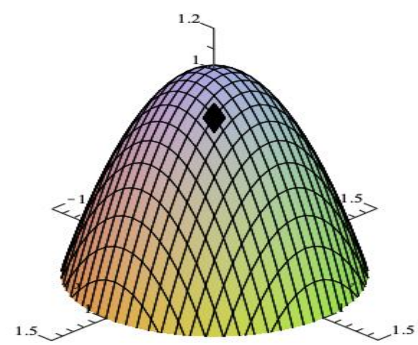
Calculus 3 - Surface Integrals

Earlier we introduced line integrals. Suppose we had a piece of wire with density $\rho(x, y, z)$ that we bent in the shape of a 3D curve $C(x, y, z)$. If we assume that the density is constant along a small piece with length ds, the mass of that piece would be $\rho(x, y, z)ds$ and then add up all the pieces so that in the limit the mass of the wire would be

$$m = \int_C \rho(x, y, z)ds. \tag{1}$$

This we called a *line integral*.





We now do the same except instead of a line, we do this with a surface. Assume that the density of a surface is given by $\rho(x, y, z)$. The shape of the surface is given by $S(x, y, z)$. If we have a small part of the surface, denoted by dS, then the mass of the little part of the surface is $\rho(x, y, z)dS$. Now add up the little pieces and in the limit we get

$$m = \iint_S \rho(x, y, z)dS. \tag{2}$$

This we call a *surface integral*.

Example 1. Evaluate

$$\iint_S (z - 3x - y) dS. \tag{3}$$

where S is the surface of the plane $2x + 5y - z = -1$ on the interval $0 \le x \le 1, 0 \le y \le 1$.

Soln.

First we need to know what dS is. Recall from surface area that the surface is $z = f(x, y)$ then

$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dA \tag{4}$$

Here the surface is $z = 2x + 5y + 1$, so calculating derivatives gives

$$f_x = 2, \quad f_y = 5 \tag{5}$$

and so

$$dS = \sqrt{1 + 2^2 + 5^2}\, dA. \tag{6}$$

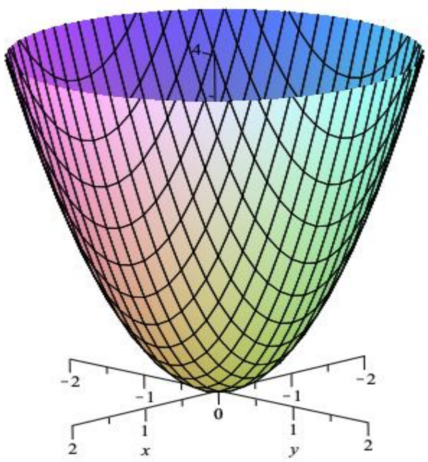
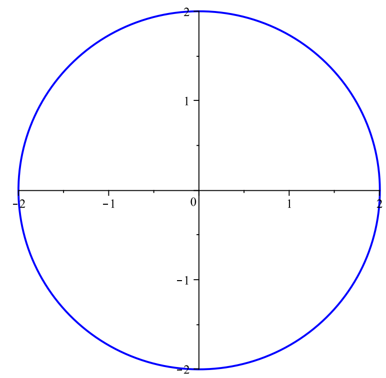
Bringing this and the surface into (3) gives

$$\begin{aligned}
&\int_0^1 \int_0^1 (2x + 5y + 1 - 3x - y) \sqrt{30}\, dy\, dx \\
=&\sqrt{30} \int_0^1 \int_0^1 (-x + 4y + 1)\, dy\, dx \\
=&\sqrt{30} \int_0^1 (-xy + 2y^2 + y)\Big|_0^1 dx \\
=&\sqrt{30} \int_0^1 (-x + 3)\, dx \\
=&\sqrt{30} \left(-\frac{1}{2}x^2 + 3x\right) \Big|_0^1 = \frac{5}{2}\sqrt{50}
\end{aligned} \tag{7}$$

Example 2. Evaluate

$$\iint_S (x^2 + y^2) dS. \tag{8}$$

where S is the surface of the paraboloid $z = x^2 + y^2$ for $0 \le z \le 4$.

Soln.

First we find dS. Since $z = x^2 + y^2$ then

$$f_x = 2x, \quad f_y = 2y \tag{9}$$

and from (4)

$$dS = \sqrt{1 + 4x^2 + 4y^2}\, dA. \tag{10}$$

and (8) is

$$\iint_R (x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\, dA \tag{11}$$

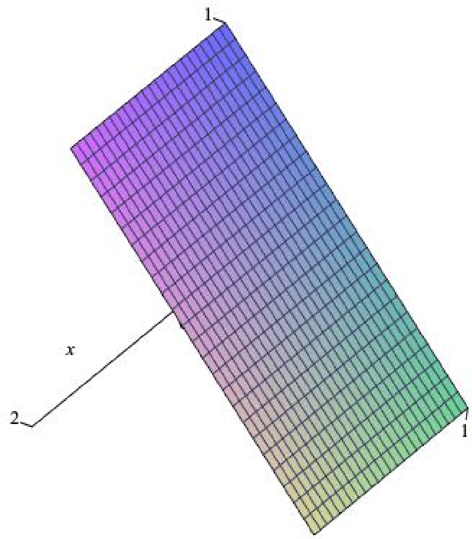
noting that once we bring in the surface, we are now projecting down into the xy plane. Since the region of integration is a circle of radius 2, we introduce polar. In doing (16) becomes gives

$$\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\, r^3 dr\, d\theta = \frac{391\sqrt{17}+1}{60}\,\pi \tag{12}$$

Example 3. Evaluate

$$\iint_S (z^4 + x) dS. \tag{13}$$

where S is the surface of the plane $y + z = 1$ for $0 \le x \le 2$.



Soln.

If we were to bring in the surface $z = 1 - y$ then

$$f_x = 0, \quad f_y = -1 \tag{14}$$

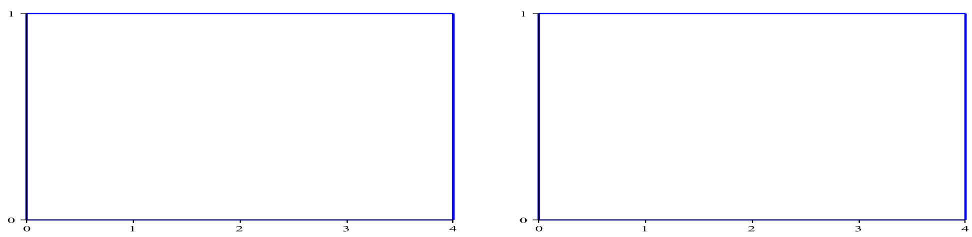
and from (4)

$$dS = \sqrt{2}\, dA. \tag{15}$$

and (13) is

$$\sqrt{2}\int_0^2\int_0^1\Big((1-y)^4+x\Big)dy\,dx. \tag{16}$$

We certainly can do this and the integration wrt y is doable, but maybe projecting in another direction is better. Instead of projecting into the xy plane (down), let's project in the xz plane (from the right)



Previously, given $z = f(x, y)$ then projection down (into the xy plane) we have

$$\iint_S F(x,y,z)dS = \iint_{R_{xy}} F(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dA_{xy} \tag{17}$$

Now if the surface is given as $y = g(x,z)$ then projected left (into the xz plane) we have

$$\iint_S F(x,y,z)dS = \iint_{R_{xz}} F(x,g(x,z),z)\sqrt{1+g_x^2+g_z^2}\,dA_{xz} \tag{18}$$

Similarly if surface is given as $x = h(y,z)$ then projection back (into the yz plane) we have

$$\iint_S F(x,y,z)dS = \iint_{R_{yz}} F(h(y,z),y,z)\sqrt{1+h_y^2+h_z^2}\,dA_{yz} \tag{19}$$

So in the example, we will project into the xz plane. So given that

$$y = 1 - z \tag{20}$$

then we have

$$dS = \sqrt{1 + 0^2 + (-1)^2}\, dxdz \tag{21}$$

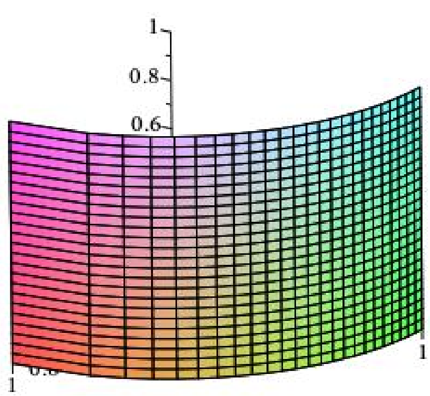
and our surface integral becomes

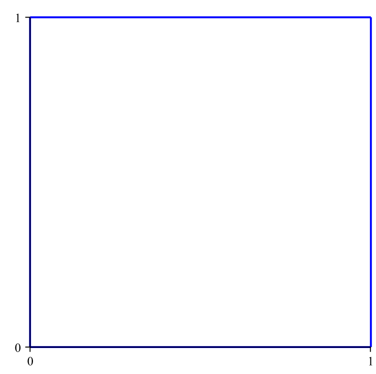
$$\sqrt{2} \int_0^1 \int_0^2 (z^4 + x) dxdz = \frac{12}{5}\sqrt{2}. \tag{22}$$

Example 4. Evaluate

$$\iint\limits_S y dS. \tag{23}$$

where S is the surface of the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 1$.





Soln.

Our choices are to project into the

1. yz plane

2. xz plane

We will set up each and then determine which is better

(i) *yz plane*

We solve the cylinder for x so $x = \sqrt{1 - y^2}$. Now dS is

$$dS = \sqrt{1 + \frac{y^2}{1 - y^2}} dA_{yz} = \frac{1}{\sqrt{1 - y^2}} dA_{yz} \tag{24}$$

The surface integral (23) becomes

$$\int_0^1 \int_0^1 \frac{y}{\sqrt{1 - y^2}} dy\, dz \tag{25}$$

(i) *xz plane*

We solve the cylinder for y so $y = \sqrt{1 - x^2}$. Now dS is

$$dS = \sqrt{1 + \frac{x^2}{1 - x^2}} dA_{xz} = \frac{1}{\sqrt{1 - x^2}} dA_{xz} \tag{26}$$

The surface integral (23) becomes

$$\int_0^1 \int_0^1 \frac{y}{\sqrt{1 - x^2}} dx\, dz = \int_0^1 \int_0^1 \frac{\sqrt{1 - y^2}}{\sqrt{1 - y^2}} dx\, dz = 1 \tag{27}$$

I think the second one is clearly easier!