# MH370 location analysis based on BFO only (version 0.3, June 4th 2015)

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## Introduction

The satellite "ping" data is crucial in the effort to locate the missing aircraft with registration 9M-MRO (carrier of missing flight MH370). The focus of the analysis here is to find a method to calculate the possible end point of the flight path as well as to reconstruct the path followed towards this end point, *without the need of assuming a (average) speed or (autopilot mode) restricted track.* It uses the so called BFO data only to reconstruct position latitude as a function of time. Only after the lat(t) function is found it is combined with BTO data to determine the full positions including longitude.

The analysis presented below can be used in MH370 flight path reconstruction if:

- The published Inmarsat data is complete and correctly represents the received/measured BFO values.
- The AES was located in 9M-MRO
- The AES was functioning correctly (including frequency compensation)

## Vector analysis

The start is eq. (6) as derived by Henrik Rydberg [1], which can be related to the measured BFO [2] assuming a zero rate of climb (ROC) during the portion of the flight we are considering.

$$D = f/(c \cdot \mathrm{p}) \cdot (\mathbf{p} \cdot \mathbf{v_s} + \mathbf{s} \cdot (\mathbf{v_p} - \mathbf{v_s})) \tag{1}$$

Here D is the (first order approximation) of the frequency compensated Doppler shift ($\Delta f_{up}$ + $\Delta f_{comp}$ for zero ROC), $f$ is the transmitter frequency, $c$ is the speed of light, $\mathrm{p}^2 = \mathbf{p} \cdot \mathbf{p}$, **p** is the aircraft position vector, **s** is the satellite position vector, and $\mathbf{v_p}$ and $\mathbf{v_s}$ are the aircraft and satellite velocity vectors respectively. Note that **p** and **s** are defined in a translated ECEF reference system, having the origin shifted to the satellite nominal position **Os** (above N0, E64.5 at geostationary height). Currently we have used **Os** = (18155, 38063, 0) in ECEF [km].

The method basically starts by realizing that the satellite mainly moves N-S about the nominal position, which gives rise to the D value because the frequency compensation algorithm ignores this satellite movement. As "Gysbreght" (@jeffwise.net) pointed out the D value numerically seems not very much dependent on aircraft longitude, nor on longitudinal velocity components. This can be explained (see error analysis) by detailed analysis of eq.(1) and allows ignoring certain x, y components in the vectors in eq. (1). The result is a differential equation (DE) having time dependent coefficient, which expresses $v_{pz}(t)$ as a function of $p_z(t)$, and can be solved for $p_z(t)$ by numerical integration. To do this for all times we need a (smoothened and continuous) parametrization of D, $\mathbf{v_s}$ and **s**. The satellite positions and velocities vs. time are well known and the parametrization currently used was given by "el-gato" (@jeffwise.net):

“the following coefficients to a polynomial of sixth degree (t^0..t^6) describe satellite position and velocity well, where t is the time in seconds since 2014-03-07T00:00:00 (r,v in ecef in m or m/s, resp.):

rx = [25390482.8946,-603.709190215,0.0202346407905,-3.51527772866e-07,3.336001796e-12,-1.62943435432e-17,3.16666502295e-23]
ry = [52828180.6019,-1335.18772729,0.0498326103051,-9.79751909045e-07,1.06955367164e-11,-6.14815083023e-17,1.45432521871e-22]
rz = [-13999389.4178,1383.34310161,-0.0643188230889,1.55178847604e-06,-1.92854573249e-11,1.17064142086e-16,-2.75704174714e-22]

vx = [589.937962045, -0.0634348200486, 2.70332163472e-06, -5.89252174093e-11, 6.97853792222e-16, -4.27753510762e-21, 1.06352790169e-26]
vy = [-1689.70257166, 0.143905678484, -4.9795915624e-06, 8.966832853e-11, -8.86971097191e-16, 4.57247710665e-21, -9.60218377943e-27]
vz = [1009.11103215, -0.103087216501, 3.97737381779e-06, -6.86620866226e-11, 5.40993469347e-16, -1.65772106007e-21, 5.90565435208e-28]“

The parametrization of D has been done in Microsoft Excel based on the measured values at 19:41, 20:41, 21:41, 22:41, 23:11, 00:11 UTC. In the rest of the analysis we define $t$ = 0 at 19:41 UTC on March 7th 2014, and work with $t$ in seconds.

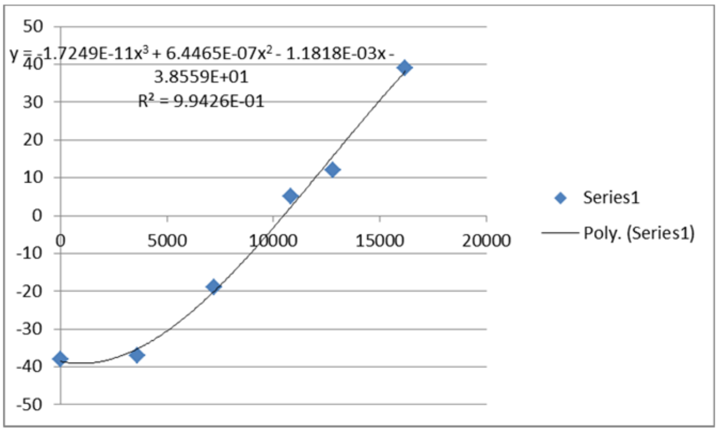



Figure 1: D vs. $t$

In the rest of the analysis we define $t$ = 0 at 19:41 UTC on March 7th 2014, and work with $t$ in seconds.

**Deriving and solving the DE**
First we rewrite eq. (1):

$$D(t) = f / c \cdot p(t) \cdot ((\mathbf{p} - \mathbf{s}) \cdot \mathbf{v_s} + \mathbf{s} \cdot \mathbf{v_p}) \tag{2}$$

The assumptions used to simplify the problem:
* For the relevant time interval $\mathbf{s} \cdot \mathbf{v_p}$ is dominated by $s_z(t) \cdot v_{pz}(t)$,
* Aircraft to (nominal) satellite distance p is relatively weakly dependent on changing aircraft position (because of the large distance of satellite to earth), so use $p(0)$.
* $(\mathbf{p} - \mathbf{s}) \cdot \mathbf{v_s}$ can well be approximated by $(p - s)_x(0) \cdot v_{sx}(t) + (p - s)_y(0) \cdot v_{sy}(t) + (p - s)_z(t) \cdot v_{sz}(t)$

The resulting DE to be solved is:

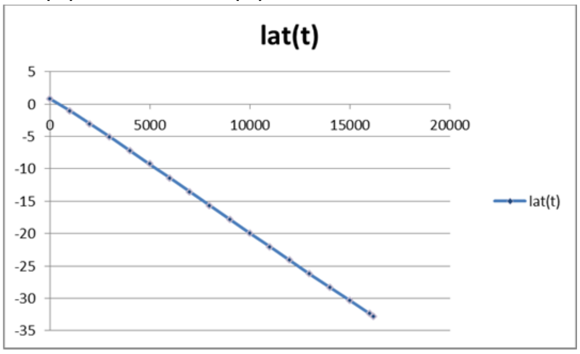
$$p_z(t)\cdot v_{sz}(t)+v_{pz}(t)\cdot s_z(t) = D(t)\cdot c\cdot\ p(t)\ /\ f + s_z(t)\cdot v_{sz}(t)+(s_x(0)-p_x(0))\cdot v_{sx}(t)+(s_y(0)-p_y(0))\cdot v_{sy}(t) \quad (3)$$

Which directly gives $v_{pz}(t)$ as a function of satellite velocity and position, aircraft z-position and residual Doppler shift. It is solved by:

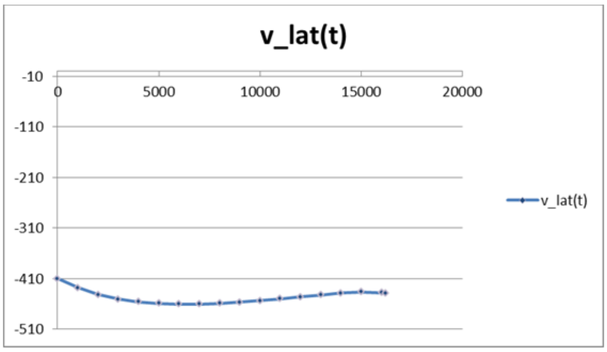
$$p_z\ (t + \Delta t) = p_z\ (t) + \Delta t \cdot v_{pz}\ (t) \quad (4)$$

**Results**

Lat(0) = N0.8, Lon(0) = E94

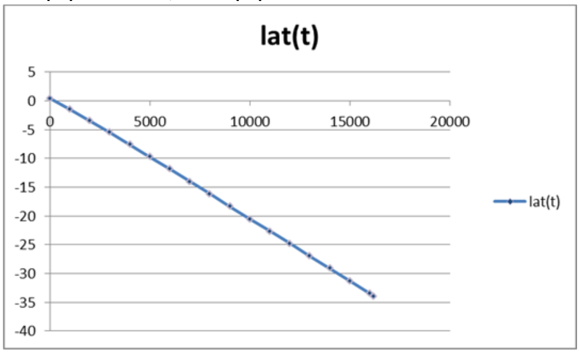



a)

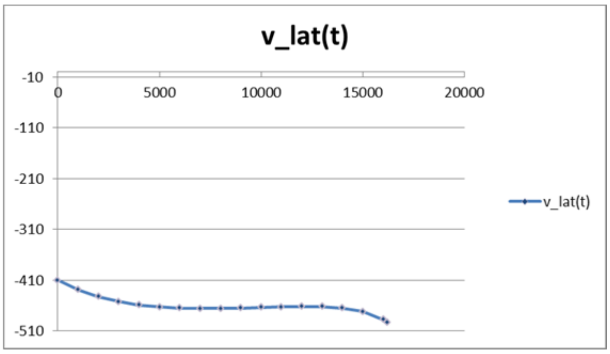



b)

Lat(0) = N0.4, Lon(0) = E94

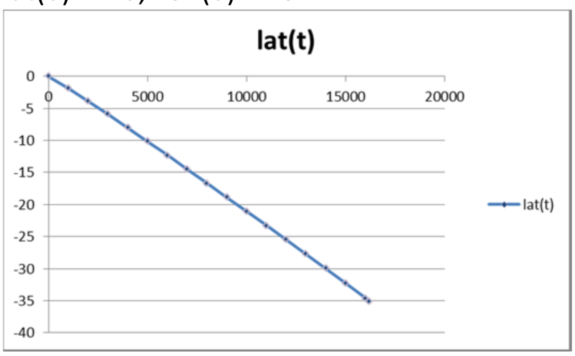



c)

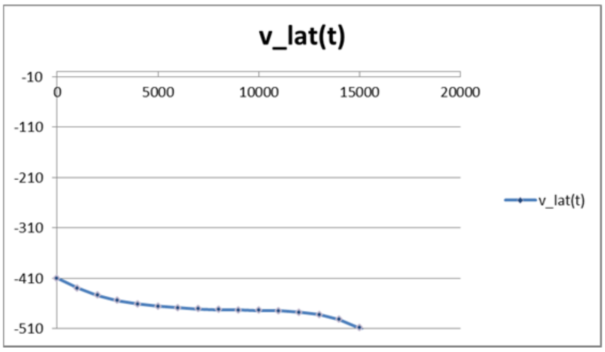



d)

Lat(0) = N0, Lon(0) = E94




e)




f)

Figure 2: Calculated lat($t$) [degrees]and $v_{lat}(t)$ [knots] for different realistic starting positions at 19:41 UTC for an early final turn south (before 18:40 UTC). Note the sensitivity of v_lat($t$) curves on slight changes in lat(0).

**Error analysis for flight paths starting at 19:41 UTC @ March 7th 2014**

Equation (2) can be rearranged into an expression for $v_{pz}$:

$$s_z \cdot v_{pz} = D(t)\, c \cdot p(t) / f - (\mathbf{p} - \mathbf{s}) \cdot \mathbf{v_s} - s_x \cdot v_{px} - s_y \cdot v_{py} \qquad (5)$$

Here we focus on estimation of possible errors introduced by fixing the aircraft longitude at the initial value, an assumption needed to find a solution for aircraft latitude as a function of time from BFO values only.

*Ignoring ($s_x \cdot v_{px} + s_y \cdot v_{py}$):*
The typical cruising speed of 490 knots corresponds to 247 m/s. For a mainly southerly oriented track we assume $v_{px}$ en $v_{py}$ < 100 m/s. Note that for an aircraft at level flight near 90 degree longitude $v_{py}$ will mostly be much smaller than this anyway. Note that during the time interval considered (19:41 UTC – 00:20 UTC) $s_y$ < 20 km and $s_x$ < 30 km. All together it means that the error budget for completely ignoring ($s_x \cdot v_{px} + s_y \cdot v_{py}$) is less than 5 km^2 / s.

*Error in p by using fixed aircraft position:*
Because of the geostationary height of the satellite, an error in aircraft position will always result in an error of less than 2% in *p* (aircraft to nominal satellite position). As D varies between -40 and +40 Hz the term $D \cdot c \cdot p / f$ varies between +-280 km^2 / s. With the error in *p* < 2% if we take p^2 = (p(0))(p(0)) constant, the error contribution of this whole term will be less than 6 km^2 / s.

*Error in ($\mathbf{p}$ - $\mathbf{s}$)$\cdot \mathbf{v_s}$ by ignoring changes in $(p - s)_x$ and $(p - s)_y$:*
Write ($\mathbf{p}$ - $\mathbf{s}$) = ($\mathbf{p}$ – $\mathbf{s}$) (0) + $\Delta$($\mathbf{p}$ - $\mathbf{s}$)(t)

Under the assumption given above $\Delta(p_i - s_i)(t)$ in x and y direction is less than 2000 km (linearly increasing in time from 0)
$v_{sx}$ is typically 0.002 km / s
$v_{sy}$ is typically 0.001 km / s
All together the error contribution by ignoring *$(p - s)_x$ and $(p - s)_y$* is less than 5 km^2 / s, linearly increasing in time from 0 at *t* = 0.

*Total absolute error budget:*
The total maximum error due to the unknown variation in aircraft longitude can thus be estimated to increase from 11 km^2 / s at 19:41 UTC to 16 km^2 / s at 00:20 UTC.

*Relative error in $s_z \cdot v_{pz}$:*
To translate the absolute error budget into a relative error budget we need to estimate the absolute value of the terms $D \cdot c \cdot p / f$ and ($\mathbf{p}$ – $\mathbf{s}$) $\cdot$ $\mathbf{v_s}$. As these terms are strongly changing with time, their estimates are plotted in figure 3:

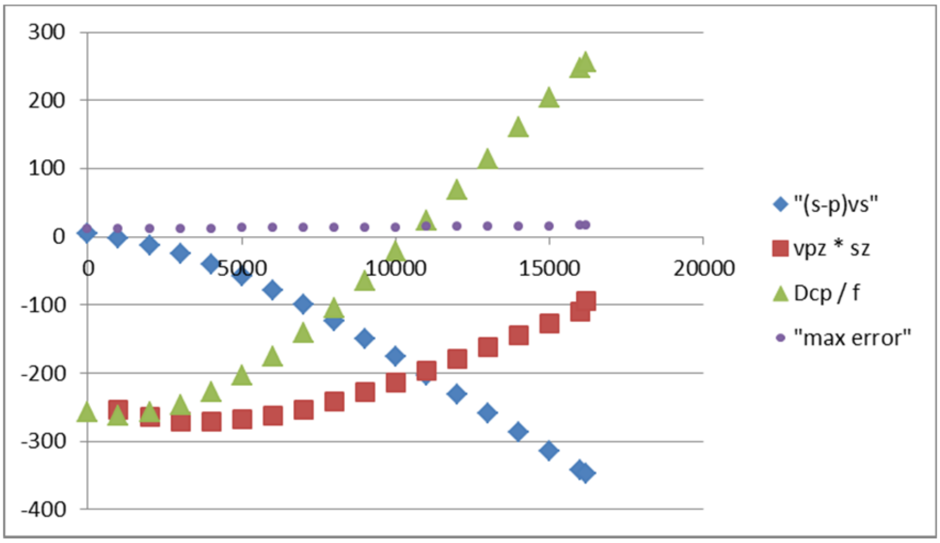



Figure 3: Estimated contributions of the two major terms in eq. (5) and there sum (the red squares) [km^2 / s ]. The sum can be compared with the estimated total max. error (purple dots) to get an estimate for the relative error in $v_{pz}$ introduced by ingoring aircraft longitude displacements. Initially the relative error is less than 5% and increases to about 10% at *t* = 13000 s. Only in the final hour the max. error is between 10% and 16%.

*Possibilities for improvement:*

The error estimates have been ample. For a mainly southerly track the error in p can be largely reduced by only fixing aircraft longitude and calculate the time varying p (taking into account the calculated changing latitude). Practically it must be possible to reduce the error in p to less than 1% (< 3 km^2 / s). This will be implemented soon. Also the estimated error in $(p - s)_x$ *and* $(p - s)_y$ is rather large. It would not be surprising if the combination with BTO analysis would point out that $\Delta(p - s)_x$ (t) *and* $\Delta(p - s)_y$ (t) are both less than 1000 km, leading to an error of less than 3 km^2 / s (linearly increasing in time) by fixing $(p - s)_x$ *and* $(p - s)_y$ to the initial value. With these modifications the total max. error would increase linearly in time from 8 to 11 km^2/s.

*From relative error in $v_{pz}(t)$ to the relative error in $p_z(t)$:*

As equation (4) points out the position $p_z(t)$ is found from $v_{pz}(t)$ by direct integration. It is tempting to think that for example a relative error in $v_{pz}(t)$ of less than 10% will lead to an error of less than 10% in $p_z(t)$ at all times. However, as $v_{pz}(t)$ depends on $p_z(t)$ there could be an exponential growth of errors introduced early in the integration time. This will be addressed next. Looking closer to eq. (5) the coefficient in the DE connecting $v_{pz}$ with $p_z$ is given by $v_{sz}$ / $s_z$. Its value changes from -1.6e-06 $s^{-1}$ at *t* = 0 s, to -9e-05 $s^{-1}$ at *t* = 12k s. This means that for t < 12k s the exponential grow of possible errors in $p_z$ is limited ($e^{\lambda t}$ where $\lambda t$ < 1). Again in the final hour of the interval considered one has to be aware of a possible multiplication effect of initial position error into the final position.

*A more global look at the errors:*
A pragmatic way to look at the error budget is by first solving for $v_{pz}(t)$ and then compare the obtained values with *max. abs. error* / $s_z(t)$. Typical values for the *max. abs. error* are between 11 and 16 km^2 / s. Typical values for $s_z(t)$ are between 1200 and 400 km. So typical values of the max. error in $v_{pz}(t)$ are between 10 and 40 m/s, mostly less than 20 m/s. For solutions that indicate that abs($v_{pz}$) > 200 m/s at all times the error is indeed most of time 10% or less. The exception is the last hour (*t* between 13k and 16.2k). There the error can go up to 20%.
With a total flight time of 16k s an estimated average 20 m/s max. error corresponds with +-300 km in z-direction, or +-4 degrees in latitude assuming a final position in the 30S – 40S range.

**Summary of initial results**
Based on 19:41 UTC latitudes of 0 – 0.8 degrees N, we find 00:11 UTC positions of S33 - S35 (degrees). In combination with the estimated error range this would indicate a 00:11 UTC position in the range of 29S – 39S. Crucial of course is the assumed initial position. This will be discussed in a follow-up document where we compare BFO-only with combined BFO/BTO analysis. This document will include an extensive discussion, conclusions and recommendation section.

**References**

[1] Henrik Rydberg, "MH370: Finding the Path ", 3 June 2014, bitmath.org/mh370/mh370-path.pdf
[2] Chris Ashton, Alan Shuster Bruce, Gary Colledge, Mark Dickinson, The Search for MH370, *Journal of Navigation* **68** (2015), 1 – 22.