Toronto Math Circles: Junior
Third Annual Christmas Mathematics Competition
Solutions

1. Eddy is trying to guess a four letter pass code. He makes the following five attempts

$$6087, 5173, 1358, 3825, 2531$$

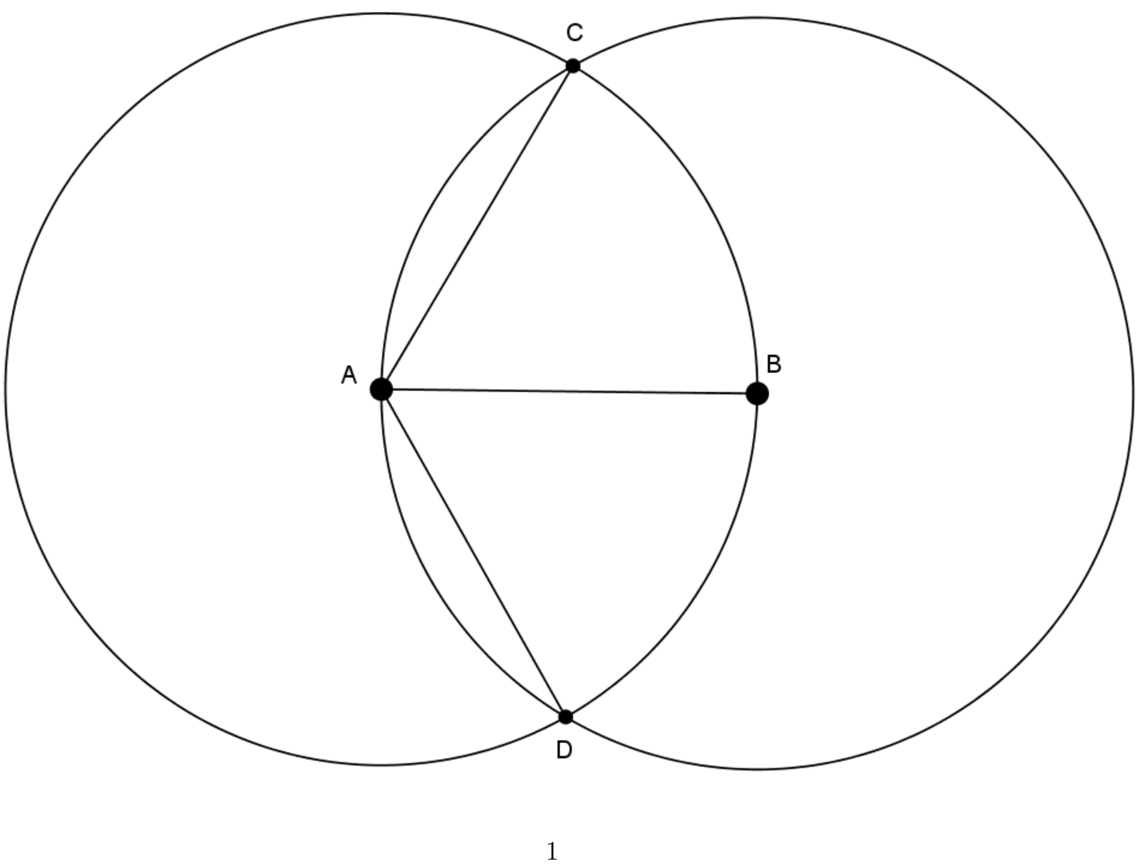
In each of his guesses, exactly two of the digits are in the correct pass code and these two correct digits are never in correct position. For example, if 1234 is a guess and 1 and 2 are in the correct pass code then 1 cannot be in the first position and 2 cannot be in the second position. Determine the correct pass code.

Solution. First, observe that 3 and 5 appears in all four positions. This means that 3 and 5 cannot be in the original pass code because if they were in the original pass code then they cannot be in any of the four positions, which is impossible. Based on 2531, two of the numbers must be 1 and 2. Based on 5173, one of the other numbers must be 7. Based on 1358, the last number must be 8. Therefore, the four digits of the pass code must be 1278, in some order. Based on 5173, 1358, and 2531, the digit in the third position must be 1. Based on 6087, 1358, and 3825, the digit in the first position must be 8. Based on 6087 and 5173 and the first position is already taken, the second digit must be 7. Finally, the digit in the fourth position must be 2. Therefore, the pass code is 8712.

2. Using only a compass and a straightedge, describe a procedure to construct an 120° angle. Be sure to explain why this angle is 120°. Be sure to include a clearly labeled sketch.

Solution. Starting with two points A and B, construct the line connecting these two points. Using the compass, draw a circle centered at A with radius AB. Similarly, draw a circle centered at B with radius AB. These two circles intersects at two points, call them C and D. Draw the two lines AC and AD. Observe that $AC = CB = AB$ and $AB = AD = BD$. Thus, $\triangle ABC$ and $\triangle ABD$ are equilateral. Therefore,

$$\angle CAD = \angle CAB + \angle BAD = 60^\circ + 60^\circ = 120^\circ$$



3. Determine if the number 1,116,428,043 is a perfect square.

   Solution. Observe that the sum of the digits is 30. Since 30 is divisible by 3 then the given number is also divisible by 3. However, since 30 is not divisible by 9 then the given number is also not divisible by 9. Therefore, if the given number is a perfect square and is divisible by 3 then it must also be divisible by 9. This is impossible. Therefore, the given number is not a perfect square.

4. In a class, there are 10 students. A teacher randomly chooses 5 students and writes their names on a list. If exactly one of the student's name is John, what is the probability that John is on this list?

   Solution 1. Consider the nine students that are not John. The number of lists with John on it, call this A, is equivalent to the number of 4 student lists out of the 9 students without John. The number of lists without John on it, call this B, is equivalent to the number of 5 student lists out of the 9 students without John. Therefore, the probability of having John on the list is

$$\frac{A}{A+B}$$

   Since there are only 9 students then the number of different 4 name lists is the same as the number of different 5 name lists. Thus, $A = B$. Therefore, the probability of having John on the list is $\frac{1}{2}$.

   Solution 2. The problem is actually the hypergeometric probability distribution. Therefore, the probability of John being on the list is

$$\frac{\binom{9}{4}\binom{1}{1}}{\binom{10}{5}} = \frac{1}{2}$$

   Remark. In contrast to the second solution, the point of the first solution is to demonstrate that there is no need to compute the number of possible lists with John on it.

5. There are n cards placed faced down in a line. A move consists of flipping a faced down card to the faced up position and also flipping the immediate card to its right. Explain why after a finite number of moves there will be no more moves to be made.

   Solution. Denote a faced down card with a 1 and a faced up card with a 0. This will turn every combination of cards into a binary number. Equivalently, a move consists of turning a block of 10 or 11 into 01 or 00, respectively. Observe that in either case, the binary number decreases. Thus, a given sequence of moves is equivalent to a strictly decreasing sequence of binary numbers. Since this sequence has a lower bound then it cannot be an infinite sequence. Therefore, there are only a finite number of moves to be made.

   Remark 1. It is also possible to prove this using induction.
   Remark 2. Two possible followup questions are "What is the shortest/longest possible sequence of moves?" and "How many possible sequences of moves are there?"
   Remark 3. A significantly more difficult problem would be to set the cards in a circle and ask if there exists a sequence of moves to achieve a desired result.

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Third Annual Christmas Mathematics Competition
Solutions

1. Denote $\lfloor x \rfloor$ to be the largest integer not greater than x. Let $\{x\} = x - \lfloor x \rfloor$. Find all triplets (x, y, z) that satisfy the system of equations

$$\begin{cases} x + \lfloor y \rfloor + \{z\} & = \frac{11}{10} \\ \{x\} + y + \lfloor z \rfloor & = \frac{11}{5} \\ \lfloor x \rfloor + \{y\} + z & = \frac{33}{10} \end{cases}$$

Answer. $\left(1, \frac{1}{5}, \frac{21}{10}\right)$
Solution. Adding the three equations and dividing by 2 yields

$$x + y + z = \frac{33}{10}$$

Subtracting each of the original equations from this yields

$$\begin{aligned} \{y\} + \lfloor z \rfloor &= \frac{11}{5} \\ \lfloor x \rfloor + \{z\} &= \frac{11}{10} \\ \{x\} + \lfloor y \rfloor &= 0 \end{aligned}$$

respectively. This implies that

$$\begin{aligned} \lfloor x \rfloor &= 1 \\ \lfloor y \rfloor &= 0 \\ \lfloor z \rfloor &= 2 \end{aligned}$$

and

$$\begin{aligned} \{x\} &= 0 \\ \{y\} &= \frac{1}{5} \\ \{z\} &= \frac{1}{10} \end{aligned}$$

Combining these yields, $(x, y, z) = \left(1, \frac{1}{5}, \frac{21}{10}\right)$.

2. There are n cards placed faced down in a line. A move consists of flipping a faced down card to the faced up position and also flipping the immediate card to its right. Show that after a finite number of moves there will be no more moves to be made.

Solution. Denote a faced down card with a 1 and a faced up card with a 0. This will turn every combination of cards into a binary number. Equivalently, a move consists of turning a block of 10 or 11 into 01 or 00, respectively. The exception is turning the right most card, which consist of turning 1 into 0. Observe that in any of these three cases, the binary number decreases. Thus, a given sequence of moves is equivalent to a strictly decreasing sequence of binary numbers. Since this sequence has a lower bound then it cannot be an infinite sequence. Therefore, there are only a finite number of moves to be made.

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Remark 2. Two possible followup questions are "What is the shortest/longest possible sequence of moves?" and "How many possible sequences of moves are there?"
Remark 3. A significantly more difficult problem would be to set the cards in a circle and ask if there exists a sequence of moves to achieve a desired result.

3. Let x be a real number such that $0 < x < \frac{\pi}{4}$. Arrange the following four numbers in ascending order.

$$(\cos x)^{(\sin x)^{\sin x}}, (\sin x)^{(\cos x)^{\sin x}}, (\cos x)^{(\sin x)^{\cos x}}, (\sin x)^{(\sin x)^{\sin x}}$$

Solution. Since $0 < x < \frac{\pi}{4}$ then $0 < \sin x < \cos x < 1$. Thus,

$$(\sin x)^{\sin x} > (\sin x)^{\cos x}$$

Therefore,

$$(\cos x)^{(\sin x)^{\sin x}} < (\cos x)^{(\sin x)^{\cos x}}$$

Similarly,

$$(\cos x)^{\sin x} > (\sin x)^{\sin x}$$

gives

$$(\sin x)^{(\cos x)^{\sin x}} < (\sin x)^{(\sin x)^{\sin x}} < (\cos x)^{(\sin x)^{\sin x}}$$

Therefore,

$$(\sin x)^{(\cos x)^{\sin x}} < (\sin x)^{(\sin x)^{\sin x}} < (\cos x)^{(\sin x)^{\sin x}} < (\cos x)^{(\sin x)^{\cos x}}$$

4. Let $\triangle ABC$ be an equilateral triangle inscribed in a circle. Let M be a point on the minor arc BC. Prove that $MA = MB + MC$. Be sure to include a clearly labeled sketch.

Solution. First, observe that $AM > CM$. Let D be a point on AM such that $AD = MC$. Since $AB = BC$, $AD = CM$, and $\angle BAM = \angle BCM$ then

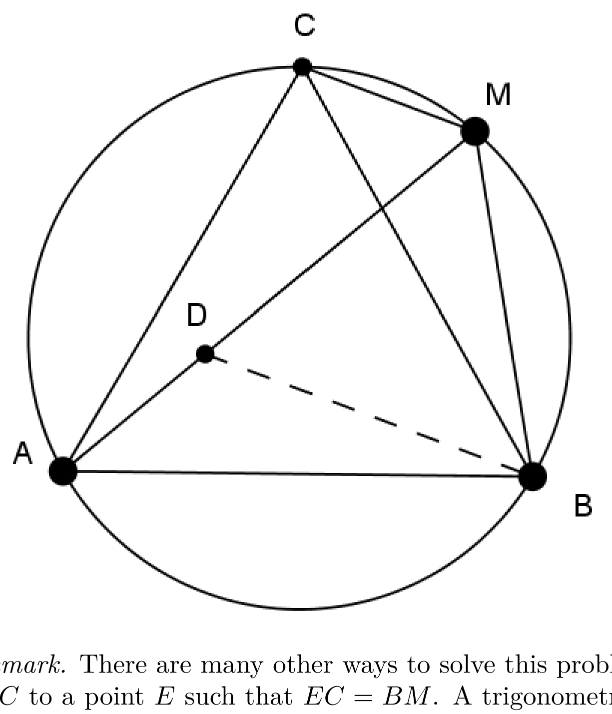
$$\triangle ABD \cong \triangle CBM$$

Thus, $BM = BD$ and $\angle ABD = \angle CBM$. Since

$$\angle DBM = \angle DBC + \angle CBM = \angle DBC + \angle ABD = \angle ABC = 60°$$

then $\triangle DBM$ is equilateral. Therefore,

$$MB = MD = MA - AD = MA - MC$$



Remark. There are many other ways to solve this problem. An alternative to creating D is to extend MC to a point E such that $EC = BM$. A trigonometry approach would be to use the sin law. This problem could also be solved with Ptolemy's Theorem.

5. A positive integer p is called a Twin Prime Pair Base (TPPB) if p and $p+2$ are both prime numbers. The Twin Prime Conjecture states that there are infinitely many values of p that are TPPB. For this problem, assume that this conjecture is true.
Let n be a positive integer. Denote $a_n = b_{n+1} + 1$ where b_n is the n^{th} smallest TPPB. Consider the polynomial
$$f(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_2x^2 + a_1x + a_0$$
Determine if this polynomial can be factored into a product of two non-constant integer coefficient polynomials.

Solution. The answer is no, it cannot be factored into a product of two non-constant integer coefficient polynomials.

Lemma. Let $\sum_{i=1}^{n} c_i x^i$ be a polynomial with integer coefficients and p is a prime such that
(1) $p | c_i$ for $i < n$
(2) $p \not| c_n$
(3) $p^2 \not| c_t$ for some t
When this polynomial is factored into a polynomial with integer coefficients, one of its factors will have degree at most t.

Proof of Lemma. Write
$$\sum_{i=1}^{n} c_i x^i = \left(\sum_{j=1}^{m} a_j x^j\right)\left(\sum_{k=1}^{l} b_k x^k\right)$$
where a_i's and b_i's are integers. Observe that
$$c_k = \sum_{i+j=k} a_i b_j$$
Since $p \not| c_n$ then $p \not| a_m$ and $p \not| b_l$. Let u be the smallest integer such that $p \not| a_u$ and v be the smallest integer such that $p \not| b_v$ then in $\mathbb{Z}_p$,
$$c_n x^n \equiv \left(\sum_{j=u}^{m} a_j x^j\right)\left(\sum_{k=v}^{l} b_k x^k\right) \mod p$$
Observe that, on the right hand side, the coefficient of x^{u+v} is $a_u b_v$, which is not divisible by p. The only term on the left hand side that is not divisible by p is c_n. Therefore, $u + v = n$, which is only possible when $u = m$ and $v = l$. Without loss of generality, assume that $m \leq l$. For the sake of contradiction, assume that $m > t$. Since $p | a_i$ for all $i \leq t$ and $p | b_j$ for all $j \leq t$ then
$$p^2 | c_t = \sum_{i+j=t} a_i b_j$$
This is a contradiction. Therefore, $m \leq t$ and the lemma is proven.

To analyze the coefficients of $f(x)$, first observe that since all b_n's are odd then all a_n's are even. Since $a_1 = 6$ then it suffice to apply the lemma from the beginning, with $p = 2$, to see that if $f(x)$ can be factored then one of the factors must be linear. Next, recall that every prime number greater than 6 can be written as $6k - 1$ or $6k + 1$ where k is a positive integer. Since b_n is a TPPB then for every integer $n > 1$, b_n can be written as $6k - 1$ where k is a positive integer. Thus, for every integer $n > 0$,
$$a_n = b_{n+1} + 1 = 6k - 1 + 1 = 6k$$
for some positive integer k. Therefore, $a_1, a_2, \ldots, a_{2n-1}$ are all divisible by 6. Since $a_0 = 4$ then, by Rational Root Theorem, if $f(x)$ has a rational root then it must be an integer, denote this as r. In $\mathbb{Z}_3$,
$$f(r) \equiv r^{2n} + 1$$

This congruent equation has no solutions because $r^2 \equiv 0, 1$. Thus, $f(x)$ has no rational root. Therefore, $f(x)$ cannot be factored into a product of integer coefficient polynomials.

Remark 1. The lemma is very similar to the Extended Eisenstein's Criterion. After some clever manipulation, there MAY be a way to apply the Extended Eisenstein's Criterion instead proving the lemma.

Remark 2. The proof of the lemma is very close to the proof of Eisenstein's Criterion itself.