Quantifier Variance and Indefinite Extensibility

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Abstract

This paper clarifies quantifier variance and uses it to provide a novel account of set theoretic quantification and indefinite extensibility. The indefinite extensibility response blocks the set theoretic paradoxes by seeing each argument for paradox as a demonstration that we have come to a different and more expansive understanding of "all sets" (or "all ordinals" or "all cardinals"). Indefinite extensibility is philosophically puzzling: extant accounts of indefinite extensibility are either metaphysically suspect—by requiring non-standard assumptions about the nature of mathematical objects—or metasemantically suspect—by requiring mysterious mechanisms of domain restriction/expansion. Happily, the view of quantifier meanings that underwrites the quantifier variance of Hilary Putnam and Eli Hirsch can be used to provide a novel account of indefinite extensibility that is both metaphysically and metasemantically satisfying. Section 1 introduces the indefinite extensibility response to the paradoxes and poses the puzzle of indefinite extensibility; section 2 develops and clarifies the metasemantic account of quantifier meanings at the heart of quantifier variance; section 3 solves section 1's puzzle of indefinite extensibility by applying section 2's account of quantifier meanings; and section 4 compares the theory developed in section 3 to several other theories in the literature.

Keywords: Indefinite Extensibility, Quantifier Variance, Metaontology, Metasemantics, Set Theory, Paradox, Quantification

1 The Puzzle of Indefinite Extensibility

Purported quantification over *all* sets, ordinal numbers, or cardinal numbers famously gives rise to paradox. One lesson that has been drawn from these so-called *logical* or *set-theoretic paradoxes* is that the concepts involved are *indefinitely extensible*. The idea goes back to Bertrand Russell:

> ...the contradictions result from the fact that...there are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question.[1]

Michael Dummett gives a similar but more involved characterization:

> What the paradoxes revealed was not the existence of concepts with inconsistent extensions, but of what may be called indefinitely extensible concepts. The concept of ordinal number is a prototypical example. The Burali-Forti paradox ensures that no definite totality comprises everything intuitively recognizable as an ordinal number, where a definite totality is one quantification over which always yields a statement that is determinately either true or false. For a totality to be definite in this sense, we must have a clear grasp of what it comprises: but, if we have a clear grasp of any totality of ordinals, we thereby have a conception of what is intuitively an ordinal number greater than any member of that totality. Any definite totality of ordinals must therefore be so circumscribed as to forswear comprehensiveness, renouncing any claim to cover all that we might intuitively recognize as being an ordinal. ... The intuitive concept of ordinal number, like those of cardinal number and of set, is an indefinitely extensible one.[2]

If the concept of set, for example, is indefinitely extensible, then we cannot quantify over absolutely all of the sets and so, *a fortiori*, we cannot quantify over absolutely everything. This inference can be challenged, but it is quite natural. In any case, clearly understanding indefinite extensibility, if it exists, is essential for understanding quantification itself.

Let's illustrate the idea of indefinite extensibility by going through an argument for Russell's paradox.[3] According to folklore, the early fathers of set theory—including, crucially, Frege—accepted, whether implicitly or explicitly, the axiom of naïve comprehension:[4]

[1] Russell (1906).
[2] Dummett (1991), page 316-317.
[3] My way of introducing indefinite extensibility follows Cartwright (1994) and the introduction to Rayo and Uzquiano (2006).
[4] See Frege (1893), (1903), and Russell (1903).

$$\mathbf{NC}: \quad \exists y \forall x (x \in y \leftrightarrow \phi(x))$$

Roughly, this amounts to assuming that for any particular formula in our language there is a set of all and only the things satisfying the formula. Russell (and Zermelo before him) showed that **NC** leads to a contradiction as follows:

1. $\exists y \forall x (x \in y \leftrightarrow x \notin x)$ [instance of **NC**]
2. $\forall x (x \in r \leftrightarrow x \notin x)$ [introducing a name witnessing (1)]
3. $(r \in r \leftrightarrow r \notin r)$ [instantiating for "$\forall x$" in (2) with "r" according to standard quantifier rules]
4. $\perp$ [from (3) using classical logic]

In essence, modern set theory responds to this paradox and others like it (such as Cantor's paradox and the Burali-Forti paradox) by rejecting **NC** in favor of more complicated—and less intuitively appealing—axioms of set existence. But the idea of indefinite extensibility allows for a more subtle response to the paradoxes.[5]

Before detailing that response, a couple of brief points concerning terminology: Dummett infamously argued that intuitionistic logic is required when reasoning with indefinitely extensible concepts, but his argument has puzzled most commentators, and isn't any part of how I'm understanding indefinite extensibility.[6] It's also worth stressing that in my usage, which is fairly standard, indefinite extensibility is a particular response to the paradoxes involving shifts in our domain of quantification, rather than a name for the mere mechanism that generates the paradoxes, whatever it may be.[7] With these clarifications made, let's detail the indefinite extensibility response.

The friend of indefinite extensibility can take the argument for paradox to fail not because **NC** is false, but because the Russell set r, introduced in (2), lies outside the range of the quantifier "$\forall x$" in (2), hence the move from (2) to (3) is invalid. As Russell and Dummett indicate in the above quotes, according to this line of thought, the attempt to talk about all of the sets (or in this case, all of the non-self membered sets) necessarily involves the introduction of

[5] For standard set theory see Jech (2003) or Kunen (1980); for a philosophical account of the iterative hierarchy of sets on which standard set theory is based, see Boolos (1971).

[6] See Dummett (1991).

[7] See Shapiro & Wright (2006) for something closer to the "mechanism of paradox" usage.

a new set not in our original totality. Of course, this is just an illustration—the friend of indefinite extensibility need not accept naïve comprehension, but the indefinite extensibility response is based on the idea that there are intuitive principles of set existence that thwart any attempt to talk about absolutely all and only the sets. Naïve comprehension is one such intuitive principle, but there are others. For example, Øystein Linnebo and others have developed versions of the set theoretic paradoxes that depend upon the interaction of set theory with principles of plural logic; and Richard Cartwright's "All-in-One" principle—holding that the objects in a domain of quantification make up a set or set-like object—can also be used for this purpose.[8] Since I'm here concerned with the nature of indefinite extensibility itself, I won't bother to catalog these approaches or endorse one of them over the others. What matters for my purposes is the general structure of any indefinite extensibility response to the set theoretic paradoxes.

This general structure can be summed up as follows: if we introduce a quantifier "$\forall^s$" that putatively quantifies over absolutely all of the sets (or set-like objects, though I will suppress this qualification in what follows), then we can—using our formal and informal principles of set theory—come to understand a new and more expansive quantifier, "$\forall^{s+}$", that has in its domain a set that was not in the domain of "$\forall^s$". Hence, contrary to what we may have originally thought, "$\forall^s$" didn't quantify over absolutely all of the sets. But things cannot end here, otherwise "$\forall^{s+}$" itself might be the sought after absolute quantifier over sets. According to friends of indefinite extensibility, this extensibility is ineliminable: the same type of argument that allowed us to go from our quantifier "$\forall^s$" to our more expansive quantifier "$\forall^{s+}$" also allows us to go from "$\forall^{s+}$" to a still more expansive quantifier "$\forall^{s++}$", and so on and so forth. The friend of indefinite extensibility will never be caught in paradox, since each argument for paradox only shows that we've reached a new and more expansive understanding of "all sets".

The general structure of the indefinite extensibility response to the paradoxes is clear enough, but any indefinite extensibility theory faces a puzzle that I call, straightforwardly enough, *the puzzle of indefinite extensibility.* Of course we can formally block the paradoxes by always insisting, when presented with any argument for paradox, that the domain of the quantifiers has been extended, but doing so without offering any explanation of both *how* our quantifiers have

[8] In Linnebo (2010); for plural logic, see Boolos (1984) and (1985); see Cartwright (1994) for the "All-in-One" principle.

been extended and *why* we are forever barred from talking about absolutely all of the sets, isn't satisfying. The puzzle arises because it is difficult to see how it is possible to provide an account of indefinite extensibility that is both metasemantically and metaphysically satisfying. Let's illustrate this by considering a couple of folkloric theories of indefinite extensibility. These folkloric theories are, of course, straw men that I have set up to be knocked down, but knocking them down is instructive.

First, a cosmic censorship account. According to this account, the sets are—as standardly assumed—timeless, necessary, independently existing, *sui generis* abstract objects. But we are—somehow, we know not how—barred from quantifying over all of them at once. It is as if some cosmic censor keeps us from talking about all of the sets. Metaphysically, this account is entirely standard, but it is metasemantically mysterious: given that the sets exist independently of our practices, there seems to be no reason that we couldn't talk about them all at once. Obviously many ordinary quantificational claims are restricted ("*All* the beer is in the fridge") but in ordinary cases we can easily unrestrict and talk about everything of the relevant kind at once ("*All* the beer that has ever or will ever exist was created by Satan"), but here we cannot. The hypothesis of cosmic censorship is dubious metasemantics.

Second, a creationist account. According to this account, the sets are created by us. More to the point, in attempting to quantify over absolutely all of the sets at once, we—somehow, we know not how—create a new set. This makes metasemantic sense of how our quantifiers manage to keep expanding, since the Russell set r couldn't have been in the range of our quantifiers before it existed. But creationist accounts involve highly nonstandard assumptions about the nature of mathematical objects like sets. For good reason, it is almost universally agreed amongst mathematicians and philosophers that sets and cardinals and ordinals and the like are not, in any sense, created by human practices.[9] The hypothesis of mathematical creationism is dubious metaphysics.

In order to have a philosophically satisfying indefinite extensibility response to the paradoxes, we need a theory that doesn't fall into either of these traps and thus solves the puzzle of indefinite extensibility. Unfortunately, though I won't argue for this here, I suspect that many extant accounts of indefinite extensibility—perhaps all—are either metaphysically or metasemantically prob-

[9] Burgess (2003) provides a nice summary of the reasons for this consensus.

lematic.[10] Fortunately, on the other hand, there is an approach to quantification that allows for an account of indefinite extensibility that avoids both the Syclla of metaphysical implausibility and the Charybdis of metasemantic mystery.

2 Quantifier Meanings

The key to solving the puzzle of indefinite extensibility is an approach to quantifier meanings that has been widely discussed in metaontology. The approach I have in mind is key to both Hilary Putnam's *conceptual relativity* and Eli Hirsch's *quantifier variance.*[11] Unfortunately, both of these views build in various elements that are detachable from the metasemantic theory of quantifier meanings that underwrites them, and the metasemantic view itself hasn't been given a name in the literature. I call the view *quantifier deflationism*; "deflationism" is an overused word in philosophy, so perhaps this name is less than ideal, but it has the advantage of suggestiveness.

Here's Putnam describing his view of quantifiers in broad brushstrokes:

> ...what logicians call "the existential quantifier." the symbol "$(\exists x)$," and its ordinary language counterparts, the expressions "there are," "there exist" and "there exists a," "some," etc. *do not have a single absolutely precise use but a whole family of uses.* These uses are not totally different; for example, in all of its uses the existential quantifier obeys the same logical laws...But these properties of the existential quantifier and the related properties of its close relative the universal quantifier "(x)" ("for all x") do not fully determine how we are to use these expressions. In particular, there is nothing in the logic of existential and universal quantification to *tell us* whether we should say that mereological sums exist or don't exist; nor is there some other science that answers this question. I suggest that we can *decide* to say either.[12]

[10] For example, I think the fictionalist account suggested on pages 34-35 of Field (2008) is metaphysically problematic, in that it rejects the existence of sets; and I think the specification account that can be extracted from Glanzberg (2004) is metasemantically problematic in that it doesn't adequately explain why we can't successfully specify a domain of all sets. Of course, this footnote doesn't suffice for showing that these theories fall to my dilemma. Several other accounts—including modal accounts of various kinds—will be discussed in more detail below.

[11] Carnap (1950) is sometimes seen as an early forerunner of these views; for discussion, see Eklund (2009); see also the remarks about quantification in Wittgenstein (1974).

[12] Putnam (2004), pages 37-38.

As Putnam notes at the end of this quote, his view leads to a brand of quantifier pluralism. Here's Hirsch applying this pluralism in discussing a particular, disputed quantificational sentence:

> This sentence would qualify as true in [one language] but, I assume, false [in the other]...The different semantic rules that would have the effect of rendering the sentence true in one language and false in the other must in some sense provide different rules for "counting what things there are in the world." If there could be these two languages they would have to embody in some sense different concepts of what it is "for there to exist something."[13]

The views of both Putnam and Hirsch have been widely and critically discussed in the literature, and it's common for critics to complain that their views are unclear.[14]

To some extent, both Putnam and Hirsch have unwittingly fostered confusion about their views with unhelpful terminological choices. Both conceptual relativity and quantifier variance involve not only the metasemantic view that I'm calling "quantifier deflationism" and its attendant pluralism, but also claims of egalitarianism and equivalence between competing quantifier meanings. In addition, both Putnam and Hirsch, in different ways, apply their views in arguing that ontological disputes are insubstantial and this has led to a close association of quantifier variance with these applications. For these and other reasons, clarity is served by making a fresh start both terminologically and philosophically. Below I'll succinctly explain quantifier deflationism, the type of quantifier pluralism that it entails, and how Putnam and Hirsch's views map over onto my framework.

Explaining quantifier deflationism requires a bit of metasemantics. In general, the explanatory relationship between sentential semantic facts and properties like truth and truth conditions and sub-sentential semantic properties like reference can be viewed in roughly two ways:

> **Bottom Up** : Sub-sentential semantic facts are explanatorily prior to sentential semantic facts (*ceteris paribus*)
>
> **Top Down** : Sentential semantic facts are explanatorily prior to sub-sentential semantic facts (*ceteris paribus*)

[13] Hirsch (2011), pages 186-187.
[14] For example, see van Inwagen (2002).

Most of the metasemantic and metaconceptual theories thus far developed fit neatly into one of these two categories. The issue between bottom up and top down accounts certainly hasn't been conclusively decided, but many theories in metasemantics fit the top down approach more closely, including use theories of meaning as endorsed by Wittgenstein, Horwich, and others; assertibility based approaches as endorsed by Dummett, Wright, and others; normative inferentialist theories as endorsed by Sellars, Brandom, and others; inferential/conceptual role theories as endorsed by Block, Harman, Field, and others; and theories of interpretation that give pride of place to charity or rationality such as those endorsed by Davidson, Dennett, Lewis, Quine, and others.[15]

Bottom up approaches to metasemantics need to provide an independent account of quantifier meanings, since quantifiers are sub-sentential expressions and according to bottom up approaches the meanings and semantic properties of sub-sentential expressions are explanatorily prior to the meanings and semantic properties of sentential expressions. So while metasemanticists of all stripes agree that quantifiers range over domains of objects, bottom up theorists face pressure to make grasp of a quantifier's domain essential to understanding sentences containing the quantifier. By contrast, top down theorists think that facts concerning a quantifier's domain are explained by facts about the truth-conditions of whole sentences involving the quantifier. This is where Putnam and Hirsch step in.

Both Putnam and Hirsch clearly see themselves as developing an account of quantifier meanings against the background of a top down metasemantics—Putnam tends to stress the close tie between linguistic meaning and sentence use, while Hirsch tends to focus on the role of charity in interpretation. In my view, and that of many commentators, these are two different sides of the same coin.[16] In any case, against the background of some version of top down metasemantics, Putnam and Hirsch go on to develop a broadly inferential account of quantifier meanings, though neither ever spells out this metasemantics in any great detail. Let's fill in some of those details: we need to be able to identify a sub-sentential expression in a given language as a quantifier using its inferential role in that language:

Quantifier Inferentialism : sub-sentential expression Q in language

[15] See Block (1986), Brandom (1994), Davidson (1984), Dennett (1987), Dummett (1973), Field (1977), Harman (1982), Horwich (1998), Lewis (1974), Quine (1960), Sellars (1953), Wittgenstein (1953), and Wright (1992).

[16] Block (1998) and Horwich (1998) each make similar points about the close relationship between interpretive charity and use-based metasemantics.

> L is an (unrestricted) type i quantifier expression just in case Q plays, in L, the inferential role of an (unrestricted) type i quantifier expression

In order to apply this to a given type of quantifier, we need to specify the inferential role of that type of quantifier.

For the standard existential and universal quantifiers, it is both natural and relatively uncontroversial to think that the relevant inferential role is given by something like the standard natural deduction introduction and elimination rules for these quantifiers.[17] Where writing square brackets around a formula indicates that the formula is an assumption/premise and adding a numerical superscript to the brackets indicate that the assumption is discharged at the line indexed by that numeral, these rules are:

$$(\exists I)\ \frac{\phi}{\exists\zeta\phi^{[\alpha/\zeta]}} \qquad (\exists E)\ \frac{\exists\zeta\phi^{[\alpha/\zeta]} \qquad \begin{array}{c}(\phi)^n\\ \vdots\\ \psi\end{array}}{\psi}\, n$$

provided that in ($\exists E$) α isn't in $\phi^{[\alpha/\zeta]}$ or ψ or any assumptions that were used in the derivation of ψ. And:

$$(\forall I)\ \frac{\phi}{\forall\zeta\phi^{[\alpha/\zeta]}} \qquad (\forall E)\ \frac{\forall\zeta\phi}{\phi^{[\zeta/\alpha]}}$$

Provided that in ($\forall I$) α isn't in any assumptions that were used in the derivation of ϕ. A quantifier in L's being unrestricted can also be accounted for inferentially: an existential quantifier $\exists_i$ is unrestricted in L just in case for any formula ϕ and existential quantifier in L, $\exists_k$, $\exists_k x\phi(x) \vdash \exists_i x\phi(x)$; and a universal quantifier $\forall_i$ is unrestricted in L just in case for any formula ϕ and universal quantifier in L, $\forall_k$, $\forall_i x\phi(x) \vdash \forall_k x\phi(x)$.

Of course, natural languages aren't formal languages, but these natural deduction rules provide a useful heuristic. The key point is that a sub-sentential expression in a given language is a universal quantifier just in case it plays the inferential role of a universal quantifier in that language. In a natural language the relevant inferential role is no doubt much messier and more difficult to specify than it is in a formal language. In addition, what matters for inferential role is the general *structure* of the inferential contribution—languages with gram-

[17] Natural deduction systems were introduced in Gentzen (1934) and Jaśkowski (1934).

mars different from our standard logical languages can still have expressions that play the role of the universal quantifier, but it will be by satisfying structurally analogous versions of the above quantifier rules.[18] With these caveats and clarifications in mind, the key point is merely that when truth conditions of sentences are determined without appeal to the semantics of sub-sentential expressions, in a top down fashion, different types of sub-sentential expressions can be identified on the basis of their inferential behavior in the language.

It is important to stress that **Quantifier Inferentialism** is distinct from what is commonly called *inferentialism* or *conceptual role semantics* in the philosophical literature. These approaches, applied to the quantifiers, sees inference rules like $(\forall I)$ and $(\forall E)$ as fixing or determining the meaning of the quantifier "$\forall$". Some extreme versions of inferentialism even do away with truth conditions and standard model theory altogether in favor of a wholly inferential approach to meaning. By contrast, all that **Quantifier Inferentialism** claims is that a sub-sentential expression counts as a universal quantifier, for example, if it plays the right inferential role — **Quantifier Inferentialism** concerns which sub-sentential expressions are quantifiers, it isn't concerned with how the meanings of the quantifiers are determined. Of course, standard inferentialism is one way of combining **Quantifier Inferentialism** with a top down approach to metasemantics, but it isn't the only way.

Seen in broadest outline, the view of quantifiers that underwrites the views of Putnam and Hirsch, that above I called "quantifier deflationism", is simply the combination of a top down metasemantics with an inferential account of quantifiers:

Top Down + Quantifier Inferentialism = Quantifier Deflationism

According to quantifier deflationism, our use of language determines meanings/truth conditions for whole sentences, and quantifiers are those sub-sentential expressions in a language that play certain inferential roles. Ultimately, the semantic facts concerning quantifiers will be explained using some top down picture, where facts about the use of whole sentences are central. This top down picture might be a version of inferentialism in the manner of the previous paragraph, or it might not, but in any case, quantifier deflationism uses inferential role to pick out the quantifier expressions from the non-quantifier expressions.

[18] We could deal with this either by specifying the structural role in detail or, more simply, by considering a language's translation or regimentation into standard first-order logic, but I won't dwell on this here.

This global approach to quantifier meanings is analogous to the local approach to singular terms in mathematics pursued by Neo-Fregeans such as Crispin Wright and Bob Hale.[19] Neo-Fregeans think the truth of a mathematical principle like Hume's Principle:

$$\mathbf{HP}: \quad \forall F \forall G(\#F = \#G \leftrightarrow F \approx G)$$

Can be explained by the fact that **HP** serves as an implicit definition of the "number of"-operator it contains (expressed by "#"). Neo-Fregeans combine this idea with acceptance of an inferential account of singular terms according to which a mathematical expression like "3" is a singular term just in case it plays the inferential role of a singular term. From this background, Neo-Fregeans offer arguments that mathematical singular terms like "3" refer.[20] Commentators have disagreed about whether Neo-Fregeanism is tacitly committed to at least a local form of quantifier deflationism, but in any case, Neo-Fregeans are offering a top down metasemantics for arithmetical language, together with an inferential account of singular termhood.[21] The approach of the quantifier deflationist is directly analogous to the Neo-Fregean approach to singular terms in mathematics, the difference is that quantifier deflationism is meant to be applied to language and quantification globally. Ultimately, I think it is of dubious coherence to adopt quantifier deflationism without endorsing a similar brand of deflationism about singular terms and predicates, but I won't stress this here.

Putnam and Hirsch have been concerned with quantifier deflationism largely because quantifier deflationism entails an interesting type of quantifier pluralism. There are many types of pluralism about quantifiers that might be endorsed, e.g., one could accept a pluralism about *kinds* of quantifiers, e.g., accepting that there are various different quantifiers in addition to the familiar $\exists$ and $\forall$—"almost all", "infinitely many", "countably many", etc. Another type of quantifier pluralism concerns which logical rules of inference the very same quantifier obeys, e.g., classical logicians and intuitionists accept slightly different rules of inference for the existential quantifier. But regardless of the relationship between quantifier deflationism and the these types of quantifier pluralism,

[19] See Wright (1983), Hale (1987), and the essays in Hale & Wright (2001); see also Dummett (1956).

[20] Quantifier deflationists will offer analogous arguments concerning the quantifiers, but since the details of such arguments will depend on the details of the top down metasemantic theory being endorsed, I pass over them in silence here.

[21] For debate about the relationship between Neo-Fregeanism and quantifier variance, see Hale (2007), Hawley (2007), Sider (2007), and Hale & Wright (2009).

quantifier deflationism leads to a type of pluralism according to which there are different and non-synonymous unrestricted quantifier meanings in different languages all obeying the very same logical rules of inference. What one language means by "some" and "all" might not be the same as what some other language means by "some" and "all". Read naïvely, this is trivial—the words "some" and "all" might mean, in some alien language, what our words "dog" and "cat" mean, but with quantifier inferentialism endorsed, we can put the point more accurately by saying that there are different unrestricted quantifier meanings in different languages.[22]

This is a type of pluralism according to which there are distinct quantifiers of each type, all of which obey the very same *logical* rules of inference but which *mean different things*. Again I must stress that this is not merely the trivial claim of the conventionality of linguistic signs. Quantifier pluralism is based on the idea that different language can have expressions that are, inferentially, unrestricted quantifiers, but which don't mean the same thing and thus can't be translated into our own unrestricted quantifier expressions. This is possible because the logical inference rules for the quantifiers by themselves fail to determine truth-conditions for many quantificational sentences in the language. In other words, when you know that a sub-sentential expression is an unrestricted existential quantifier, you know very little about how it contributes to the truth conditions of sentences in which it appears. From this simple fact it follows that there are many distinct and inequivalent unrestricted quantifier meanings. We can put things a little bit more explicitly:

> **Quantifier Pluralism** : There are languages L and K with expressions Q_L and Q_K, respectively, such that (1) Q_L and Q_K are both unrestricted quantifier meanings of the same type and (2) Q_L and Q_K mean different things, i.e., Q_L and Q_K cannot correctly be translated into each other

Whether or not this is true or not depends upon what it means to say that "there are languages" meeting these conditions. Perhaps all of the languages that exist are human languages and perhaps all of these languages have equivalent quantifiers? I doubt that this is the case—I find it hard to believe that when early cavemen talked about "everything" their quantifier was ranging over inaccessible cardinals, as ours arguably does when we talk about everything. In any case, the more interesting reading of this claim is *modal*—according to

[22] See section 4 of my (2014b) for a general account of this kind of conceptual pluralism.

this reading **Quantifier Pluralism** is true if it is possible that there be such languages as L and K (the non-modal reading and the modal reading collapse if "languages" here are simply abstract objects).[23] On the interesting reading, **Quantifier Pluralism** is true because there clearly could be communities that used language in ways distinct from us while still having universal and existential quantifiers that differ in meaning from our own.

Putnam and Hirsch have gone beyond the mere claim of quantifier pluralism, which merely states the existence of multiple and non-synonymous quantifier meanings, and endorsed the claim that some of these distinct meanings are *equally good from any objective point of view.* In fact, both Putnam's *conceptual relativity* and Hirsch's *quantifier variance* explicitly build in this idea of equivalence. To illustrate, Putnam says of conceptual relativity:

> ...conceptual relativity always involves descriptions which are cognitively equivalent...but which are incompatible if taken at face value...[24]

Putnam's notion of cognitive equivalence is a relation between theories: two theories are cognitively equivalent just in case they are mutually relatively interpretable in a way that preserves goodness of scientific explanation.[25] And Hirsch says of quantifier variance:

> ...truth-conditionally equivalent languages are of equal metaphysical merit. That is the doctrine of quantifier variance. The doctrine says that there is no uniquely best ontological language with which to describe the world.[26]

Where two languages are truth-conditionally equivalent when they can express all and only the same coarse-grained truth-conditions, which can helpfully be modeled as functions from contexts of utterance to sets of possible worlds.[27]

We can now see that both conceptual relativity and quantifier variance go beyond quantifier deflationism and quantifier pluralism in advocating a kind of *egalitarian* quantifier pluralism. The basic idea, put somewhat metaphorically, is that theories couched in languages with different and non-synonymous

[23] I won't get too specific about the nature of the modality here, but standard physical possibility suffices—for more on this see the discussion of modality in section 4 below.

[24] Putnam (2004), page 48.

[25] See Putnam (1983); the logical notion of relative interpretability is introduced and explained in Tarski, *et al.* (1953); applying relative interpretability to the cases of interest to Putnam actually requires a generalization of the usual notion, see the appendix to my (2014a).

[26] Hirsch (2011), page xii.

[27] See Kaplan (1989) where these are called "characters".

unrestricted quantifiers can be equally good in that both are able to—in their own way—express all and only the same facts. We can formulate this kind of egalitarian quantifier pluralism, schematically, as follows:

> **Egalitarian Quantifier Pluralism** : There are languages L and K with expressions Q_L and Q_K, respectively, such that (1) Q_L and Q_K are both unrestricted quantifier meanings of the same type; (2) Q_L and Q_K mean different things, i.e., Q_L and Q_K cannot correctly be translated into each other; (3) the languages L and K are objectively equivalent

Roughly then, Putnam and Hirsch's terminology maps over to my own in the following way:

> **Conceptual Relativity** = **Egalitarian Quantifier Pluralism** understood according to Putnam's account of equivalence[28]
>
> **Quantifier Variance** = **Egalitarian Quantifier Pluralism** understood according to Hirsch's account of equivalence

I hope it is now clear why I didn't follow either Putnam or Hirsch in terminology, despite their influence. Their terminology obscures the fact that quantifier pluralism itself can do interesting theoretical work, even apart from any equivalence claims. In addition, so much is packed into their terminology that it can be difficult to isolate disputes and objections.

Accepting the standard Quinean analysis of ontological claims as existentially quantified claims, Putnam and Hirsch have gone on to, in different ways, apply their views to provide deflationary accounts of ontological disputes.[29] These are the applications of quantifier variance with which readers are most likely familiar. Much of the discussion of quantifier variance in the literature on metaontology has concerned its anti-ontological applications. We don't have to address any of these applications here, since quantifier deflationism and its attendant pluralism can both be accepted without accepting the anti-ontological uses to which Putnam and Hirsch have put the views (e.g., both Cian Dorr and Theodore Sider can be read as accepting something like quantifier pluralism

[28] Putnam's inter-theory notion of equivalence can give rise to an inter-language notion of equivalence, as needed for filling out **Egalitarian Quantifier Pluralism**, in the following way: two languages are equivalent just in case for every theory formulable in one language there is a cognitively equivalent (in Putnam's sense) theory formulable in the other and vice-versa.

[29] Quine's influential analysis comes from his (1948); see the essays in Hirsch (2011) for his metaontological views.

while rejecting the idea that this acceptance is incompatible with substantive ontology).[30] I have elsewhere defended quantifier deflationism/pluralism from what I think is the most important objection, here I am going to apply the view, rather than defend it.[31]

Quantifier deflationism can be used to solve the puzzle of indefinite extensibility. In saying this, I am not speaking for either Putnam or Hirsch: Hirsch has steadfastly refused to apply his views to disputes concerning abstract objects and while Putnam once, in a little read paper, applied conceptual relativity to set theory, in more recent work he has, seemingly, gone back on this.[32] The application is novel, but I believe it is both less controversial and more promising than other extant applications of quantifier pluralism.[33]

3 Solving the Puzzle

According to the indefinite extensibility response to the paradoxes, in trying to—for instance—come up with a conception of all non-self membered sets, we come to a new understanding of "all sets" that is broader than the conception we started with, on pain of inconsistency. On bottom up approaches, this is puzzling, but quantifier deflationism makes good sense of it. On a top down, quantifier deflationist picture, indefinite extensibility is an instance of quantifier pluralism. Let me explain this in more detail.

In attempting to form a conception of the set of all non-self membered sets, we do various things, including introducing a term "r" for the Russell set. In so doing, we move from our language L to an expanded language, $L+$; the quantifier deflationist claims that the meaning of "everything" or "$\forall$" has shifted in the move from L to $L+$ so that from the perspective of $L+$, "$\forall$" in L can be seen as a restriction of "$\forall$" in $L+$. That is, the deflationist claims that L and $L+$'s quantifiers are an instance of quantifier pluralism. To establish this, simply note that (i) "$\forall$" has the same logical inferential role in both L and $L+$ and, by hypothesis, neither quantifier is restricted (more on this in a second) and (ii) the argument for Russell's paradox shows that treating "$\forall$" in L as extensionally

[30]See Dorr (2005) and Sider (2009) and (2011).

[31]See my (2014a).

[32]Putnam (2000) is the bit where he says it, essay 11 in Putnam (2012) is the bit where he takes it back.

[33]The somewhat cryptic approaches to absolute generality in Hellman (2006) and Rayo (2012) are, I suspect, best understood as versions of quantifier pluralism, though I won't insist upon this. All other potential anticipations I am aware of will be discussed below in section 4.

equivalent (having the same domain as) to "$\forall$" in $L+$ results in a contradiction, so "$\forall$" in L isn't extensionally equivalent to "$\forall$" in $L+$, so they obviously mean different things and so can't be translated into each other. We don't need to endorse any egalitarianism about this instance of quantifier pluralism for the approach to go through.

It might be thought odd, or even inconsistent, that I claimed both (a) that "$\forall$" in L was unrestricted and (b) from the perspective of $L+$, "$\forall$" in L is a restriction of "$\forall$" in $L+$. But the incoherence here is only apparent, (a) and (b) don't contradict each other when properly understood. (a) merely claims that there is no quantifier *in* L, for which "$\forall$" in L is a restriction, but that is different than the claim that there is no expansion of L from the perspective of which "$\forall$" is seen as restricted. (b) contradicts this latter claim, but not (a). Let's say that a quantifier Q in a language K is *absolutely unrestricted* if there is no possible language $K+$ from whose perspective Q is restricted. The indefinite extensibility theorist argues that no quantifier is absolutely unrestricted, but this is compatible with thinking various quantifiers, including "$\forall$" in both L and $L+$, are non-absolutely unrestricted.

In general, for a quantifier deflationist, once we realize that our rules and principles for reasoning about sets always allow us to come to understand a more expansive quantifier, in the above sense, all that truly required an explanation has been explained. This is because in accepting a top down metasemantics, as quantifier deflationists do, it is the rules and principles for reasoning with our set theoretic language that explain the facts about domains and expansions of quantifiers. These facts about how our usage changes are undisputed, but according to quantifier deflationism they are freestanding, explanatorily speaking. On other approaches, our use must answer to a priorly given domain of objects, accessible from all perspectives, and so those who start with a quantifier "$\forall$" and come to understand a more expansive quantifier "$\forall^{+}$" must have unrestricted their initial quantifier, lifting the veil provided by the cosmic censor. Pleasingly, the very facts that generate mysteries on other accounts, according to quantifier deflationism, end the matter with no residual mystery remaining.

Those familiar with the literature on quantifier variance might question this, since standard applications of quantifier pluralism, are much different. Standard applications are both *static* and *inter*personal—they concern different imagined language communities or different speakers with stable linguistic dispositions. Most familiarly, both Putnam and Hirsch have typically illustrated their views by appeal to cases where, e.g., community A accepts composite objects while community B does not.[34] By contrast, my application of quantifier pluralism to explain indefinite extensibility is both *dynamic* and *intra*personal-–it concerns a change in the language of a single speaker or community of speakers from one moment to another. Given the differences between standard applications and the present application, it might be wondered how dynamic, intrapersonal change of this kind can possibly result in a change of the meaning of "all", however subtle. Presumably, we typically don't think that we have changed the meaning of "all" in running through the reasoning of the Russell paradox.

I dispute this presumption. Obviously it has long been realized that in going through the Russell paradox, our set theoretic quantifiers have shifted in some fashion. This recognition is precisely what generates the paradox. It matters little whether speakers would intuitively describe this as a change in the meaning of "all". Theoretical matters of metasemantics are not to be decided by consulting highly theoretical intuitions about meaning whose very existence is dubious. And, in any case, as the discussion of section 2 shows, there is both a sense in which "all" changes its meaning when we run through the Russell paradox or other similar paradoxes, and a sense in which it does not. Meaning change occurs in the sense that "$\forall$" in L, can be seen, from the perspective of $L+$, to be restricted, and any attempt to treat the unrestricted universal quantifier of L as equivalent to that of $L+$ results in a contradiction. Meaning change does not occur in the sense that the logical inferential role of "$\forall$", as summed up in ($\forall I$) and ($\forall E$), is roughly the same in both L and $L+$. The most accurate thing to say, informally, is that in cases of indefinite extensibility, we come to a slightly but not radically different conception of "all".

After discovering the paradox, we come to understand quantifiers that contain our initial Russell set r in their domains; this is manifested in our linguistic practice in the way that terms for r can and cannot be instantiated for our quantifiers. This is why the logical inferential roles of "$\forall$" in L and $L+$ are *roughly* but not *exactly* the same—they are structurally the same, but since the languages differ, the substitution instances of the schematic letters in ($\forall I$) and ($\forall E$) differ. There is nothing mysterious about this: adding various terms and rules to our language often alters the meanings of old terms in drastic ways. If we add the tonk rules to our language, we'll have changed the meaning of virtually everything.[35] And if we start calling snow "rain" in a systematic manner, then "rain" will come to mean what "rain or snow" means in our current lan-

[34] See again Putnam (2004) and the essays in Hirsch (2012).
[35] See my (2015) for the fullest discussion of tonk and its implications for metasemantics.

guage. As examples like these show, it is uncontroversial that as our language expands and develops, the meanings of expressions change in various ways. Indefinite extensibility, understood using quantifier pluralism, is simply a hitherto unrecognized and somewhat subtle version of this familiar phenomenon.

Let's call this the *deflationary* theory of indefinite extensibility. The deflationary theory manages to solve the puzzle of indefinite extensibility by avoiding both metasemantic mystery and metaphysical implausibility. Metasemantically, for deflationists, there is no cosmic censor who swoops in to stop us from quantifying over what is independently, "out there". Instead, it is our changing linguistic practice that is explanatorily prior to facts about the domain of quantification, exactly as a top down metasemantics requires. When confronted with an argument for paradox, we face a choice point in our use of language: either accept inconsistency, or accept that our quantifier's domain has expanded. But according to quantifier deflationism, in making this latter choice, as surely we must, we, in effect, make it the case that the choice is correct, and that our quantifier has expanded.

A natural worry is that we pay for the lack of metasemantic mystery in the coin of metaphysical implausibility. In particular, it is natural to worry that the deflationary theory implies that we have *created* a new set, so deflationism is just a more complicated version of creationism. The mistake behind this objection is simple: quantifier deflationism says nothing about whether the objects in the range of our new quantifier are newly created, it concerns only the creation of a new *concept* of existence, in the move to this new understanding of the quantifier. When we come to have a new concept of "everything", perhaps it is true and makes good sense to say that we "created" this new concept out of the old, but to move from this claim about our concepts to a claim about objects in the world is a horrific *nonsequiter.* It is no better than arguing that because we created our concept of cows, we must have created cows. To think otherwise is akin to a use-mention error.

The association of creationist-like views with quantifier deflationism is a mistake, but it is one that both Hilary Putnam and Eli Hirsch have had to deal with many times. Unfortunately, a major reason for the close association of Putnam and Hirsch's views with creationism is that Putnam himself originally presented his version of quantifier deflationism as part of a package of anti-realist views including a general epistemic view of truth and, at times, something like global creationism.[36] Since 1990, Putnam has officially rejected all of these anti-realist doctrines while retaining conceptual relativity and thus quantifier deflationism, but the damage has been done.[37] By contrast, in all of his presentations of quantifier variance, Hirsch has labored mightily to free his view from whatever anti-realist associations linger from Putnam's presentations.[38] My impression is that this effort has largely been successful. Quantifier deflationism, applied to indefinite extensibility, can be freely coupled with a standard metaphysical view of sets as *sui generis*, necessarily existing, abstract objects.

The deflationary theory of indefinite extensibility is metaphysically standard (in the relevant sense) and metasemantically non-mysterious, hence section 1's puzzle of indefinite extensibility is solved by the deflationary approach. Our rules for reasoning with set theoretic expressions force us, given a putatively absolute quantifier over sets, to come to understand a new quantifier that, on pain of inconsistency, must include everything included by our old quantifier and more besides. These facts about how we use "$\forall$" and related expressions themselves provide a top down explanation of how and why and in what sense we have come to understand a more expansive quantifier. That we cannot come to understand an absolute and final quantifier is explained similarly: as we "go on in the same way" we continue to accept analogs of the principles that allow us, when given any quantifier meaning as input, to come to understand a more expansive quantifier meaning.[39] So there can be no ultimate, final, and absolute quantifier, given the rules we accept in our set theoretic language.

4 Related Approaches

The deflationary theory of indefinite extensibility is novel, but it has certain affinities with several extant theories of indefinite extensibility and set theoretic quantification. I have already cited work by Hellman, Rayo, and Putnam that I think might be best understood using quantifier deflationism, but it will be useful to say something about the relationship of the deflationary theory to linguistic and modal approaches to indefinite extensibility more generally.

One central aspect of the deflationary theory is that it is *linguistic* in a

[36] See Putnam's "Realism and Reason" in his (1978) for his initial "internal realism" view, which includes a description of conceptual relativity.

[37] See his (1990) for this change of mind.

[38] See Hirsch (2002); see also Searle (1995), pages 160-167 and Boghossian (2006), pages 32-41.

[39] The phrase "going on in the same way" is drawn from Wittgenstein (1953).

certain sense—indefinite extensibility results from our subtly changing our language. The *interpretationism* approach pioneered by Timothy Williamson and developed in greater detail by Gabriel Uzquiano is also linguistic, but in a different way.[40] According to interpretationism, indefinite extensibility arises through an endless process of reinterpretation of predicates like "set", "ordinal", and "cardinal". Williamson explains the approach in a way that makes its affinities to the present proposal clear:

> ...given any reasonable assignment of meaning to the word "set" we can assign it a more inclusive meaning while feeling that we are going on in the same way...[41]

The crucial difference between the deflationary theory and interpretationism is that deflationists think that the domain of our quantifiers has expanded, in a certain sense, after we have altered our language, while interpretationists do not. In other words, the interpretationist doesn't think the quantifiers themselves are reinterpreted, but only our set theoretic predicates.

This difference is important in a number of ways. Interpretationists think that we start with an abundant collection of non-sets, and at each stage in an iterative process of reinterpretation, more and more non-sets get reinterpreted as "sets". As such, the quantifiers at each stage in the process of further and further interpretation range over the same domain as the quantifiers at any other stage of the process. In order for this process to not run out of objects to be re-interpreted as "sets", we need to assume that there are a great number of non-sets at the initial stage. In particular, we must assume at every stage that there are more things *simpliciter* than could be members of any set. Of course, anyone who accepts standard set theory accepts, in some sense, that the objects number more than any set, but on the interpretationist account this posit is a *precondition* for set theory rather than something we accept *because* we accept set theory or some other rich mathematical theory. This cosmological posit also has ramifications for the type of set theory we're free to adopt, e.g., interpretationists can't accept a urelemente set axiom, saying that there is set of all non-sets.[42]

Forcing a substantive cosmological posit of this kind is a metaphysical drawback to any philosophical theory. In addition, interpretationists must reject the

[40] Williamson (1998) and Uzquiano (2013).
[41] Quoted from Williamson (1998), page 20.
[42] This is also pointed out by Shapiro (2003); see Uzquiano (2013) for discussion. McGee (1997) uses a set urelemente axiom in proving a categoricity result for set theory.

standard view of sets as *sui generis* objects, and this is in tension with our intuitive metaphysical picture of sets. For these reasons, I don't think interpretationism manages to solve the puzzle of indefinite extensibility, despite its similarities to the deflationary theory.

Interpretationism has been given a modal formulation by Gabriel Uzquiano where the modality is interpretational—it concerns subsequent reinterpretations of the set theoretic terminology. And modal approaches to indefinite extensibility and set theoretic quantification have been popular since Hilary Putnam gave the first modal treatment of set theory in the late 1960s. In addition to Uzquiano and Putnam, modal approaches to set theoretic quantification have been developed by Charles Parsons, Geoffrey Hellman, Øystein Linnebo, and Kit Fine among others.[43] These account differ in various respects, but they all depend upon some modal translation of standard set theoretic claims:

> **Modal Translation** : define the modal translation $\phi^{\Diamond}$ of any set theoretic sentence ϕ recursively with the following clauses: (i) $(\forall x\phi)^{\Diamond} = \Box\forall x\phi^{\Diamond}$; (ii) $(\exists x\phi)^{\Diamond} = \Diamond\exists x\phi^{\Diamond}$; (iii) if ϕ is quantifier-free, $\phi^{\Diamond} = \phi$

This type of translation neatly pairs with the standard, iterative conception of set.[44] Intuitively, the modality is defined over the iterative hierarchy so that, $\ulcorner\Box\phi\urcorner$ is true at rank α, just in case $\ulcorner\phi\urcorner$ is true for all ranks $\beta \geq \alpha$, and $\ulcorner\Diamond\phi\urcorner$ is true at rank α, just in case $\ulcorner\phi\urcorner$ is true for some rank $\beta \geq \alpha$. This is suggestive, but if taken as an explanation of the modality being used it is problematic for the following simple reason: these truth conditions for the box and diamond used non-modal quantification over all ranks of the iterative hierarchy—that is, over all sets *simpliciter*—but that is exactly what we were trying to explain by introducing these modal notions. Understood in this way, the modal translation only sheds light on set theoretic quantification by assuming a background grasp of set theoretic quantification.

Most proponents of modal approaches to set theory weren't directly concerned with indefinite extensibility itself, but no matter the purpose to which a

[43]See Putnam (1967), Parsons (1977), Hellman (1989), Linnebo (2010), and Fine (2006).

[44]This simple type of translation is based on the treatment in Linnebo (2010). Not all modal approaches use this simple translation scheme, e.g., Putnam's original (1967) account involved forming a conjunction of the axioms of set theory, SET, letting ϕ^* be just like ϕ except all mathematical terminology is replaced with schematic letters of the appropriate type, and translating a set-theoretic sentence "p" as "$\Box(SET^* \rightarrow p^*)$" ("$\Diamond SET^*$" might need to be added as a conjunct to ward off triviality). Putnam-style treatments can be used in the service of nominalism, see Hellman (1989).

modal treatment is to be put, we need some independent account of the modal notions involved in order to have a philosophically satisfying theory. I've already mentioned that Uzquiano has given an interpretationist account of the modality, but this is far from the only approach in the literature. Since sets are necessarily existing abstract objects, the modality can't be a standard metaphysical modality, explained standardly. Putnam is explicit that the modality involved in his approach is a primitive kind of mathematical possibility, but it isn't at all clear what this primitive notion involves and Putnam never does much to explain it. Every proponent of a modal account of set theoretic quantification has recognized the need to say something about the type of modality involved, but most content themselves with a few informal—and often cryptic—remarks. Two of these authors have said things that suggest to me that they *might* be assuming something like the quantifier deflationist's picture, at least tacitly.

The first is Øystein Linnebo.[45] In his modal theory, the modality involved is explained in terms of a "process of individuating mathematical objects", i.e., "to provide it with clear and determinate identity conditions". Linnebo doesn't say much more about this, so it's difficult to understand exactly what he means. Uncharitably, critics could take him to be endorsing a version of creationism, but given the close tie between identity and quantification, perhaps Linnebo is instead suggesting something like an inchoate version of quantifier deflationism? He doesn't say enough for me to be sure, and he might be better understood as endorsing a version of interpretationism.

The second is Kit Fine.[46] Fine's modal account is developed against the background of an approach to mathematical existence that he calls *procedural postulationalism*.[47] According to procedural postulationism, under certain conditions we can successfully "postulate" the existence of mathematical objects. Fine has coupled this approach to mathematical existence with a modal approach to set theoretic quantification appealing to *postulational* modalities. In essence, $\ulcorner \Diamond\phi \urcorner$ will be true if we could go on to postulate objects that would make it the case that $\ulcorner \phi \urcorner$ is true. It is far from clear to me what it means to "postulate" an object into existence, but Fine has explicitly disavowed creationist interpretations. Hirsch has interpreted Fine as endorsing a restricted form of quantifier variance, but Fine's own attempts to explain his position remain

[45]In Linnebo (2010).
[46]In Fine (2006).
[47]See Fine (2005).

elusive, at least to me.[48]

Regardless of whether any extant modal approaches can be squared with quantifier deflationism, just as Uzquiano has used Williamson's interpretationism to explain set theoretic modality, we can use quantifier deflationism to ground a similar modal approach. Section 3's account of indefinite extensibility involved appeal to well-behaved expansions of our current set theoretic language. An expansion of our language adds some new vocabulary and perhaps new axioms and rules of inference to our language, so that language L expands language K when L contains all of the rules, axioms, and vocabulary of K and more besides. It is difficult or impossible to say, in full detail, what it means for an expansion of our language to be "well-behaved", but minimally, it would seem to require that the expansion not be inconsistent and that the meanings of the old terms in our language not be radically altered. The notion of a an expansion of our language is intuitively modal, since we are concerned with the space of all possible expansions of our language, and it is most natural to take the relevant type of possibility here to be physical possibility—we aren't concerned with expansions of our language that would only be "possible" if we were able to break the laws of physics, for example.

I sympathize with readers who still feel that they lack a completely firm grip of the space of all possible well-behaved expansions of our language, but I don't think any of the literature's more model theoretic accounts of linguistic expansion would be helpful to us in this context.[49] Let's press on with out intuitive, somewhat hazy grasp of well-behaved expansions and see where it gets us.

Using the notion of a well-behaved expansion of our language, we can explain the modal operators "$\Box$" and "$\Diamond$" as follows: $\ulcorner \Box\phi \urcorner$ is true in L just in case $\ulcorner \phi \urcorner$ is true in every well-behaved expansion of L and $\ulcorner \Diamond\phi \urcorner$ is true in L just in case $\ulcorner \phi \urcorner$ is true in some well-behaved expansion of L. The set theoretic principles that are "necessary", in this sense, will be those principles that will continue to hold in every well-behaved expansion of our language; and the set theoretic principles that are "possible", in this sense, will be those principles that hold in some well-behaved expansion of our language. In this way, the deflationist can make some sense of our open-ended commitments to certain set theoretic principles—those principles hold not just in our current language, but also in

[48] Hirsch (2011), page xv, citing Fine (2009).

[49] I'm thinking of the account given in McGee (2000), for example.

all well-behaved expansions of our language.[50]

Even at this level of detail, I think that the deflationists's modal treatment of set theoretic quantification compares favorably with some of the earlier modal accounts that involved hazy or mysterious explanations of the nature of the modality involved. However, the modal notions carry with them the threat of an extended paradox, at least for the quantifier deflationist. Here's how the threat goes: once we've accepted the modal notions into our language, we can define a quantifier "$\exists^A$" as follows: $\ulcorner \exists^A \xi \phi \urcorner$ is true in L just in case $\ulcorner \Diamond \exists \xi \phi \urcorner$ is true in L, i.e., just in case $\ulcorner \exists \xi \phi \urcorner$ is true in some well-behaved expansion of L. In effect, this quantifier quantifies, in our current language, over every object quantified over in some well-behaved expansion of our language. Consider the dual of this quantifier, "$\forall^A$": for quantifier deflationists, it must be admitted that "$\forall^A$" really is an unrestricted universal quantifier, since it obeys the inferential rules of an unrestricted quantifier. But now we can run a version of the standard argument for paradox, using "$\forall^A$", and in the process come to understand an even more expansive universal quantifier, "$\forall^{A+}$". Yet this shouldn't be possible, for given the definitions of "$\forall^A$" and our modal notions, the domain of "$\forall^A$" already includes any object we could come to quantify over in any possible expansion of our language!

Despite appearances, I think this argument for extended paradox only poses serious problems for us if we cling hard to the old way of thinking about metasemantics to which quantifier deflationism is opposed. If we think of the meaning of "$\forall^A$", including its domain, as being in some way explanatorily prior to our understanding of how to use "$\forall^A$" in whole sentences, then this extended paradox is indeed puzzling. But if we reject this, and see the domain of "$\forall^A$" as being explained by how we use "$\forall^A$", then the extended paradox is just as puzzling as the standard paradoxes and no more. The meaning of "$\forall^A$", is, like all other quantifiers, fixed by how we use it, rather than by some explanatorily prior collection (in this case the collection of all possible well-behaved expansions of our language). So just like the sets, the ordinals, the cardinals, and whatnot, in trying to quantify over absolutely all well-behaved expansions of our language,

[50] To provide a few more details: points in a frame for this modal logic would be languages related to our current language by the relation of well-behaved expansion. We can say that Rab if and only if language b expands language a. Presumably, this will give rise to an upward branching tree where each branch is infinite (arguably), as is the vertical branching at each point (again, arguably). R so defined (as improper expansion) will be reflexive, transitive, and non-Euclidean, so the modal logic involved will be at least as strong as $S4$ but weaker than $S5$. Although I stress that attempting to get even this precise about the modality involved requires certain idealizations and assumptions that arguably render the entire exercise unhelpful.

we find ourselves in a well-behaved expansion that wasn't in our initial domain of quantification.

The definition of "$\forall^A$" was given in terms of "$\Box$", which was itself defined using a standard universal quantifier over well-behaved expansions of our current language. In effect, the analysis of "$\Box$" itself turns out to be indefinitely extensible. This isn't a refutation of the quantifier deflationist's approach, it is simply yet another reminder that the modal notions here, though a useful heuristic, shouldn't be seen as providing us a way to reach a super-duper, absolutely unrestricted sense of "everything", for there is no such sense of "everything". The entire point of the indefinite extensibility approach is that there can be no such final stopping point. In characterizing our modal notions we used standard quantifiers, over well-behaved expansions of our language, so it is no surprise to find that those quantifiers too, are indefinitely extensible. It's turtles all the way down (up?), I'm afraid.

The deflationary theory of indefinite extensibility isn't a formal theory of sets, it is instead a philosophical account of the indefinite extensibility of our set theoretic quantifiers. There are a number of formal paths to blocking the paradoxes in line with the ideas of this paper, but the real answer to the paradoxes is not the formalism, but rather the philosophical analysis that explains *how* and *why* indefinite extensibility arises. As a famous methodological sermon once said, there is no mathematical substitute for philosophy.[51] But conversely, there is no philosophical substitute for mathematics, and the task of developing mathematical theories of sets and exploring their features remains before us, now with mysteries removed.[52]

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