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Research Article

Algebra Automorphism Modulus Additive Conditions for Surjective Maps on Unital C*-Algebras

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Abstract

In the present work, authors considered norm preserver conditions for automorphisms on unital C*-algebras. This is done by first establishing sufficient conditions for which a surjective map between unital C*-algebras is an algebra automorphism.

Keywords: Automorphism; C*-algebras; Surjective map; Unital.

Introduction

In preserver problems, one of the most basic questions we seek to answer is whether an operator between two spaces with the same structure is a homomorphism [1]. In particular, we ask whether the operator $T : V \to W$ preserves the operation in this spaces, that is , we ask if $T(m * n) = T(m) * T(n)$, where $*$ is the operation in V in the first case and the operation in Y in the second case [2]. If in case T does preserve the operation, then f is called a homomorphism and we can apply any of the results we know concerning the spaces V and W and homomorphisms between them.

One of the first interesting preserver problem was by [3] in which It was required that the operator T be surjective and to preserve the zero element and the distances between elements. The last property is merely the definition of an isometry, so it was not surprising that the conclusion was that T is an isometric transformation thus bringing in linearity hence making a statement about norms. Although Mazur-Ulam Theorem [4] does not assume that a map is linear from the beginning, many of the classical results in the area of preserver problems assume the map to be linear and to preserve some other property that then leads to a conclusion categorizing such maps.

The Mazur-Ulam Theorem [5] goes against the usual way of doing things by first verifying a norm condition then concluding that the map is linear. In linear preserver problems, the maps are between algebras and the maps are assumed to be linear. A part from the maps being linear, they also preserve some other property that then leads to a conclusion classi- fying such maps. An example of one such result is the Gleason-Kahane-Zelazko Theorem [6]. When the mapping in Kahane-Zelazko Theorem was strengthened to be surjective from a uniform algebra to a uniform algebra and preserves the spectra of algebra elements, the results goes beyond concluding that the mapping is multiplicative to preserving the distance between algebra elements and the structure of the algebra.

Banach-Stone Theorem [7] is another example of such result that establishes an isometric algebra isomor- phism between spaces of continuous functions on compact spaces. The Gleason-Kahane- Zelazko Theorem [8], apart from having a spectral condition, it also required the mapping $T : U \longrightarrow Y$ to be a linear operator. There are several other results that require preservation of all or part of the spectra of the elements of the algebra or a subset of the elements of the algebra but do not require the mapping T to be linear. The first of such results was Kowalski and Slodkowski [9] which demanded that the spectrum of the difference between algebra elements be preserved in order to have the mapping preserve the algebraic structure as well as the distance between algebra elements. The spectral condition in the results implied that $\|Tm -$

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$Tn\| = \|m - n\|$, $\forall m, n \in G$, that is, T preserves distances be- tween the algebra elements. This spectral condition brought the isometry conclusion which was not a surprise. We also see that the Mazur-Ulam Theorem [10] implies that T is an R−linear mapping, so the additivity requirement for an isomorphism is met. Unital operators are mappings that preserve the unit element that is the mapping $T : V \longrightarrow W$ between unital algebras has the property $T(1_V) = 1_W$. Spectral preserver problems started taking a multiplicative direction where the unit element was to be preserved. One such result was from [11] which was extended by Rao and Roy to surjective self-maps from any uniform algebra to itself and for an arbitrary compact Hausdorff set P .

The results in [2] were significantly improved, one year later, by [3]. This was done by allowing T to be an operator between any two uniform algebras instead of requiring it to be a self -map and by only requiring the preservation of a subset of the spectra (the peripheral spectra) of products of algebra elements. For algebra elements m and n, if $\sigma(m) = \sigma(n)$ then $\sigma\pi(m) = \sigma\pi(n)$ but not vice versa. Later [6] extended this theorem to standard operator algebras. Luttman and Tonev were joined with lambert to show that instead of the preservation of the peripheral spectra of products of algebra elements, T need only preserve at least one element of the peripheral spectra of products. The requirement that T be unital was removed and added the requirement that T preserve the peripheral spectra of all algebra elements. However, this requirement is not more than the previous results because the theorem requires that T be unital, in which case $\sigma\pi(Tm) = \sigma\pi(TmT\ 1) = \sigma\pi(m \cdot 1) = \sigma\pi(m)$, so a map that satisfies the hypothesis of theorem does in fact preserve the peripheral spectra of algebra elements. The proofs of these theorems largely depend on variations of the classical result by [2] which was refined by [5]. A stronger version of the lemma is found in [9]. In [8] the authors took the additive direction by showing that a surjection that preserves the peripheral spectra of sums of algebra elements as well as the sup-norms of the sums of the moduli of algebra elements will preserve the distances between algebra elements as well as the structure of the algebra.

Research methodology

As we have seen, the set of linear multiplicative functionals on a commutative Banach algebra and the set of maximal ideals for that algebra are in bijective correspondence, so we can make the following definition.

Definition 2.1.

Let D be a commutative Banach algebra with unit. The set MD of all nonzero linear multiplicative functionals of D is called the maximal ideal space of D. Though the space MD does not possess a natural algebraic structure, we can equip it with the weak−∗topology it inherits as a subset of D^*, the collection of all bounded linear functionals on D. When applied to the maximal ideal space, we call this toplolgy the Gelfand topology. We recall that under this topology, a net of elements φ_α in M_D tends to $\varphi \in M_D$ if and only if $\varphi_\alpha(m) \to \varphi(m)$, $\forall m \in D$. Thus, under the Gelfand topology, convergence of functionals in M_D is point wise convergence. A weak−* limit of linear multiplicative functionals is itself a non-zero linear multiplicative functional because $(\lim \varphi_\alpha)(1) = \lim \varphi_\alpha(1) = 1$. We also note that the space MD is compact in the weak−*topology by the Banach-Alaoglu theorem.

Definition 2.2.

Let m be an element in a commutative Banach algebra D. The Gelfand transform of m is the function $\hat{m}$ on M_D defined by $\hat{m}(\varphi) = \varphi(m)$, $\forall\varphi\varphi \in M_D$. The Gelfand transform of m is clearly continuous on M_D with respect to the Gelfand topology since if $\varphi_\alpha \to \varphi$, then $\varphi_\alpha(m) \to \varphi(m)$, which implies that $\hat{m}(\varphi_\alpha) \to \hat{m}(\varphi)$.

Results and discussion

In this section, we present our results. We use the fact that if $X \subset C(P)$ and $Y \subset C(S)$ will be uniform algebras on compact sets P and S respectively and Since X and Y are sub algebras of the unital C^*-algebra A and also C(S) being isomorphic to the unital C^*-algebra A, the results discussed in this Chapter concerning uniform algebras can be extended to unital C^* -algebra and isomorphism changed to automorphism since the transformation will be in the same space. The following proposition gives sufficient conditions under which surjective maps, in unital C^*-algebra, are automorphisms.

Proposition 3.1.

If $\phi : S \longrightarrow P$ is homeomorphism and if $T : X \subset A \longrightarrow C(S) \cong A$ is a surjection defined by $Tm = m \circ \phi \forall f \in X \subset A$, then T is linear, multiplicative, injective, and continuous then it is an isometric algebra isomorphism.

Proof. Let $m, n \in A$ and $\lambda,\mu \in C$. Then $T(\lambda m + \mu n = (\lambda m + \mu n) \circ \phi) = \lambda(m \circ \phi) + \mu(n \circ \phi) = \lambda Tm + \mu Tn$, so T is linear. Also, T is multiplicative because $T(mn) = (mn) \circ \phi = (m \circ \phi)(n \circ \phi) = TmTn$. Because ψ is a homeomorphism, it is surjective, so T is injective. Finally, the continuity of T follows from the linearity of T and the inequality

$$\|Tm\| = \sup_{y\in S} |m(\phi(y))| \leq \|f\|.$$

If $Tm = m \circ \phi$, then we call T a ϕ−composition operator. If $T : X \longrightarrow Y$ is a ϕ−composition operator, then T satisfies the equation $\||Tm| + |Tn|\| = \||m| + |n|\|$ for every $m, n \in A$ since the fact that ϕ is a homeomorphism implies that $\||Tm| + |Tn|\| = \||m \circ \phi| + |n \circ \phi|\| = \||m| + |n|\|$. The map T also satisfies the equations $\|Tm + Tn\| = \|m + n\|$ and $\|\lambda Tm + \mu Tn\| = \|\lambda m + \mu n\|$, for every $m, n \in A$ and $\lambda,\mu \in C$ since

$$\|\lambda Tm + \mu Tn\| = \|\lambda m \circ \phi + \mu n \circ \phi\| = \|\lambda m + \mu n\| \ \forall \lambda,\mu \in C$$

In particular, for $\lambda = \mu = 1$,proving that T satisfies $\|Tm + Tn\| = \|m + n\|$.

We also have the following preservation of relationships among the peripheral spectra:$\sigma_\pi(Tm) = \sigma_\pi(m \circ \phi) =\sigma_\pi(Tm + Tn) = \sigma_\pi(m + n)$. This completes the proof.

Next, we show that a surjective operator $T : X \subset A \longrightarrow Y \subset A$ that satisfies certain conditions naturally induces a homeomorphism between the Choquet boundary of X and the Choquet boundary of Y .

Definition 4.2.

An operator $T : X \subset A \longrightarrow Y \subset A$ is norm-additive in modulus if it satisfies

$$\max_{x\in P} |(Tm)(x)| + |(Tn)(x)| = \max_{x\in P} |m(x)| + |n(x)|, \forall m, n \in X.$$

Example 4.3.

The operator $T : X \subset A \longrightarrow Y \subset A$ for which $Tm = im$ is a norm-additive in modulus because $\||Tm|+ |Tn|\|= \||im|+|in|\|= |i| \||m| + |n|\|= \||m| + |n|\|$. Where the operation $T : X \subset A \longrightarrow Y \subset A$ for which $Tm = -m$ is similarly norm-additive in modulus. In fact , all operators $T : X \subset A \longrightarrow Y \subset A$ such that $Tm = \alpha m$ with $\alpha\alpha \in X \subset A$ and $|\alpha(x)| = 1, \forall \in P$ are norm-additive in modulus since $\||Tm| + |Tn|\| = \||\alpha m| + |\alpha n|\|= |\alpha| \||m| + |n|\|= \||m| + |n|\|$.

Example 4.4.

The operator $T : X \subset A \longrightarrow Y \subset A$ defined by $Tm = \|m\| , \forall m \in X \subset A$ is also norm-additive in modulus: $\||Tm| + |Tn|\| = \||\|m\|| + |\|n\||\|= \||m| + |n|\|$.

We note that this operator does not preserve $|m|$ unless m is a constant function. Clearly, for any norm-additive in modulus operator, we have $T 0 = 0$ since $0 = \||0|+ |0|\|= \||T 0| + |T 0|\|= 2 \||T 0|\|$ implies that $|T 0| = 0$. Also, an operator that is norm-additive in modulus is norm-preserving since

$$\|Tm\| = \||Tm| + |T 0|\|= \||m|+ |0|\|= \|m\|$$

Another example of norm-additive in modulus operators is given by the next proposition

Lemma 4.5.

An operator $T : X \subset A \longrightarrow Y \subset A$ that satisfies $\|Tm + \alpha Tn\| = \|m + \alpha n\| , \forall m, n \in X \subset A$ and α with $|\alpha| = 1$ is norm-additive in modulus.

Proof. If $T : X \subset A \longrightarrow Y \subset A$ satisfies $\|Tm + \alpha Tn\| = \|m + \alpha n\| , \forall m, n \in X \subset A$ and $\alpha = 1$ is norm-additive in modulus, then we can choose an α with $|\alpha| = 1$ such that $\||Tm| + |Tn|\| = \||Tm| + |\alpha Tn|\|= \||Tm+ \alpha Tn|\|= \||m +\alpha n|\|\leq \||m| + |\alpha n|\|= \||m| + |n|\|$. Similarly, $\||m| + |\alpha n|\| \leq \||Tm| + |Tn|\|$, so T is norm-additive in modulus.

Definition 4.6.

An operator $T : X \subset A \longrightarrow Y \subset A$ is monotone increasing in modulus if the inequality $|m(x)| \leq |n(x)|$ on ∂X implies that $|(Tm)(y)| \leq |(Tn)(y)|$ on $\partial Y, \forall m, n \in X$.

Example 4.7.

The operators $Tm = \alpha m$ for $\alpha \in X$ with $|\alpha| = 1$ given in Example ... as norm-additive in modulus are also monote increasing in modulus since if $|m(x)| \leq |n(x)|$, then

$|(\alpha m)(x)| = |\alpha(x)|\ |m(x)| = |m(x)| \leq |n(x)| = |\alpha(x)||n(x)| = |(\alpha n)(x)|$.

The next proposition provides a connection between monotone increasing in modulus operators and norm-additive in modulus operators.

Proposition 4.8.

A norm-additive in modulus operator is monotone increasing in modulus.

Proof. Let $T : X \subset A \longrightarrow Y \subset A$ be a norm-additive in modulus operator. If $|m(x)| \leq |n(x)|$ on ∂X, then clearly $\||m| + |p|\| \leq \||n| + |\alpha p|\|$ for any $p \in X \subset A$. Because T is norm-additive in modulus, we have that $\||Tm| + |Tp|\| = \||m| + |p|\| \leq \||n| + |p|\| = \||Tn| + |Tp|\|$.

Assume that there is some $b_o \in \partial Y$ such that $|(Tm)(b_o)| > |(Tn)(b_o)|$.

Because δY is dense in ∂Y, we may assume that $b_o \in \delta Y$. choose a $\omega > 0$ such that

$$|(Tn)(b_o)| < \omega < |(Tm)(b_o)|$$

and an open neighborhood N of b_o in $Y \subset A$ such that $|(Tn)(b_o)| < \omega$ on N. Let r be a real number greater than 1 such that $\|Tm\|$, $\|Tn\| \leq r$ and let $Tp \in P_{bo}(Y)$ be a peaking function for Y with $E(Tp) \subset N$, so $(Tp)(b_o) = 1$ and $|(Tp)(b)| < 1$ for any $b \in \partial Y \setminus N$. By replacing Tp with a sufficiently high power of Tp, we have

$$|(Tn)(b)| + |r(Tp)(b)| < r + \omega, \forall b \in \partial Y \setminus N.$$

This inequality also holds on N because $|(Tn)(b)| < \omega, \forall b \in N$ and $|(Tp)(b)| \leq 1$ for all $b \in Y$. Thus we have that $|(Tn)(b)| + |r(Tp)(b)| < r + \omega, \forall b \in \partial Y$.

$|(Tm)(b_o)| + r = |(Tm)(b_o)| + r\,|(Tp)(b_o)| \leq \||Tm| + r\,|Tp|\| \leq \||Tn| + r\,|Tp|\|$

Therefore, $|(Tm)(b_o)| < \omega$, which is a contradiction. Hence, $|(Tm)(b)| \leq |(Tn)(b)|$, $\forall b \in \partial Y$.

This holds for every p = 1, ..., n, so $E(Tn) \subset E(Th_j)$. Hence the family $E(Th) : h \in \varepsilon_x(X)$ has the finite intersection property, as claimed. Because each E(Th) is a closed subset of S, a compact set, the family $E(Th) : h \in \varepsilon_x(X)$ must have a non-empty intersection. We observe that the set $E(Tm) = (Tm)^{-1}(\sigma_\pi(Tm))$ is a union of peak sets because $(Tm)^{-1}(u)$ is a peak set for any $u \in \sigma_\pi(Tm)$. Thus, every $b \in E_x$ belongs to an intersection $F \subset E_x$ of peak sets of Y .

Therefore, F meets δY and thus $E_x \cap \delta Y \neq \emptyset$. We note that [2] considered sets similar to E_x that involve peaking functions instead of C−peaking functions but also require T to preserve the peripheral spectra of all algebra elements.

Lemma 4.9.

Let $T : X \subset A \longrightarrow Y \subset A$ be a norm-additive in modulus, R^+−homogeneous, surjective operator. If $a \in \delta X$ and $b \in E_x \cap \delta Y$, then $T^{-1}(\varepsilon_b(Y)) \subset \varepsilon_a(X)$.

Proof. Let $a \in \delta X$. If T is R^+−homogeneous, surjective, and norm-additive in modulus, then T is monotone increasing in modulus and norm-preserving, as we have seen, so $E_x = \emptyset$. Let $b \in E_x$, fix a $p \in \varepsilon_b(Y)$, and let $h \in T^{-1}(p)$. In order to prove that $h \in \varepsilon_a X$, we must show that $|h(a)| = \|h\|$. Let N be an open neighborhood of a and let $k \in \|h\| \cdot P_a(X)$ be a C−peaking function such that $E(k) \subset N$. Because $b \in E_x = {}_{m \in \varepsilon a(X)} E(Tk) \subset E(Tk)$, we have that $|(Tk)(b)| = \|Tk\|$, which implies that $Tk \in \varepsilon_b(Y)$ Because T preserves the norms, $|p(b)| = \|p\| = \|h\| = \|k\| = \|Tk\|$. Thus, because T is norm-additive in modulus,

$$\|h\| + \|k\| \geq \||h| + |k|\| = \||p| + |Tk|\| \geq |p(b)| + |(Tk)(b)| = \|p\| + \|Tk\| = \|h\| + \|k\|.$$

Therefore , $\||h| + |k|\| = \|h\| + \|k\|$, so there must be an $a_N \in \partial X$ such that $|h(a_N)| = \|h\|$ and $|k(a_N)| = \|k\|$. Hence , $a_N \in E(k) \subset N$ and any neighborhood N of a must contain a point a_N with $|h(a_N)| = \|h\|$. Because h is continuous, we must have $|h(a)| = \|h\|$, which implies that $h \in \varepsilon_a(X)$. Thus, $T^{-1}(\varepsilon_b(Y)) \subset \varepsilon_a(X)$.

Theorem 4.10.

If $T : X \subset A \longrightarrow Y \subset A$ is a norm-additive in modulus, R^+−homogeneous surjection, then the set E_x is a singleton that belongs to δY for any generalized peak point $a \in \delta Y$.

Proof. Let $b \in E_x$ and suppose there is a $r \in E_x \setminus b$. Then there is a function $p \in \varepsilon_b(Y)$ such that $|p(r)| < \|p\|$. For any $h \in T^{-1}(p) \subset \varepsilon_a(X)$, we have $E(p) = E(Th) \supset E_x$, which implies that the function $|p| = |Th|$ is a constant on E_x with value $\|p\|$. This is a contradiction to $|p(r)| < \|p\|$. Hence, the set E_x contains only the point b. We see that T induces an associated mapping $\tau : \delta X \subset A$

$\longrightarrow \delta Y \subset A$ such that $a \longrightarrow \tau(a)$. From this mapping τ we will obtain the homeomorphism that will allow us to conclude that a map T satisfies certain of the conditions is an algebra automorphism.
We see that $\varepsilon_{\tau(a)}(Y) = \varepsilon_b(Y) \subset T(\varepsilon_a(X))$. If $h \in \varepsilon_a(X)$, then we have that $\tau(a) = E_x \subset E(Th)$. Hence, $|(Th)(\tau(a))| = \|Th\| = \|h\| = |h(a)|$, for any $h \in \varepsilon_a(X)$. We note that if $h \in C \cdot P_a(X)$ and T preserves the peripheral spectrum of h, that is $\sigma_\pi(Th) = \sigma_\pi(h)$, then $|(Th)(\tau(a))| = \|Th\| = \|h\| = |h(a)|$ implies that $(Th)(\tau(a)) = h(a)$ since the peripheral spectra are singletons. We also note that, in [4], the authors considered a mapping similar to τ that mapped each $a \in P$ to the singleton set $_{h\in P_a(X)} E(Th)$. In that paper, the operator $T : X \subset A \longrightarrow Y \subset A$ was assumed to be peripherally-additive that is $\sigma_\pi(Tm + Tn) = \sigma_\pi(m + n)$, $\forall m, n \in X$ and thus preserved the peripheral spectrum for every $m \in X$.

Corollary 4.11.

If $T : X \subset A \longrightarrow Y \subset A$ is a norm-additive in modulus, R^+−homogeneous surjection, then $T(\varepsilon_a(X)) = \varepsilon_{\tau(a)}(Y)$.
Proof. Let $h \in \varepsilon_a(X)$ for some $a \in \delta X \subset A$ and let $p = Th$. Then equation gives

$$|p(\tau(a))| = |(Th)(\tau(a))| = |h(a)| = \|h\| = \|p\|$$

This implies that $p \in \varepsilon_{\tau(a)}(Y)$, so $T(\varepsilon_a(X)) \subset \varepsilon_{\tau(a)}(Y)$ and, frrom Lemma 4.1.7, we have that $T(\varepsilon_a(X)) \supset \varepsilon_{\tau(a)}(Y)$.
The next proposition shows that when T is a norm-additive in modulus, R^{+1}−homogeneous surjection, (4.10) holds for every $m \in X \subset A$ and $a \in \delta X \subset A$, not merely for functions that take their maximum modulus at a.

Proposition 4.12.

If $T : X \subset A \longrightarrow Y \subset A$ is a norm-additive in modulus , R^+−homogeneous surjection, then the associated mapping τ that T induces is continuous and the equation $|(Tm)(\tau(a))| = |m(a)|$,holds for every $a \in \delta X \subset A$ and all $m \in X \subset A$. If, in addition, T is bijective, then τ is a homeomorphism from $\delta X \subset A$ onto $\delta Y \subset A$, and if $\psi : \delta Y \subset A \longrightarrow \delta X \subset A$ is the inverse mapping of τ , then $|(Tm)(b)| = |m(\psi(b))|$ for every $b \in \delta Y \subset A$

Proof. We first show that $|(Tm)(\tau(a))| = |m(a)|$, $\forall a \in \delta X \subset A$ and $\forall m \in X \subset A$. Let $a \in \delta X \subset A$, $m \in X \subset A$ and r be a real number greater than 1. If $h_o \in r\|m\| \cdot P_a X$ is a function as in the strong Additive Bishop's lemma, then $\|Th_o\| = \|h_o\| = r\|m\| = r\|Tm\|$. So, $Th_o \in r\|Tm\| \cdot \varepsilon_{\tau(a)}(Y)$. Because T is norm-additive in modulus, it implies that

$$r\|m\| + |m(a)| = \inf_{h\in\varepsilon_a(X),\|h\|=r\|m\|} \||h| + |m|\|$$
$$= \inf_{h\in\varepsilon_a(X),\|h\|=r\|m\|} \||Tm(b)| + |Th|\|$$
$$= \inf_{p\in\varepsilon_{\tau(a)}(X),\|h\|=r\|m\|} \||Tm| + |p|\|$$
$$= \|m\| + |(Tm)(\tau(a))|$$

Consequently, $|(Tm)(\tau(a))| = |m(a)|$, as claimed. To show the continuity of τ , we let $a \in \delta X$ and $k \in (0, 1)$. Choose an open neighborhood N of $\tau(a)$ in δY and a peaking function $p \in P_{\tau(a)}(Y)$ such that $E(p) \subset N$ and $|p(b)| < k$ on $\delta Y \setminus N$. If $h \in T^{-1}(p)$, then $h \in \varepsilon_a(X)$, and , according to , we have $|h(a)| = |(Th)(\tau(a))| = |p(\tau(a))| = 1 > k$.
Therefore, the open set $W = \xi \in \delta X \subset A : |h(\xi)| > k$ contains a. The first part of the proof shows that for every $\xi \in W$, we have $|p(\tau(\xi))| = |(Th)(\tau(\xi))| = |h(\xi)| > k$, which implies that $\tau(\xi) \in N$ since $|p(\eta)| < k$ for $\eta \in \delta Y \setminus N$. Consequently, $\tau(W) \subset N$, so τ is continuous. Now suppose that T is bijective. Then T^{-1} is R^+−homogeneous, and because the equation $\||Tm| + |Tn|\| = \||m| + |n|\|$ is symmetric with respect to m and Tm, it must also hold for the operator $T^{-1} : Y \longrightarrow X$. Now T^{-1} induces an associated map $\phi : \delta Y \subset A \longrightarrow \delta X \subset A$ that is continuous and satisfies $|(T^{-1}p)(\phi(\eta))| = |p(\eta)|$, $\forall \eta \in \delta Y \subset A$ and for any $p \in \varepsilon_{\phi(\eta)}(Y)$. Let $a \in \delta X \subset A$ and $b = \tau(a) \in \delta Y \subset A$. If $h \in \varepsilon_a(X)$, then $p = Th \in \varepsilon_b(Y)$, so
$|h(\phi(b))| = |(T^{-1}(p))(\phi(b))| = |p(b)| = |(Th)(b)| = |(Th)(\tau(a))| = |h(a)| = \|h\|$.
Hence, $\phi(b) \in E(h)$ for any $h \in \varepsilon_a(X)$. Because $\bigcap_{h\in\varepsilon_a(X)} E(h) = x$, we see that

$$\phi(\tau(a)) = \phi(b) = a, \forall a \in \delta X \subset A.$$

Similarly, $\tau(\phi(b)) = b$, $\forall b \in \delta Y \subset A$. Thus, τ and ϕ are both bijective and $\phi = \tau^{-1}$, so τ is a homeomorphism. The rest is clear.

Conclusions

In summary, we have determined sufficient conditions for which a surjective map between unital C^*-algebras is an algebra automorphism, we have shown that If A is a unital C^*-algebra which is commutative then it is isomorphic to the space $C(P)$ of all continuous functions on a compact set P and uniform algebra is a sub algebra of the space $C(P)$. Therefore if $X \subset C(P)$ and $Y \subset C(S)$ are uniform algebras with Choquet boundary δX and δY, it is shown that if $T : X \longrightarrow Y$ is a surjection that preserves the norm of the sums of the moduli of algebra elements, then T induces a homoemorphism ϕ between the Choquet boundaries of X and Y such that $|Tm| = |m \circ \phi|$ on the Choquet boundary of Y. If, in addition, T preserves the norms of all linear combi- nations of algebra elements and either preserves both i and 1 or the peripheral spectra of C-peaking function, then T is a composition operator and thus an algebra automorphism.

Conflicts of interest

Authors declare no conflict of interest.

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