Three-Dimensional goal-oriented finite element forward modeling for magnetotelluric using model decomposition parallel approach

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Summary

A three-dimensional parallel goal-oriented finite element forward modeling method for magnetotelluric has been developed. To adapt complex structures and topography and satisfy continuity requirements of electric fields, the computational domain was discretized into tetrahedral elements, and lowest order Nédélec element was used. We reformulate the resulting complex linear system to its real form and utilize a block-diagonal pre-conditioner. The CG method with Auxiliary Space Maxwell pre-conditioner was used to solve the two smaller real-valued equation required by application of the block-diagonal pre-conditioner. We use a goal-oriented error estimator to drive the local mesh refinement procedure. Our algorithm was parallelized using model decomposition method. The mesh was partitioned into several parts, each computational process is only responsible for its own part. The resulting linear system was stored using parallel compressed row storage format and then solved by cooperation of all processes. Finally, a topography model with spherical hill and valley demonstrates the ability to simulate complex structures of our code.

Keywords: magnetotelluric, parallel, model decomposition, goal-oriented, finite element

Introduction

Efficient and accurate three dimensional magnetotelluric forward modeling method has attracted increasing attention in recent years. The most popular numerical modeling techniques applied to forward modeling of magnetotelluric are finite-difference (FD) and finite-element (FE) methods. FD technique has been widely used in 3D problems (Mackie, Madden, & Wannamaker, 1993; Newman & Alumbaugh, 2002) because it is easier to implement and has smaller computational complexity compared with FE technique. For problems containing complex structures, the FE method is generally preferred because it has an advantage of naturally supporting unstructured meshes.

The FE method has long been used in 3D EM geophysical problems (Nam et al., 2007; Grayver & Bürg, 2014). The accuracy of FE method is largely determined by the mesh density distribution and the number of elements. To get more accurate solutions, the meshes are often refined in regions where electrical conductivity difference are large such as the material interfaces or where high accuracy is required such as close to the receivers. For complicated realistic problems, the number of unknowns of the linear system arising from discretization of the partial difference equation (PDE) is often on the order of millions or even tens of millions, which imposes high computational effort including high computer memory and runtime costs and even impossible to run using a single node.

There are several parallel schemes to overcome the computational bottleneck. One of them is data decomposition method (Key, 2016). The main idea behind data decomposition method is first partitioning the input data into smaller independent groups and then each task is running independently in parallel. Domain decomposition method (Ren, Kalscheuer, Greenhalgh, & Maurer, 2014) is another approach that solves several smaller problems by partitioning the large domain into smaller subdomains instead of the whole big problem. In domain decomposition method, a interface problem need to be solved. The interface problem involves solving dense linear systems, which is difficult and does not scale well.

In this work we introduce a parallel goal-oriented edge-based finite element method for three dimensional MT forward modeling. The parallel strategy we used here is model decomposition method. A numerical example was taken to show the effects of topography and the efficiency of our parallel scheme.

Goal-oriented finite element modeling

Governing equations

For magnetotelluric problem, the electromagnetic fields are governed by the Maxwell's equations. In frequency domain, assuming the time harmonic difference is $e^{i\omega t}$ and neglecting displacement currents, the second order vector Helmholtz equation has the form of

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu\sigma\mathbf{E} = 0, \quad (1)$$

where ω is angular frequency, μ is the magnetic permeability, σ is the electric conductivity. To guarantee that eq. (1) has unique solution, Dirichlet boundary conditions are used. In this case, the tangential components of the electric fields are prescribed to be some certain values at boundaries, i.e.,

$$\mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{E}_0, \quad (2)$$

where $\mathbf{n}$ is the outward pointing normal vector, $\mathbf{E}_0$ is the plane wave solution of a layered model that can be computed analytically.

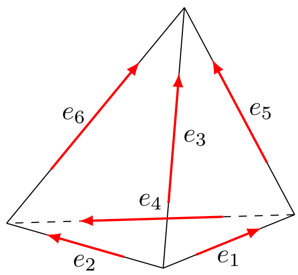

Figure 1: Nédélec element of lowest order on tetrahedron

Finite element discretization

Let $\mathbf{V}$ be a test function that belongs to save space with $\mathbf{E}$, i.e. $\mathbf{V}, \mathbf{E} \in H(\mathbf{curl}, \Omega)$, $H(\mathbf{curl}, \Omega)$ is defined as $H(\mathbf{curl}, \Omega) = \{\mathbf{u} \in L^2(\Omega) : \nabla \times \mathbf{u} \in L^2(\Omega)\}$. Taking the inner product of both side of eq. (1) with $\mathbf{V}$ and integrating by parts, the equivalent weak form of eqs (1) and (2) can be stated as find $\mathbf{E} \in H(\mathbf{curl}, \Omega)$ such that

$$\int_\Omega \nabla \times \mathbf{V} \cdot \nabla \times \mathbf{E} + i\omega\mu\sigma \int_\Omega \mathbf{V} \cdot \mathbf{E} = 0 \quad (3)$$

for all $\mathbf{V} \in H(\mathbf{curl}, \Omega)$.
The computational domain, Ω, is discretized into a set of none-overlapping tetrahedra. For the discretization of this function space, we choose Nédélec element of lowest order. The 6 shape functions of tetrahedral element (shown in Fig. 1), one for each edge, are continuous across cell boundaries in its tangential components, but discontinuous in its normal component, therefore satisfy all continuity requirements of electric fields.

Linear solvers

Apply eq. (3) to each tetrahedron in Ω and assemble them together, we get a linear system

$$\mathbf{K}\mathbf{e} = \mathbf{s}, \quad (4)$$

where $\mathbf{K}$ is a large complex symmetric matrix, its condition number is large especially for low frequency. It is difficult to solve using iterative method with generic pre-conditioners, and it need too much memory to use a direct solver.

Here we introduce a optimal block pre-conditioner by reformulating the complex linear system into its equivalent real form. Let $\mathbf{K} = \mathbf{K}_r + i\mathbf{K}_i$, $\mathbf{e} = \mathbf{e}_r + i\mathbf{e}_i$, $\mathbf{s} = \mathbf{s}_r + i\mathbf{s}_i$, eq. (4) can be written as

$$\underbrace{\begin{bmatrix} \mathbf{K}_r & -\mathbf{K}_i \\ -\mathbf{K}_i & -\mathbf{K}_r \end{bmatrix} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_i \end{bmatrix} = \begin{bmatrix} \mathbf{s}_r \\ -\mathbf{s}_i \end{bmatrix}}_{\mathbf{Ax}=\mathbf{b}}. \quad (5)$$

We use a block-diagonal pre-conditioner

$$\mathbf{P} = \begin{bmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{bmatrix}, \quad (6)$$

where $\mathbf{B} = \mathbf{K}_r + \mathbf{K}_i$. It can be proved that the condition number of $\mathbf{P}^{-1}\mathbf{A}$ has an upper bound of $\sqrt{2}$ (Chen, Chen, Cui, & Zhang, 2010), which indicate that using $\mathbf{P}$ as pre-conditioner will get optimal convergence rate independent of matrix size, frequency and conductivity. We use FGMRES (Saad, 1993) method to solve eq. (5), our tests show that linear solver converges in less than 25 iterations in all cases.

Inside FGMRES, we need to calculate the product of the inverse of $\mathbf{P}$ with an arbitrary vector. The product involves the solution of linear system

$$\mathbf{B}\mathbf{y} = \mathbf{c}, \quad (7)$$

which can be solved using CG method with Auxiliary Space Maxwell pre-conditioner (Hiptmair & Xu, 2007). Once the linear system eq. (4) is solved, the electromagnetic field can be computed, then we can calculate the impedance tensor as well as apparent resistivity ans phase.

Goal-oriented error estimator

For magnetotelluric method, the electromagnetic fields are only measured at receivers, therefore we only need the accurate solution at some certain points in the mesh. This is done by solving equation on a series of subsequently locally refined meshes of the initial coarse mesh. A posteriori error estimator was used to choose which cell need to be refined. Let $\mathbf{E}$

be a solution of eq. (1), error indicator $\eta^2(\mathbf{E})$ is calculated as follows:

$$\eta_K^2(\mathbf{E}) = \sum_{f=1}^{4} h_f \|\mathbf{n}_f \cdot \sigma\mathbf{E}\|^2, \tag{8}$$

where K is tetrahedral element, f is the face of K, h_f and $\mathbf{n}_f$ are the diameter and normal vector of the face f, respectively. This indicator measures the jump of current density across the face and will vanish for the true solution, thus it turns out to be meaningful error indicator to guide mesh refinement.

In order to get accurate solution at receiver locations, a dual problem with fictitious sources at the receiver locations is solved, and its solution $\mathbf{E}^D$ is used to calculate the dual error indicator $\eta_K^2(\mathbf{E}^D)$ which can be used to weight the global error indicator. The weighted error indicator is expressed as

$$\eta_K^2 = \eta_K^2(\mathbf{E})\eta_K^2(\mathbf{E}^D). \tag{9}$$

Our algorithm begins with a coarse mesh. We solve the original problem and the dual problem and use their solutions to calculate the weighted error indicator. A certain portion of the elements with the largest errors are refined according to the weighted error indicator. Equations are then solved on the refined mesh and again the elements with largest errors are refined. This procedure is repeated until the error decreases to a certain level.

Model decomposition method and implementation

The computation domain is discretized into tetrahedra using the tetrahedral mesh generator Tetgen (Si, 2015). Then the mesh need to be partitioned. Each part of the partition should have roughly the same number of elements so that the computational task can be equally divided and distributed to a number of processors. Besides, the interface between distinct partition should be as small as possible to reduce the amount of interprocess communication. Here, we use a multilevel recursive-bisection algorithm implemented in METIS (Karypis & Kumar, 1998) to partition the mesh into N_{cpu} subdomains, where N_{cpu} is the number of CPU cores to be used.

After mesh partition, the element matrices and vectors of the subdomains are calculated concurrently, and assembled into global matrix and vector. The resulting linear system is stored in parallel. We utilize the open source library PETSc which provides a large suite of parallel matrix/vector formats and linear solvers, pre-conditioners. For the auxiliary space maxwell pre-conditioner, HYPRE (Kolev & Vassilevski, 2009) library is used through PETSc interface.

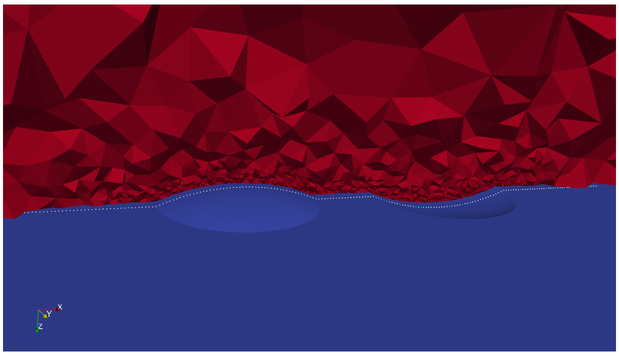

Figure 2: The topography model with a spherical hill and valley. The white dots indicate the profile along which apparent resistivity and phase values are computed.

Numerical examples

To illustrate the adaptation of the tetrahedral element and investigate the topographic effects on MT responses, we consider a model with spherical hill and valley (shown in Fig. 2). The hill is 0.5 km high and 5 km wide and the valley is 0.5 km deep and 5 km wide. The earth is a homogeneous 100 Ωm half-space.

The initial mesh we used for this model consists of 184004 tetrahedra and 214119 edges. After 10 adaptive refinements, the mesh consists of 1315800 tetrahedra and 1528654 edges, 1050284 tetrahedra and 1220800 edges, 719040 tetrahedra and 838882 edges for frequency 1.0, 0.1 and 0.01, respectively. The computation platform is a PC cluster, each node has two quad-core Intel Xeon E5530 2.4 GHz processors and 16 GB memory. The run time for the largest problem is about 103 seconds using 32 CPU cores.

Figure 3 and 4 show the apparent resistivities and phases of XY and YX mode, respectively. At the base of the hill, ρ_{xy} is higher than the background resistivity, while it is lower over the hill. In contrary, ρ_{xy} is lower than the background resistivity at the margin of the valley while it is higher over the valley. The response of YX mode is similar to XY mode, but its abnormal amplitude is slightly smaller. The results we obtained here is similar to that shown in (Nam et al., 2007).

Conclusion

We have presented a numerical scheme for 3D MT forward modeling using edge-based finite element method. The tetrahedral element is capable to to deal with complex structures and topography. To overcome the computational bottleneck, we have also implemented a model decomposition method. Since our work is still in progress, further efforts will be made to investigate the scalability of our parallel approach.

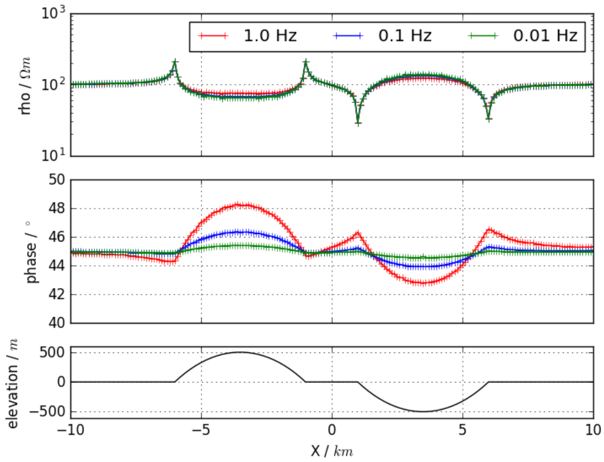

Figure 3: The apparent resistivities (top) and phases (middle) for the topography model of XY mode. The figure on bottom shows the elevation of the receivers.

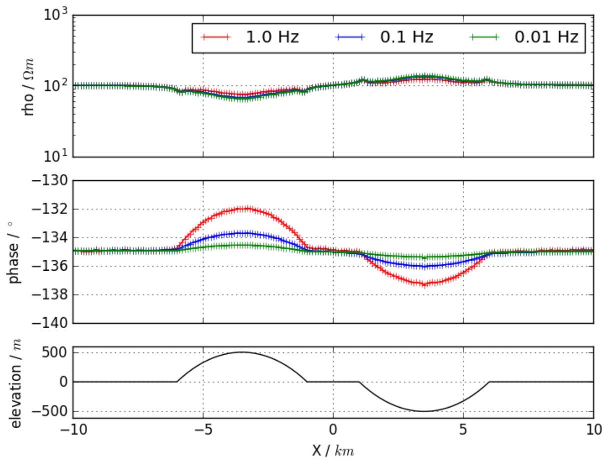

Figure 4: The apparent resistivities (top) and phases (middle) for the topography model of YX mode. The figure on bottom shows the elevation of the receivers.

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