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Research Article

Characterization of Numerical Ranges of Posinormal Operator

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Abstract

Let H be a complex Hilbert space equipped with the inner product $\langle .,. \rangle$; and let $B(H)$ be the algebra of bounded linear operators acting on H. The numerical range of a bounded linear operator A on a complex Hilbert space H is the set $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$. The numerical radius of $A \in B(H)$ is given by $r(A) = sup\{|\lambda| : \lambda \in W(A)\}$. In this paper we investigate the numerical range of an operator acting on a complex Hilbert space. In particular, we characterize the numerical range of a posinormal operator on an infinite dimensional complex Hilbert space. The present paper shows that for a posinormal operator A, W(A) is nonempty, always and is an ellipse whose foci are the eigenvalues of A.

Keywords: Numerical range; Linear operator; Posinormal operator; Hilbert Space.

Introduction

Let H be a Hilbert space equipped with the inner product $\langle .,. \rangle$; and let $B(H)$ be the algebra of bounded linear operators acting on H. The numerical range of a bounded linear operator A on a complex Hilbert space H is the set $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$ [1].

The numerical radius of $A \in B(H)$ is given by $r(A) = \sup\{|\lambda| : \lambda \in W(A)\}$. It follows that $W(A)$ is the image of the unit circle in H under the quadratic form $f(x) = \langle Ax, x\rangle$ from H to $\mathbb{C}$ and $r(A)$ is the smallest radius of a circular disk centered at the origin which contains $W(A)$ [2]. The study of numerical ranges has along and extensive history. Toeplitz was the first to introduce the notion of the numerical range [3].

The famous Toeplitz- Hausdorff theorem gives the most fundamental property of the numerical range that the numerical range of all bounded linear operators is always convex [4]. It implies that the quadratic form $f: H \to \mathbb{C}$, maps the unit circle $\|x\| = 1$ of H to a subset of $\mathbb{C}$ with all its interior filled up[5]. In this article we will concentrate in characterizing the numerical range of a posinormal operator. An operator $A \in B(H)$ is said to be posinormal if there exists a positive operator $P \in B(H)$ such that $AA^* = A^*PA$; where P is called the interrupter [6]. $P(H)$ denotes the set of all posinormal operators on H. A is said to be coposinormal if A^* is also posinormal. The general problem being considered is, given a posinormal operator $A \in B(H)$ what can be said about its numerical range $W(A)$? [7].

Preliminaries

In the present section, some basic properties of numerical ranges of operators which are to be used in later discussions are given.

Proposition 2. 1: Let $A, B \in B(H)$. Then by [8]

1. $W(A^*) = \overline{W(A)}$
2. $W(A)$ contains all of the eigenvalues of A.
3. $W(A)$ is contained in the closed disk of radius $\|A\|$ around the origin.
4. If $a, b \in \mathbb{C}$ then $W(\alpha A + bI_H) = \alpha W(A) + b$.
5. If $U \in B(H)$ is unitary then $W(UAU^*) = W(A)$.
6. $W(A) \subseteq \mathbb{R}$ if and only if A is self-adjoint.
7. $W(A + B) \subseteq W(A) + W(B)$.

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Theorem 2.2: For $A \in M_2(\mathbb{C})$, either

i. If $A = \lambda I_2$, then $W(A) = \{\lambda\}$,

ii. If the eigenvalues of A are equal and A is not a multiple of identity, $W(A)$ is a non-trivial closed disk centered at the eigenvalues of A.

Theorem 2.3: Let $A \in M_2(\mathbb{C})$ have trace zero. Then A is unitarily equivalent to a matrix whose diagonal entries are all zero.

Theorem 2.4: (The Folk Theorem) Let $A \in B(H)$ be such that $\lambda \in \partial W(A)$. If no closed disk of $W(A)$ contains λ, then λ is an eigenvalue of A.

Proposition 2.5: For operators A and B on spaces H and K respectively, the following hold:

i. W(A) is nonempty bounded convex subset of $\mathbb{C}$. If H is finite dimensional, then W(A) is even compact.

ii. $W(aA + bI) = \alpha W(A) + b$ for any complex a and b.

iii. $W(U^*AU) = W(A)$ for any unitary operator U on H.

iv. If A is unitarily equivalent to an operator of the form $\begin{pmatrix} B & * \\ * & * \end{pmatrix}$ (that is, A is a dialation of B or B is a compression of A), then $W(B) \subseteq W(A)$.

v. A complex number z is in W(A) if and only if A is unitarily equivalent to an operator of of the form $\begin{pmatrix} z & * \\ * & * \end{pmatrix}$

vi. $\frac{\|A\|}{2} \leq r(A) \leq \|A\|$.

Methodology

For a successful completion of this paper, background knowledge of Functional analysis, the operator theory, especially normal operators, self adjoint operators, hyponormal operators on a Hilbert space, numerical range and the spectrum of operators on a Hilbert space is vital. We have stated some known fundamental principles which shall be useful in our research. The methodology involved the use of known inequalities and techniques like Cauchy - Schwartz inequality, Minkowski's inequality, parallelogram law and the polarization identity. Lastly, we used the technical approach of tensor products in solving the stated problem. Also the methodology involved the use of known inequalities and techniques like the polarization identity [9]. We have also used the technical approach of derivatives of fuzzy sets to characterize the numerical range of posinormal operators [10]. Spectral theory of linear operators on Hilbert spaces is a pillar in several developments in mathematics, physics and quantum mechanics. Its concepts like the spectrum of a linear operator, eigenvalues and vectors, spectral radius, spectral integrals among others have useful applications in quantum mechanics, a reason why there is a lot of current research on these concepts and their generalizations. Spectral theory is described as a rich and important theory as it relates perfectly with other areas including measure and integration theory and theory of analytic functions.

Results and discussions

Lemma 4.1: Let H be a complex Hilbert space and B(H) the algebra of all bounded linear operators on H. Let $A \in B(H)$ be posinormal then W(A) is an ellipse whose foci are the eigenvalues of A [11].

Proof. Choose A such that $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with λ_1 and λ_2 as the eigenvalues of A. Now if $\lambda_1 = \lambda_2 = \lambda$, we have

$A - \lambda I = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$.

Let $x = (x_1, x_2)$, then

$(A - \lambda I)x = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{x_2} \\ 0 \end{pmatrix} = a \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$.

Therefore,

$\|A - \lambda I\| = \sup\{\|a(x_2, 0)\| : |x_1|^2 + |x_2|^2 = 1\} = |a|$

Hence the radius is $\frac{1}{2}|a|$. Therefore the numerical range $W(A) = \left\{z : |z| \leq \frac{|a|}{2}\right\}$. It thus follows that $W(A)$ is a circle with the center at λ and radius $\frac{|a|}{2}$. [12]

Now if $\lambda_1 \neq \lambda_2$ and $a = 0$ we have $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. If $x = (x_1, x_2)$, then

$Ax = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{pmatrix}$.

Therefore taking the inner product $\langle Ax, x \rangle$ we get

$\langle Ax, x \rangle$

$= (\lambda_1 x_1 \quad \lambda_2 x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\lambda_1 x_1 \bar{x}_1 + \lambda_2 x_2 \bar{x}_2) = (\lambda_1 |x_1|^2 + \lambda_2 |x_2|^2)$

So $\langle Ax, x \rangle = \lambda_1 |x_1|^2 + \lambda_2 |x_2|^2$.

Now letting $t = |x_1|^2$, we therefore write the above equation as follows.

$\langle Ax, x\rangle = t\lambda_1 + (1-t)\lambda_2$, since

$|x_1|^2 + |x_2|^2 = 1$. So W(A) is the set of convex combinations of λ_1 and λ_2 and is the segment joining them. If $\lambda_1 \neq \lambda_2$ and $a \neq 0$ we choose λ such that it lies between λ_1 and λ_2. We therefore have,

$$A - \frac{\lambda_1+\lambda_2}{2} I = \begin{pmatrix} \frac{\lambda_1-\lambda_2}{2} & a \\ 0 & \frac{\lambda_2-\lambda_1}{2} \end{pmatrix}$$

In this case, we let $z = re^{-i\theta}$, $\frac{\lambda_1-\lambda_2}{2} = re^{-i\theta}$ and $\frac{\lambda_2-\lambda_1}{2} = -re^{-i\theta}$ so,

$$e^{-i\theta}\left(A - \frac{\lambda_1+\lambda_2}{2}\right) = \begin{pmatrix} \frac{\lambda_1-\lambda_2}{2} & a \\ 0 & \frac{\lambda_2-\lambda_1}{2} \end{pmatrix} = A'.$$

Here we see that W(A') is an ellipse with the center at (0,0) and the minor axis $|a|$, and foci at (r,0) and (-r,0). Thus, W(A) is an ellipse with foci at λ_1 and λ_2 and the major axis has an inclination of θ with the real axis.

The following example, illustrates on how to calculate the numerical range of any given operator A on a Hilbert space H.

Example 4.2: In $\mathbb{C}^2$ let A be the operator defined by matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Take $x \in \mathbb{C}^2$, $x \in (f,g)$, $\|x\|^2 = |f|^2 + |g|^2 = 1$ with $\|x\| = 1$ $Ax = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}$ and $\langle Ax, x\rangle = (g,0)\begin{pmatrix} f \\ g \end{pmatrix} = g\bar{f}$.

Taking the absolute values on both sides we have

$$|\langle Ax, x\rangle = |f||g|| = \frac{1}{2}(|f|^2 + |g|^2) = \frac{1}{2}$$

So $W(A) \subset \left\{z\colon |z| \leq \frac{1}{2}\right\}$, a circle of radius $\frac{1}{2}$ centered at $(0,0)$.

Alternatively, given the operator A defined by matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

We then have the characteristic polynomial given by

$$A - \lambda I = \begin{pmatrix} 0-\lambda & 1 \\ 0 & 0-\lambda \end{pmatrix}$$

and hence finding the characteristic equation we see that $\lambda^2 = 0$. Therefore, $\lambda^2 = 0$ is the eigenvalue. Since for the norm we have $\frac{1}{2}\|A\|$ and therefore normalizing the vector x we see that $\left\|\frac{x}{\|x\|}\right\| = 1$.

Now we have $A(f,g) = (g,0)$. That is[14]

$$Ax = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}.$$

This implies that $\|(f,g)\| = \|(g,0)\| = \|g\|$.

From the definition of an operator norm,

$\|A\| = \sup\{\|A(f,g)\| \colon \|(f,g)\| = 1\}$

$= \sup\{\|A(f,g)\| \colon \sqrt{f^2+g^2} = 1\}$

$= \sup\{\|g\| \colon f^2 + g^2 = 1\}$

$= 1.$

Therefore, $\frac{1}{2}\|A\| = \frac{1}{2}(1) = \frac{1}{2}$.

Hence, W(A) is a circle of radius $\frac{1}{2}$ centered at (0,0).

Lemma 4.3: Let H be a complex Hilbert space and B(H) the algebra of all bounded linear operators on H. Let $A \in B(H)$ be posinormal then W(A) is nonempty.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be orthonormal sequence of vectors in H. For $\{x_n\}_{n=1}^{\infty}$ to exist in H then

$\log_{n\to\infty}\langle Ax_n, x_n\rangle = a$

The sequence $\{\langle Ax_n, x_n\rangle\}_{n=1}^{\infty}$ is bounded and $\|x\| = 1$ because x_n has norm 1. Now, using $A=A^*$ (because all posinormal operators are self-adjoint and positive) we have

$\langle Ax_n - \|A\|x_n\rangle\langle Ax_n - \|A\|x_n\rangle = \langle Ax_n, Ax_n\rangle - \langle Ax_n, \|A\|x_n\rangle - \langle \|A\|x_n, Ax_n\rangle + \langle \|A\|x_n, \|A\|x_n\rangle$

$= \|Ax_n\|^2 - 2\|A\|\langle Ax_n, x_n\rangle + \|A\|^2\|x_n\|^2$

$\leq \|A\|^2\|x_n\|^2 - 2\|A\|\langle Ax_n, x_n\rangle + \|A\|^2\|x_n\|^2$

$= 2\|A\|^2\|x_n\|^2 - 2\|A\|\langle Ax_n, x_n\rangle$

$\Rightarrow 2\|A\|^2\|x_n\|^2 - 2\|A\|^2\|x_n\|^2$

$= 0$

Therefore, as $n\to\infty$ the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to 0 in H such that $\log_{n\to\infty}\langle Ax_n, x_n\rangle = a$. Thus x is an eigenvector for the eigenvalue $\|A\|$. This implies that W (A) is nonempty.

The next result due to Toeplitz and Hausdorff shows that the numerical range of all bounded linear operators acting on the Hilbert space H is always convex. We give its proof for completion.

Theorem 4.4: Let H be a complex Hilbert space and $A \in B(H)$ be posinormal, then W (A) is always convex.

Proof. Let $\lambda_1, \lambda_2 \in W(A)$, $\lambda_1 = \lambda_2$. We prove that $(1-t)\lambda_1 + t\lambda_2 \in W(A)$ whenever $t \in [0,1]$. If $B = \alpha I + \beta A$, where $\alpha, \beta \in \mathbb{C}$ are such that $0 = \alpha + \beta\lambda_1$ and $1 = \alpha + \beta\lambda_2$ it is sufficient to show that $t \in W(B)$for all $t \in [0,1]$. Let us fix unit vectors $x, y \in H$ such that $0 = (Bx|x)$,

$1 = (By|y)$ and define $g: \mathbb{R} \to \mathbb{C}$ by $g(t) = (Bx|y)e^{-it} + (By|x)e^{it}$, $t \in \mathbb{R}$.
Moreover, there exists $t_0 \in [0, \pi]$ such that $\text{Im}\, g(t_0) = 0$. Since $\text{Im}\, g(0) = -\text{Im}\, g(t_0) = 0$.
Now observe that the vectors x and y are linearly independent. Otherwise $x = \alpha\hat{y}$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and
$0 = (Bx|x) = |\alpha|^2(B\hat{y} - \hat{y}) = (By|y) = 1.$
Define continuous functions z and f by
$z(b) = \frac{(1-b)x+by}{\|((1-b)x+by)\|}$, $t \in [0,1]$ and
$f(b) = (Bz(b)z|z(b))$, $b \in [0,1]$.
A straight forward calculation shows that f is real-valued function with $f(0) = 0$ and $f(1) = 1$.
Implying that
$t \in [0,1] \subset f([0,1]) \subset [0,1] \subset W(B)$ as required.

Corollary 4.5: Let $A \in B(H)$ be posinormal then $0 \in W(A)$.

Proof. Since A is bounded, then every eigenvalue of A that lies on the boundary of W(A) is a normal eigenvalue. An eigenvalue λ is said to be normal for an operator A if
$\text{Ker}(A - \lambda I) = \text{Ker}(A^* - \bar{\lambda} I)$
Let us assume without loss of generality that $\lambda = 0$. Suppose there is a unit vector f for which $Af = 0$ but $A^*f \neq 0$. Let $g = \frac{A^*f}{\|A^*f\|}$.
Because $\langle f, A^*f\rangle = \langle Af, f\rangle = \langle 0, f\rangle = 0$ the pair (f, g) is orthonormal in H, and therefore spans a two dimensional subspace M. It follows that W(A) contains the numerical range of the compression A_M of A to M. It is enough to show that 0 is in the interior of $W(A_M)$. Now the matrix of A_M with respect to the orthonormal basis (f, g) of M is of the form $\begin{pmatrix} 0 & a \\ 0 & * \end{pmatrix}$, where $a = \langle A_M g, f\rangle$. We need to show that $a \neq 0$, this will establish $W(A_M)$ as a non-degenerate elliptical disk with one focus at 0, and therefore complete the proof. Now,
$a = \langle A_M g, f\rangle = \langle PAg, f\rangle = \langle Ag, f\rangle = \langle g, A^*f\rangle$,
Where the term on the right, upon recalling that $g = g = \frac{A^*f}{\|A^*f\|}$, is just
$$\frac{\langle A^*f, A^*f\rangle}{\|A^*f\|} = \|A^*f\| \neq 0$$
as desired.

Conclusions

Hilbert space operators have been studied by many mathematicians including Hilbert, Weyl, Neumann, Toeplitz, Hausdorff, Rhaly, Mecheri, Shapiro among others. These operators [15] are of great importance since they are vital in formulation of principles of mathematical analysis and quantum mechanics. The operators include normal operators, posinormal operators, hyponormal operators, normaloid operators among others. Posinormal operators have not been exhaustively characterized despite the fact that they possess interesting properties. Certain properties of posinormal operators have been characterized for example continuity and linearity but numerical ranges and spectra of posinormal operators have not been considered. Also the relationship between the numerical range and spectrum has not been determined for posinormal operators. Therefore the objectives of this study were: to investigate numerical ranges of posinormal operator, to investigate the spectra of posinormal operators and to establish the relationship between the numerical range and spectrum of a posinormal operator. The methodology involved use of known inequalities like Cauchy- Schwartz inequality, Minkowski's inequality, the parallelogram law and the polarization identity to determine the numerical range and spectrum of posinormal operators and our technical approach shall involve use of tensor products. The results obtained shall be used in classification of Hilbert space operators and shall be applied in other fields like quantum information theory to optimize minimal output entropy of quantum channel; to detect entanglement using positive maps ;and for local distinguishability of unitary operators.The numerical range of a bounded posinormal operator A acting on a complex Hilbert space H is an ellipse whose foci are the eigenvalues of A. For a posinormal operator $A \in B(H)$, the following properties hold: The numerical range of A is nonempty, i.e $W(A) \neq \phi$; $W(A)$ is always convex; Zero is contained in the numerical range of A, i.e $0 \in W(A)$. The norm of A is a subset of the closure of W(A), $\|A\| = \overline{W(A)}$. The numerical radius of A is equal to the norm of A, i.e $r(A) = \|A\|$.

Conflict of interest

Authors declare there are no conflicts of interest.

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