Math 4315 - PDE's

1st order Linear PDE

$$a u_x + b u_y = cu + d$$

and so BC / IC.

If $u_s = u_x x_s + u_y y_s$

pick $x_s = a$, $y_s = b$ so $u_s = cu + d$

Now we let a, b, c, d be functions of (x, y)

Then if $a(x,y) u_x + b(x,y) u_y = c(x,y) u + d(x,y)$

then still

$$\left.\begin{aligned} x_s &= a(x,y) \\ y_s &= b(x,y) \\ u_s &= c(x,y) u + d(x,y) \end{aligned}\right\} \text{ these we solve!}$$

ex! $\quad x u_x - 2y u_y = u$

if $\quad u_s = u_x x_s + u_y y_s$

pick $\quad x_s = x$

$\qquad\quad y_s = -2y$

$\qquad\quad u_s = u$

In general if $\dfrac{dy}{dx} = ky$ then $y = c e^{kx}$

so

$$
\left.
\begin{aligned}
x_s = x &\Rightarrow x = a(r) e^{s} \\
y_s = -2y &\Rightarrow y = b(r) e^{-2s} \\
u_s = u &\Rightarrow u = c(r) e^{s}
\end{aligned}
\right\}
$$

now get rid of s

(1) $\left.\begin{aligned} x^2 &= a^2(r) e^{2s} \\ y &= b(r) e^{-2s} \end{aligned}\right] \quad x^2 y = a^2(r) b(r)\, e^{2s} \cdot e^{-2s}$

$$= A(r)$$

(2) $\quad \dfrac{u}{x} = \dfrac{a(r) \cancel{e^{s}}}{c(r) \cancel{e^{s}}} = B(r)$

$r = A^{-1}(x^2 y) \Rightarrow \dfrac{u}{x} = B(A^{-1}(x^2 y)) \Rightarrow u = x f(x^2 y)$

suppose B.C. $u(x,1) = x^3 + x$

then $u(x,1) = x f(x^2,1) = x^3 + x$

$\Rightarrow \quad f(x^2) = x^2 + 1$

$\Rightarrow \quad f(\lambda) = \lambda + 1$ $\Leftarrow$ now we know the form of f

Sol^n $\quad u = x[x^2 y + 1]$

$= x^3 y + x$

$u_x = 3x^2 y + 1$

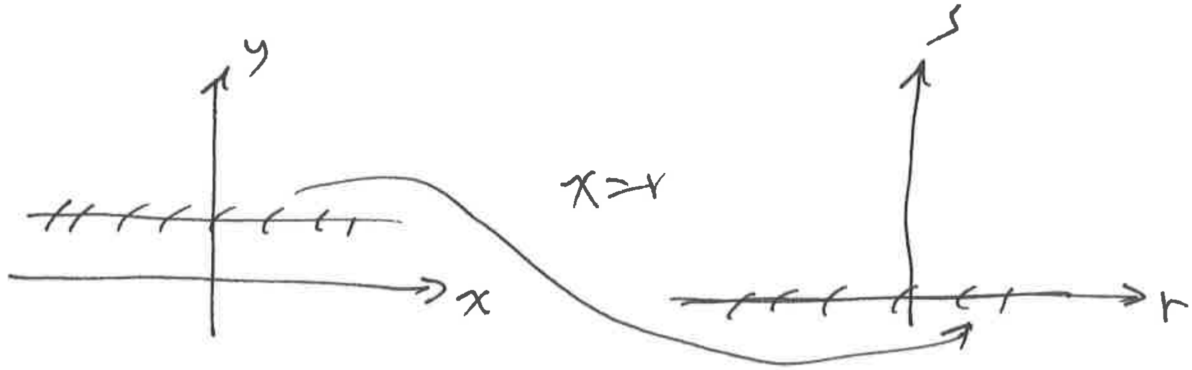
$u_y = x^3$

sub $\quad x u_x - 2y u_y = x(3x^2 y + 1) - 2y \cdot x^3$

$= x^3 y + x$

$= u$ ✓

Now pass the boundary condition to the (r, s) plane



so solve

$x_s = x$

$y_s = -2y$

$u_s = u$

subject to

$s = 0$

$x = r$

$y = 1$

$u = r^3 + r$

so $x_s = x \Rightarrow x = a(r)e^{s}$ $\quad s=0 \;\; x = r \Rightarrow a(r) = r$

$$\boxed{x = re^{s}}$$

$y_s = -2y \Rightarrow y = b(r)e^{-2s}$ $\quad s = 0 \;\; y = 1 \Rightarrow b(r) = 1$

$$\boxed{y = e^{-2s}}$$

$u_s = u \Rightarrow u = c(r)e^{s}$ $\quad s = 0 \;\; u = r^3 + r \Rightarrow c(r) = r^3 + r$

$\Rightarrow u = r^3 e^{s} + re^{s}$

$= (re^{s})^3 e^{-2s} + re^{s} = x^3 y + x$ same

Ex 2 $\quad u_x + (x-y)u_y = 1 \qquad u(x,x) = 0$

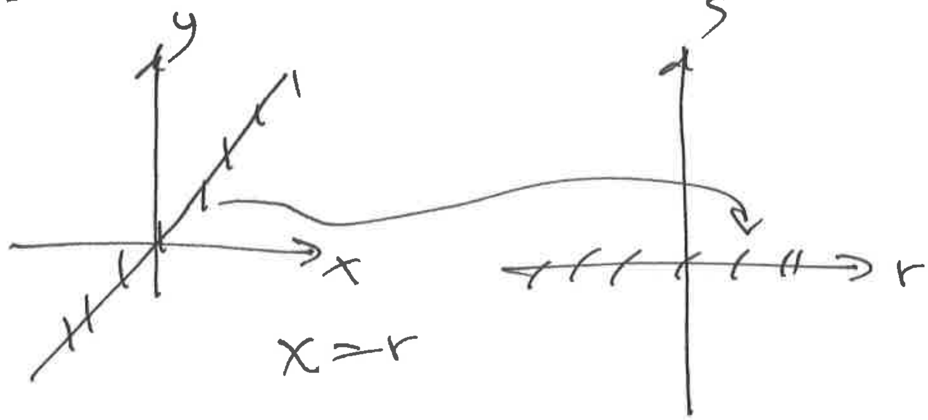
if $u_s = u_x x_s + u_y y_s$

pick

$x_s = 1$

$y_s = x - y$

$u_s = 1$



pick new boundary $\quad s = 0$

$x = r, \quad y = x = r, \quad u = 0$

(1) $\quad x_s = 1 \Rightarrow x = s + a(r) \qquad s = 0,\ x = r \Rightarrow a(r) = r$

so $\boxed{x = s + r}$

(2) $\quad u_s = 1 \Rightarrow u = s + b(r) \quad s = 0,\ u = 0 \Rightarrow b(r) = 0$

so $\boxed{u = s}$

(3) $\quad y_s = x - y \Rightarrow \dfrac{dy}{ds} + y = (s + r) \qquad \mu = e^{s}$

$$\frac{\partial}{\partial s}\left(e^{s} y\right) = s e^{s} + r e^{s}$$

$$e^{s} y = (s-1)e^{s} + r e^{s} + c(r)$$

so we use $s=0,\ y=r$ now

$$\cancel{r} = -1 + \cancel{r} + C(r) \Rightarrow C(r) = 1$$

$$e^{s} y = (s-1)e^{s} + re^{s} + 1$$

$$\boxed{y = s - 1 + r + e^{-s}}$$

so $x = s + r$

$y = s - 1 + r + e^{-s} = s + r - 1 + e^{-s}$

$u = s$

so $y = x - 1 + e^{-s}$

$\Rightarrow \quad y - x + 1 = e^{-s} = e^{-u}$

$$u = -\ln|y - x + 1|$$