Calculus II Sample Final Solutions

1. (i) $\displaystyle\int \sin^2 x\, \cos^3 x\, dx$

If
$$\int \sin^2 x\, \cos^2 x\, \cos x\, dx$$
let $u = \sin x$ so $du = \cos x\, dx$. Therefore
$$\int u^2(1-u^2)\, du = \int u^2 - u^4 du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + c = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c.$$

1. (ii) $\displaystyle\int x \ln x\, dx$

If
$$u = \ln x \qquad v = \frac{x^2}{2}$$
$$du = \frac{dx}{x} \qquad dv = x\, dx,$$
so $\int x \ln\, x\, dx = \frac{x^2}{2}\, \ln x - \int \underbrace{\frac{x^2}{2}\frac{1}{x}}_{=x/2}\, dx = \frac{x^2}{2}\, \ln x - \frac{x^2}{4} + c.$

1.(iii) $\displaystyle\int x \sin 2x\, dx$

$$u = x \qquad v = -\frac{1}{2}\cos 2x$$
$$du = dx \;\; dv = \sin 2x\, dx,$$
so $\displaystyle\int x \sin 2x\, dx = -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x\, dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c.$

1.(iv) $\displaystyle\int \frac{dx}{x^2+3x+2} = \int \frac{dx}{(x+1)(x+2)}$

First we use partial fractions
$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \quad \Rightarrow \quad A(x+2) + B(x+1) = 1$$

so

$$A + B = 0, \quad 2A + B = 1 \quad \Rightarrow \quad A = 1, \quad B = -1.$$

Therefore

$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx = \ln|x+1| - \ln|x+2| + c.$$

1. (v) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

If $x = \sin\theta$ then $dx = \cos\theta\, d\theta$. For the limits $x = 0 \;\Rightarrow\; \theta = 0$, $x = 1/2 \;\Rightarrow\; \theta = \pi/6$. On substitution, we get

$$\int_0^{\pi/6} \frac{\sin\theta \cancel{\cos\theta} d\,\theta}{\cancel{\cos\theta}} = \int_0^{\pi/6} \sin\theta d\,\theta = -\cos\theta\big|_0^{\pi/6} = -\frac{\sqrt{3}}{2} + 1.$$

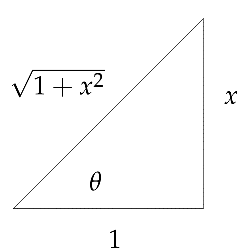
1. (vi) $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$

$$\lim_{b\to\infty} \int_0^b xe^{-x^2}\,dx = \lim_{b\to\infty} -\frac{1}{2}\,e^{-x^2}\Big|_0^b = \lim_{b\to\infty} \frac{1}{2} - \frac{1}{2}e^{-b^2} = \frac{1}{2}.$$

1. (vii) $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$

If $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$ so

$$\int \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int \cos\theta\, d\theta = \sin\theta + c = \frac{x}{\sqrt{1+x^2}} + c.$$



1. (viii) $\displaystyle\int \frac{x\,dx}{(x-1)(x-2)^2}$

First we use partial fractions

$$\frac{x}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \quad \Rightarrow$$

$$A(x-2)^2 + B(x-1)(x-2) + C(x-1) = x$$

expanding and isolating coefficients of x gives

$$4A + 2B - C = 0, \quad -4A - 3B + C = 1, \quad A + B = 0,$$

$$\Rightarrow \quad A = 1, \quad B = -1 \quad C = 2.$$

Therefore

$$\begin{aligned} \int \frac{dx}{(x-1)(x-2)^2} &= \int \frac{1}{x-1}\,dx - \int \frac{1}{x-2}\,dx + \int \frac{2}{(x-2)^2}\,dx \\ &= \ln|x-1| - \ln|x-2| - \frac{2}{x-2} + c. \end{aligned}$$

1. (ix) $\displaystyle\int \frac{dx}{x\,(x^2+1)}$

First we use partial fractions

$$\frac{1}{x\,(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad \Rightarrow$$

$$A\left(x^2+1\right) + (Bx+C)x = 1$$

expanding and isolating coefficients of x gives

$$A + B = 0, \quad C = 0, \quad A = 1,$$

$$\Rightarrow \quad A = 1, \quad B = -1 \quad C = 0.$$

Therefore

$$\int \frac{dx}{x\,(x^2+1)} = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + c.$$

1. (x) $\displaystyle\int x\,e^{-3x}\,dx$

If

$$u = x \qquad v = -\frac{1}{3}e^{-3x}$$

$$du = dx \qquad dv = e^{-3x}dx$$

so $\displaystyle\int x\,e^{-3x}\,dx = -\frac{1}{3}x\,e^{-3x} - \int -\frac{1}{3}e^{-3x}dx = -\frac{1}{3}x\,e^{-3x} - \frac{1}{9}e^{-3x} + c.$

1. (xi) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

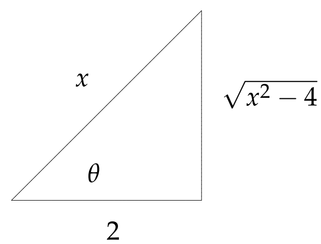
This is an improper integral!

$$\lim_{b\to 1}\int_0^b \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1} \sin^{-1} x\Big|_0^b = \lim_{b\to 1}\sin^{-1} b = \frac{\pi}{2}.$$

1. (xii) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$

If $x = 2\sec\theta$, then $dx = 2\sec\theta\ \tan\theta\, d\theta$ so

$$\int \frac{\cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta}{4\sec^2\theta\,\cancel{2}\cancel{\tan\theta}} = \frac{1}{4}\int \cos\theta\, d\theta = \frac{1}{4}\sin\theta + c = \frac{1}{4}\frac{\sqrt{x^2-4}}{x} + c.$$



2. Do the following converge

2.(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

Compare with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} \Big/ \frac{1}{n} = \lim_{n\to\infty}\frac{n^3}{n^3+1} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic) then by the limit comparison test (LCT), the original series diverges.

2.(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{3^n}$

If $a_n = \dfrac{n^2}{3^n}$, then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^2}{3^{n+1}} \Big/ \frac{n^2}{3^n} = \lim_{n\to\infty}\frac{(n+1)^2}{3n^2} = \frac{1}{3} < 1,$$

then by the ratio test, the original series converges.

2.(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

We first consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$. Since this converges ($p = 3$), the the original series converges absolutely.

2.(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

We first consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ and compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{1}{n^2+1}./\frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1,$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p = 2$), then by direct comparison $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges so the original series converges absolutely.

2.(v) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

Since

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0,$$

then by the n^{th} term test, the series diverges.

2.(vi) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

Let $f(x) = \frac{1}{x \ln x}$. Clearly $f(x) > 0$ and $f'(x) = -\dfrac{\ln x + 1}{(x \ln x)^2}$ for $x \geq 3$ showing that $f(x)$ is decreasing so that the integral test may be used. Consider

$$\int_3^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_3^b \frac{dx}{x \ln x} = \lim_{b\to\infty} \ln \ln x|_3^b = \infty.$$

Since the integral diverges, then by the integral test, the series does as well.

2.(vii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

We first consider $\sum_{n=1}^{\infty} \frac{1}{n+1}$. Since this diverges (it's harmonic), we then check the two conditions for conditional convergence. If we let $a_n = \frac{1}{n+1}$, then clearly

$$\lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Next, we need to show $a_{n+1} < a_n$. To do so we let

$$f(x) = \frac{1}{x+1} \quad \text{then} \quad f'(x) = -\frac{1}{(x+1)^2} < 0$$

so by the alternating series test (AST), the series converges conditionally.

2.$(viii)$ $\quad \sum_{n=1}^{\infty} \frac{2^n}{n!}$

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} / \frac{2^n}{n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$ so by ratio test, the series converges.

2.(ix) $\quad \sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n$

Since

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2} < 1,$$

then by the root test, the series converges.

3. Calculate the 4^{th} degree Taylor polynomial with remainder for the following. Expand about the point c that is given.

3.(i) $\quad f(x) = \sin x$ about $x = \pi$.

$$\begin{aligned} f(x) &= \sin x & f(\pi) &= 0, \\ f'(x) &= \cos x & f'(\pi) &= -1, \\ f''(x) &= -\sin x & f''(\pi) &= 0, \\ f'''(x) &= -\cos x & f'''(\pi) &= 1, \\ f^{(4)}(x) &= \sin x & f^{(4)}(\pi) &= 0, \\ f^{(5)}(x) &= \cos x \;\; (R) & & \end{aligned}$$

The Taylor polynomial is

$$P_4(x) = -(x-\pi) + \frac{1}{3!}(x-\pi)^3$$

and the remainder is

$$R_4(x) = \cos c\, \frac{(x-\pi)^4}{4!}$$

where c is between π and x.

3.(ii) $f(x) = \dfrac{1}{x+2}$ about $x = 1$.

$$\begin{aligned}
f(x) &= \frac{1}{x+2} & f(1) &= \frac{1}{3}, \\
f'(x) &= \frac{-1}{(x+2)^2} & f'(1) &= -\frac{1}{3^2}, \\
f''(x) &= \frac{2}{(x+2)^3} & f''(1) &= \frac{2}{3^3}, \\
f'''(x) &= \frac{-3!}{(x+2)^4} & f'''(1) &= -\frac{3!}{3^4}, \\
f^{(4)}(x) &= \frac{4!}{(x+2)^5} & f^{(4)}(1) &= \frac{4!}{3^5}, \\
f^{(5)}(x) &= \frac{-5!}{(x+2)^6} \quad (R)
\end{aligned}$$

The Taylor polynomial is

$$\begin{aligned}
P_4(x) &= \frac{1}{3} - \frac{1}{3^2}\frac{(x-1)}{1!} + \frac{2!}{3^3}\frac{(x-1)^2}{2!} - \frac{3!}{3^4}\frac{(x-1)^3}{3!} + \frac{4!}{3^5}\frac{(x-1)^4}{4!} \\
&= \frac{1}{3} - \frac{(x-1)}{3^2} + \frac{(x-1)^2}{3^3} - \frac{(x-1)^3}{3^4} + \frac{(x-1)^4}{3^5}.
\end{aligned}$$

The remainder is

$$R_4(x) = \frac{-5!}{(c+2)^6}\,\frac{(x-1)^5}{5!}$$

where c is between 1 and x.

4. Determine the interval of convergence of the following.

4 (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{(n+1)!}$,

Choosing

$$u_n = \frac{4^n x^n}{(n+1)!}$$

then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}x^{n+1}}{(n+2)!}\Big/\frac{4^n x^n}{(n+1)!}\right| = \lim_{n\to\infty}\frac{4}{n+2}|x| = 0 < 1$$

so the series converges for all x

4 (ii) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n\,3^n}$,

Choosing

$$u_n = \frac{(x-2)^n}{n\,3^n}$$

then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)3^{n+1}}\Big/\frac{(x-2)^n}{n3^n}\right| = \lim_{n\to\infty}\frac{|x-2|\,n}{3(n+1)} = \frac{|x-2|}{3} < 1$$

So $|x-2| < 3$ or $-1 < x < 5$. Checking the endpoints gives

$$x = -1 \quad \sum_{n=1}^{\infty}\frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \quad \text{which converges by AST}$$

$$x = 5 \quad \sum_{n=1}^{\infty}\frac{3^n}{n3^n} = \sum_{n=1}^{\infty}\frac{1}{n} \quad \text{which diverges - harmonic}$$

Therefore the interval of convergence is $-1 \le x < 5$.

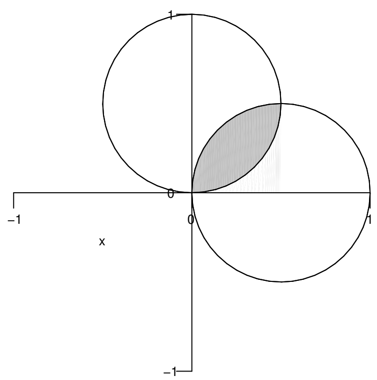
6. Polar Areas

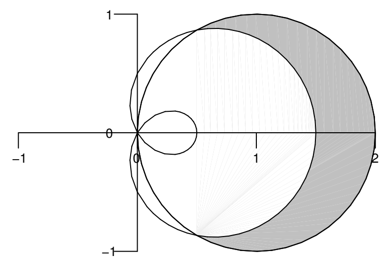
(i) The curves intersect at $\theta = \pi/4$ (set $\sin\theta = \cos\theta$ and solve for θ). We will use symmetry and the curve $r = \sin\theta$. Therefore, the area is

$$\frac{2}{2}\int_0^{\pi/4}\sin^2\theta\,d\theta = \frac{\pi}{8} - \frac{1}{4}$$

(ii) Setting the curves equal to each other give $2\cos\theta = 1/2 + \cos\theta$ or $\cos\theta = 1/2$ giving $\theta = \pi/3$. Again, we use symmetry. The area is given by

$$\frac{2}{2}\int_0^{\pi/3}(2\cos\theta)^2 - \left(\frac{1}{2} + \cos\theta\right)^2 d\theta = \frac{5\pi}{12} - \frac{\sqrt{3}}{8}$$





8. Planes and Lines

(i) As the plane contains the lines

$$x = -1 + t, \quad x = 2 - s$$
$$y = 1 + t, \quad y = s$$
$$z = 2t, \quad z = 2$$

then it contains the two vectors

$$\overrightarrow{u} =< 1, 1, 2 >, \quad \overrightarrow{v} =< -1, 1, 0 > .$$

Next, cross the two vectors

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = < -2, -2, 2 > .$$

Now pick a point on any line. If $s = 0$ then a point is $(2, 0, 2)$ and the equation of the plane is given by

$$-2(x - 2) - 2(y - 0) + 2(z - 2) = 0.$$

(ii) If the points are $P(1, 1, 3)$, $Q(-2, 4, -3)$ and $R(3, -4, 4)$ then we construct two vectors, $\overrightarrow{PQ} =< -3, 3, -6 >$ and $\overrightarrow{PR} =< 2, -5, 1 >$. The cross product will give the normal

$$\overrightarrow{n} = \begin{vmatrix} i & j & k \\ -3 & 3 & -6 \\ 2 & -5 & 1 \end{vmatrix} = < -27, -9, 9 > .$$

The equation of the plane is given by

$$3(x - 1) + (y - 1) - (z - 3) = 0.$$

(iii) As the line follow the normal, then its direction is in $< 3, 1, -1 >$ and the equation of the line through $(1, 2, 3)$ is

$$x = 1 + 3t, \quad y = 1 + t, \quad z = 3 - t.$$

(iv) As the line follow the vector $\vec{PQ}$ then its direction is $< -3, 3, -6 >$ and the equation of the line through $(1, 1, 3)$ is

$$x = 1 - 3t, \quad y = 1 + 3t, \quad z = 3 - 6t.$$

9. Vector Projections

(i) If

$$\vec{u} =< -1, 3 >, \quad \vec{v} =< 2, 2 >,$$

then

$$\vec{u} \cdot \vec{v} = -2 + 6 = 4, \quad \vec{v} \cdot \vec{v} = 4 + 4 = 8,$$
$$\text{proj}_{\vec{v}} \, \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v} = \frac{4}{8} < 2, 2 > = < 1, 1 >$$

The orthogonal complement is given by

$$\text{ortho}_{\vec{v}} \, \vec{u} = \vec{u} - \text{proj}_{\vec{v}} \, \vec{u} =< -1, 3 > - < 1, 1 > = < -2, 2 > .$$

9. (ii) If

$$\vec{u} =< 5, 5 >, \quad \vec{v} =< 1, 2 >,$$

then

$$\vec{u} \cdot \vec{v} = 5 + 10 = 15, \quad \vec{v} \cdot \vec{v} = 1 + 4 = 5,$$
$$\text{proj}_{\vec{v}} \, \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v} = \frac{15}{5} < 1, 2 >=< 3, 6 >$$

The orthogonal complement is given by

$$\text{ortho}_{\vec{v}} \, \vec{u} = \vec{u} - \text{proj}_{\vec{v}} \, \vec{u} =< 5, 5 > - < 3, 6 > = < 2, -1 > .$$