Matrix Games with Atanassov's Intuitionistic Fuzzy Variables

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Abstract In real life situations, the decision makers' outlook is more precisely expressed in terms of membership and non-membership expressions. Therefore, intuitionistic fuzzy terms are best suited to deal with such information to avoid any loss of information. This paper aims to investigate the matrix games with the payoffs matrix represented by Atanassov's intuitionistic fuzzy terms (IFTs). The motive behind this is to study extensively the important properties of such games and to establish the mathematical programming methodology of the IVI2TLT games. The methodology proposed develops the solution concept following Inuiguchi et al. [18] approach, to solve such games; the crisp equivalent problems of respective players' are free from binary variables. It has been established that solving such a fuzzy game is equivalent to solving a pair of (crisp) multi-objective linear programming problems using an indeterminacy function for each goal. Finally, the mathematical models are reduced to nonlinear programming problems to acquire the optimal strategies for the players. An example is illustrated to validate and applicability of the proposed technique.

Keywords intuitionistic fuzzy set · matrix games · multi-objective linear programming problems · non-linear programming problems

1 Introduction

The matrix games with single Intuitionistic fuzzy goal of each of the two players, has been studied earlier [1,14,20]. However, we can have a game theoretic model of a real problem with multiple objectives (like, costs, productivity, time, etc.) by making a one to one correspondence of each of the objectives for pay-offs. Since each player has multiple goals, the concept of vector optimization appears to be more suitable. But comparing the pay-offs of the players in two person zero-sum multi-objective games is much more difficult than comparing them in scalar game, and the classical optimal solution concept is no longer applicable. For this reason, a new solution concepts of Pareto-optimal security strategy has been proposed in [15].

One of the earliest study to analyze the maxmin and minmax values of two person zero-sum multi-objective games is due to Zeleny [25]. By introducing a parameter vector λ, the game reduces to a parametric linear programming problem. He then discussed the concept of Pareto-optimal solutions and ideal points for two person zero-sum multi-objective games. Further, Cook [12] introduced a goal vector and formulated such games as goal programming problems, while Corley [13] presented the necessary and sufficient condition for optimal mixed strategies for the same. Ghose and Prasad [15] introduced the concept of Pareto-optimal Security Strategies (POSS) for multi-objective two person zero-sum games and obtained it by scalarization of the original game. Fernandez and Puerto [16] studied the same game model as that of [15] and established an equivalence between POSS and efficient solution of a pair of multi-objective programming problems.

Though single objective two person zero-sum fuzzy matrix games have been studied extensively in the literature ([22],[9]), the results on multi-objective scenario are rather scarce. The main contribution in this direction has been the work of Sakawa and Nishizaki [22]. Their approach was to associate a fuzzy goal with respective payoff matrix and define the solution in terms of maximizing the degree of minimal goal attainment for each player. Further, Aggarwal et al. [3] has also studied multiobjective two-person zero-sum games but with different approach.

1.1 Preliminaries

For all the notations we shall be following [5–7, 14]. Using the Hurwicz's optimism-pessimism criterion [17], for a fixed $\lambda,\ \lambda \in [0,1]$, an I-fuzzy set $\tilde{A}$ is transformed into a fuzzy set $\tilde{A}$ whose membership function is described by

$$f_{\tilde{A}}(\lambda, x) = (1-\lambda)\mu_{\tilde{A}}(x) + \lambda(1-\nu_{\tilde{A}}(x)),\ x \in X.$$

This function is called as **indeterminacy resolving function of** $\tilde{A}$. The parameter λ depicts the outlook of the decision maker towards resolving indeterminacy; $\lambda = 0$, means that the decision maker resolves indeterminacy fully in favor of membership (complete optimism in resolving indeterminacy), while $\lambda = 1$ indicates that the decision maker resolves indeterminacy fully in negation of the non-membership function (complete pessimism in resolving indeterminacy).

Consider a multi-objective optimization problem with l goals and w constraints. Let the set of goals be $G_r,\ r = 1, 2, ..., l$ and let the set of constraints be $C_w, k = 1, 2, ..., w$, each of which can be characterized as an I-fuzzy set on the universal set X. Angelov [4] used the Bellman and Zadeh's extension principle [11] and defined the I-fuzzy decision as follows:

$$\tilde{D} = (\tilde{G_1} \cap \tilde{G_2} \cap\ ... \cap \tilde{G_l}) \cap (\tilde{C_1} \cap \tilde{C_2} \cap ... \cap\ \tilde{C_w})$$

with

$$\tilde{D} = \{\langle x, \mu_{\tilde{D}}(x), \nu_{\tilde{D}}(x)\rangle | x \in X\},$$

where

$$\mu_{\tilde{D}}(x) = \min_{r,k}\{\mu_{\tilde{G}_r}(x), \mu_{\tilde{C}_k}(x)\}$$

and

$$\nu_{\tilde{D}}(x) = \max_{r,k}\{\nu_{\tilde{G}_r}(x), \nu_{\tilde{C}_k}(x)\}$$

Angelov [4] associated a value function with $\tilde{D}$ as $V_{\tilde{D}(x)} = \mu_{\tilde{D}}(x) - \nu_{\tilde{D}}(x),\ x \in X$, and the optimal solution is defined in the sense of finding an $x^* \in X$ such that $V_{\tilde{D}}(x^*) = \max_{x \in X} V_{\tilde{D}}(x)$. Dubey et al. [14] implemented Yager's [23] idea of resolving indeterminacy in the interval uncertainity represented by I-fuzzy sets in optimization problems. It was observed that this approach can yield a better optimal value for decision making problem than the one proposed in [4]. We briefly describe decision making approach in I-fuzzy environment as given in [14]. Let $\lambda \in [0,1]$ be fixed. Associate a fuzzy set $\tilde{D}$, having membership function explained as

$$f_{\tilde{D}}(\lambda, x) = \min_{r,k}\{f_{\tilde{G}_r}(\lambda, x), f_{\tilde{C}_k}(\lambda, x) | x \in X\},$$

where $f_{\tilde{G}_r}(\lambda, x)$ and $f_{\tilde{C}_k}(\lambda, x)$ are the indeterminacy resolving functions of the I-fuzzy sets representing the r^{th} goal and the k^{th} constraint, respectively. Then, $x^* \in X$ is an *optimal decision*, if $f_{\tilde{D}}(\lambda, x^*) = \max_{x\in X} f_{\tilde{D}}(\lambda, x)$, that is $f_{\tilde{D}}(\lambda, x^*) \geq f_{\tilde{D}}(\lambda, x)$, $\forall\ x \in X$. Hence, solving an optimization problem with Atanassov's I-fuzzy goal is equivalent to solving the following optimization problem:

$$\max \quad \alpha$$

$$\begin{aligned} \text{subject to } & f_{\tilde{G}_r}(\lambda, x) \ \geq\ \alpha,\ r = 1, 2, ..., l \\ & f_{\tilde{C}_k}(\lambda, x) \ \geq\ \alpha,\ k = 1, 2, ..., w \\ & 0 \leq \alpha \leq 1,\ x \in X. \end{aligned}$$

where $\alpha = \min f_{\tilde{D}}(\lambda, x)$. Though there is no unique way to define an I-fuzzy inequality $a^T x \gtrsim^{IF} b$, for any $a, b \in \mathbb{R}^n$, the n-dimensional real space, but two natural approaches are 'the optimistic approach' and 'the pessimistic approach' as explained in details by [1,17] and [14]. For a given acceptance tolerance $\hat{p} > 0$, the linear membership function associated with this inequality is described as follows:

$$\mu(a^T x) = \begin{cases} 1; & a^T x \geq b \\ 1 - \frac{b - a^T x}{\hat{p}}; & b - \hat{p} \leq a^T x \leq b \\ 0; & a^T x \leq b - \hat{p}. \end{cases}$$

Let $\hat{q}$ $(0 < \hat{q} < \hat{p})$ be the tolerance in rejection of the I-fuzzy inequality $a^T x \gtrsim^{IF} b$. The linear nonmembership function in optimistic and pessimistic approaches are defined respectively as follows:

$$\begin{aligned} \nu(a^T x) = & \nu_{optimistic}(a^T x) \\ = & \begin{cases} 0; & a^T x \geq b \\ 1 - \frac{a^T x - b + \hat{p} + \hat{q}}{\hat{p} + \hat{q}}; & b - \hat{p} - \hat{q} \leq a^T x \leq b \\ 1; & a^T x \leq b - \hat{p} - \hat{q}. \end{cases} \end{aligned}$$

$$\begin{aligned} \nu(a^T x) = & \nu_{pessimistic}(a^T x) \\ = & \begin{cases} 0; & a^T x \geq b - \hat{p} + \hat{q} \\ 1 - \frac{a^T x - b + \hat{p}}{\hat{q}}; & b - \hat{p} \leq a^T x \leq b - \hat{p} + \hat{q} \\ 1; & a^T x \leq b - \hat{p}. \end{cases} \end{aligned}$$

The I-fuzzy inequality $a^T x \lesssim^{IF} b$ is treated equivalent to $(-a)^T x \gtrsim^{IF} (-b)$.

1.2 I-Fuzzy Multi-objective Two Person Zero-sum Game with I-Fuzzy goals

Let V_o^r and W_o^r be the scalars representing the aspiration levels of players I and Players II corresponding to rth pay-offs $(A^r,\ r = 1, 2, ..., l)$, respectively. The I-fuzzy multi-objective matrix game with I-fuzzy goals, denoted by *IFMOMG*, is defined as

$$IFMOMG = (S^m, S^n, A^r, {V_o}^r, \gtrsim^{IF}_{p_o^r, q_o^r}\ ,\ W_o^r,\ \lesssim^{IF}_{s_o^r, t_o^r}),$$

$r = 1, 2, ..., l$, where p_o^r and q_o^r are the tolerance levels associated with the acceptance and rejection of the aspiration level V_o^r for Player I. Similarly, s_o^r and t_o^r are the tolerance associated with the acceptance and rejection of the aspiration level ${W_o}^r$ for Player II ($\forall\ r = 1, 2, ..., l$). Now Player I problem is to find $x \in S^m$ such that $x^T A^r y \gtrsim^{IF}_{p_o^r, q_o^r} {V_o}^r,\ \forall\ y \in S^n$, and Player II problem is to find $y \in S^n$ such that $x^T A^r y \lesssim^{IF}_{s_o^r, t_o^r} W_o^r,\ \forall x \in S^m,\ r = 1, 2, ..., l$. In other words, the Player I problem, associated with the

r^{th} pay-off matrix is

(IFP-I) Find $x \in S^m$ such that

$$x^T A_j^r \gtrsim^{IF}_{p_o^r, q_o^r} V_o^r, \quad j = 1, 2, ..., n.$$

Similarly, the Player II problem, associated with the r^{th} pay-off matrix is

(IFP-II) Find $y \in S^n$ such that

$$A_i^r y \lesssim^{IF}_{s_o^r, t_o^r} W_o^r, \quad i = 1, 2, ..., m.$$

Definition 1 Security level of satisfaction for Player I For a strategy $x \in S^m$, the security level of satisfaction for Player I corresponding to r^{th} pay-offs is

$$\alpha_r(x) = \min_{1 \le j \le n} f_j{}^r(\lambda, x^T A_j{}^r)$$

Therefore, the security level for Player I is an l-tuple vector, given by

$$\alpha(x) = [\alpha_1(x), \alpha_2(x), ..., \alpha_l(x)].$$

Definition 2 Security level of satisfaction for Player II For a strategy $y \in S^n$, the security level of satisfaction for Player II corresponding to r^{th} pay-offs is

$$\beta_r(x) = \min_{1 \le i \le m} g_i{}^r(\lambda, A_i^r y)$$

Therefore, the security level for Player II is an l-tuple vector, given by

$$\beta(y) = [\beta_1(y), \beta_2(y), ..., \beta_l(y)].$$

Definition 3 Pareto-optimal security strategy for Player I A strategy $x^* \in S^m$ is a Pareto-optimal security strategy (POSS) for Player I if there is no $x \in S^m$ such that

$$\alpha(x^*) \le \alpha(x) \text{ and } \alpha(x^*) \ne \alpha(x)$$

Definition 4 Pareto-optimal security strategy for Player II A strategy $y^* \in S^n$ is a Pareto-optimal security strategy (POSS) for Player II if there is no $y \in S^n$ such that

$$\beta(y^*) \le \beta(y) \text{ and } \beta(y^*) \ne \beta(y)$$

If x^* is a POSS for player I, then his security level is given by $\alpha^* = \alpha(x^*)$. Similarly, if y^* is a POSS for player II, then his security level is given by $\beta^* = \beta(y^*)$

2 Proposed Approach

2.1 Model in Optimistic Framework

Let p_o^r and q_o^r be the tolerances pre specified by Player I for accepting and rejecting the aspiration level V_o^r in $(IFP-I)$ for all $r = 1, 2, ..., l$. Let $f_j^r(\lambda, \; A_j x)$, $j = 1, 2, ..., n$, be the indeterminacy resolving functions for $r = 1, 2, ..., l$. Next, let s_o^r and t_o^r be the tolerances pre specified by Player II for accepting and rejecting the aspiration level W_o^r in $(IFP-II)$ for all $r = 1, 2, ..., l$. Let $g_i^r(\lambda, A_i y)$, $i = 1, 2, ..., m$, for all $r = 1, 2, ..., l$. The membership and the non-membership functions for Player I in optimistic view with tolerances p_o^r and q_o^r for all $r = 1, 2, ..., l$ are as follows:

$$\mu_j{}^r(x^T A_j^r) = \begin{cases} 1; & x^T A_j^r \ge V_o^r \\ 1 + \frac{x^T A_j^r - V_o^r}{p_o^r}; & V_o^r - p_o^r \le x^T A_j^r \le V_o^r, \\ 0; & x^T A_j^r \le V_o{}^r - p_o^r. \end{cases}$$

and

$$\nu_j{}^r(x^T A_j^r) = \begin{cases} 1; \; x^T A_j^r \le V_o^r - p_o^r - q_o^r, \\ 1 - \frac{x^T A_j^r - (V_o^r - p_o^r - q_o^r)}{p_o^r + q_o^r}; \\ \qquad V_o^r - p_o^r - q_o^r \le x^T A_j^r \le V_o^r, \\ 0; \; x^T A_j^r \ge V_o{}^r. \end{cases}$$

The indeterminacy functions for ${f_j}^r(\lambda, x^T A_j^r),\ j = 1, 2, ..., n$ for Player I are as follows:

$$
{f_j}^r(\lambda, x^T A_j^r) = \begin{cases} 0;\ x^T A_j^r \le V_o^r - p_o^r - q_o^r, \\ f_{1j} = \frac{\lambda(x^T A_j^r - (V_o^r - p_o^r - q_o^r))}{p_o^r + q_o^r}; \\ \qquad V_o^r - p_o^r - q_o^r \le x^T A_j^r \le V_o^r - p_o^r, \\ f_{2j} = 1 + (x^T A_j^r x - V_o)\frac{p_o^r + (1-\lambda)q_o^r}{p_0^r(p_o^r + q_o^r)}; \\ \qquad V_o^r - p_o^r \le x^T A_j^r \le V_o^r, \\ 0;\ \ x^T A_j^r \ge {V_o}^r. \end{cases}
$$

Similarly, for Player II, the membershiip and the non-membership functions with tolerances s_o^r and t_o^r are

$$
{\mu_i}^r(A_j^r y) = \begin{cases} 1;\ \ A_i^r y \le W_o^r \\ 1 + \frac{W_o^r - A_i^r y}{s_o^r};\ \ W_o^r \le A_i^r y \le W_o^r + s_o^r, \\ 0;\ \ A_i^r y \ge {W_o}^r + s_o^r, \end{cases}
$$

and

$$
{\nu_i}^r(A_i^r y) = \begin{cases} 0;\ \ A_i^r y \le W_o^r, \\ 1 + \frac{A_i^r - (V_o^r - s_o^r - t_o^r)}{s_o^r + t_o^r}; \\ \qquad W_o^r \le A_i^r y \le W_o^r + s_o^r + t_o^r, \\ 1;\ A_i^r \ge W_o^r + s_o^r + t_o^r, \end{cases}
$$

respectively. The indeterminacy resolving functions $g_i^r(A_i^r y)$, for $i = 1, 2, .., m$, are as follows:

$$
{g_i}^r(\lambda, A_i^r y) = \begin{cases} 1;\ \ A_i^r y \le W_o^r, \\ g_{i1} = 1 - (A_i^r y - W_o)\frac{s_o^r + (1-\lambda)t_o^r}{s_o^r(s_o^r + t_o^r)}; \\ \qquad W_o^r \le A_i^r y \le W_o^r + s_o^r, \\ g_{i2} = \frac{\lambda(W_o^r + s_o^r + t_o^r - A_i^r y)}{s_0^r + t_o^r)}; \\ \qquad W_o^r + s_o^r \le A_i^r y \le W_o^r + s_o^r + t_o^r, \\ 0;\ \ A_i^r y \ge W_o^r + s_o^r + t_o^r. \end{cases}
$$

In the absence of any information about the attitude of the decision maker towards resolving indeterminacy, we continue to take $\lambda = \frac{1}{2}$ only. In this case $f_j^r(\lambda, A_j^r x)$ and $g_i^r(\lambda, A_i^r y))$ respectively, for all $r = 1, 2, ..., l,\ j = 1, 2, ..., n$ and $i = 1, 2, ..., m$, the indeterminacy resolving functions are described as:

$$
{f_j}^r(x^T A_j^r) = \begin{cases} 0;\ \ x^T A_j^r \le V_o^r - p_o^r - q_o^r, \\ f_{1j} = \frac{(x^T A_j^r - (V_o^r - p_o^r - q_o^r))}{2(p_o^r + q_o^r)}; \\ \qquad V_o^r - p_o^r - q_o^r \le x^T A_j^r \le V_o^r - p_o^r, \\ f_{2j} = 1 + (x^T A_j^r x - V_o)\frac{2p_o^r + q_o^r}{2p_0^r(p_o^r + q_o^r)}; \\ \qquad V_o^r - p_o^r \le x^T A_j^r \le V_o^r, \\ 0;\ \ x^T A_j^r \ge {V_o}^r. \end{cases}
$$

and

$$
{g_i}^r(A_i^r y) = \begin{cases} 1;\ \ A_i^r y \le W_o^r, \\ g_{i1} = 1 - (A_i^r y - W_o)\frac{2s_o^r + t_o^r}{2s_o^r(s_o^r + t_o^r)}; \\ \qquad W_o^r \le A_i^r y \le W_o^r + s_o^r, \\ g_{i2} = \frac{(W_o^r + s_o^r + t_o^r - A_i^r y)}{2(s_0^r + t_o^r)}; \\ \qquad W_o^r + s_o^r \le A_i^r y \le W_o^r + s_o^r + t_o^r, \\ 0;\ \ A_i^r y \ge W_o^r + s_o^r + t_o^r. \end{cases}
$$

respectively. It is important to note that for all $r = 1, 2, ..., l,\ f_j^r(x^T A_j^r),\ j = 1, 2, ..., n$ and $g_i^r(A_i^r y)$, $i = 1, 2, ..., m$, are piecewise linear S-shaped functions with convex type break points. We follow Inuiguchi et al. [18] algorithm to convert them into piecewise linear functions with only concave break points. The procedure transformed $f_j^r(x^T A_j^r),\ j = 1, 2, ..., n$ and for all $r = 1, 2, ..., l$ into piecewise linear functions $f_j^{r'}(x^T A_j^r),\ j = 1, 2, ..., n$ with concave break points only. Therefore, following Yang et al. [24] method for $(IFP - I)$ for Player I is equivalent to solving the following program Equivalent Optimistic Problem $(EOP - I)$:

$$
\begin{aligned}
(EOP - I) \qquad & \max(\alpha_1, \alpha_2, ..., \alpha_l) \\
\text{subject to} \qquad & f_j^{r'}(x^T A_j^r) \ge \alpha_r,\ j = 1, 2, ..., n,\ r = 1, 2, ..., l \\
& e^T x = 1, \\
& x \ge 0,\ \alpha_r \in [0, 1].
\end{aligned}
$$

Theorem 1 *The strategy x^* and vector α^* are POSS and security level of satisfaction, respectively for Player-I, iff the pair (x^*, α^*) is an efficient solution to the multiobjective problem $(EOP - I)$.*

Proof Let x^* be a POSS for Player-I. Then there is no $x \in S^m$ such that

$$\alpha(x^*) \leq \alpha(x),\ \alpha(x^*) \neq \alpha(x).$$

Therefore, for all $x \in S^m$, either

$$(\alpha_1(x^*), \alpha_2(x^*), \ldots, \alpha_l(x^*)) = (\alpha_1(x), \alpha_2(x), \ldots, \alpha_l(x))$$

or there exists an index p, $1 \leq p \leq l$, depending on x such that $\alpha_p(x) < \alpha_p(x^*)$. i.e for any $x \in S^m$ and for all $r = 1, 2, \ldots, l$, either

$$\min_{1\leq j\leq n} f_j^r(\lambda, x^{*T} A_j^r) = \min_{1\leq j\leq n} f_j^r(\lambda, x^T A_j^r)$$

or there exists an index p, $1 \leq p \leq l$, such that

$$\min_{1\leq j\leq n} f_j^p(\lambda, x^T A_j^r) < \min_{1\leq j\leq n} f_j^p(\lambda, x^{*T} A_j^r)$$

Hence, by the definition of efficient solution, x^* is an efficient solution of the multiobjective programming problem : $\max\{\min_{1\leq j\leq n}(f_j^1(\lambda, x^T A_j^1)), \min_{1\leq j\leq n}(f_j^2(\lambda, x^T A_j^2)), \ldots, \min_{1\leq j\leq n}(f_j^l(\lambda, x^T A_j^l))\}$ where $\alpha_r(x) = \min_{1\leq j\leq n}(f_j^r(\lambda, x^T A_j^r))$, for all $r = 1, 2, \ldots, l$. Further, using the representation of various memberships functions $\mu_j^r(x^T A_j^r), \nu_j^r(x^T A_j^r)$ and $f_j^r(x^T A_j^r)$ for all $r = 1, 2, \ldots, l$ and using the above algorithm, we get

$$(EOP - I) \quad \max(\alpha_1, \alpha_2, ..., \alpha_l)$$

subject to

$$f_j^{r'}(x^T A_j^r) \geq \alpha_r,\ j = 1, 2, ..., n,\ r = 1, 2, ..., l$$

$$e^T x = 1,$$

$$x \geq 0,\ \alpha_r \in [0, 1].$$

where $\alpha_r = \alpha_r(x)$, for all $r = 1, 2, \ldots, l$. Conversely, suppose that an efficient solution $(x^*, \alpha^* = \alpha(x^*)$ of $(EOP - I)$ is not a POSS for Player I. Then, there exists $x \in S^m$, such that

$$\alpha(x^*) \leq \alpha(x),\ \alpha(x^*) \neq \alpha(x). \tag{2.1}$$

By definition of $\alpha_r(x)$ for $r = 1, 2, \ldots, l$ *and* $j = 1, 2, \ldots n$, $(x, \alpha(x))$ is the feasible solution of $(EOP - I)$. Thus (2.1) contradicts the assumption that (x^*, α^*) is an efficient solution of $(EOP - I)$.

Similarly, following the same algorithm [18] and then following Yang et al. [24] method for $(IFP - I)$ for Player II, is equivalent to solving the following program Equivalent Optimistic Problem $(EOP - II)$:

$$(EOP - II) \quad \max(\beta_1, \beta_2, ..., \beta_l)$$

subject to

$$g_i^{r'}(A_i^r y) \geq \beta_r,\ i = 1, 2, ..., m,\ r = 1, 2, ..., l$$

$$e^T y = 1,$$

$$y \geq 0,\ \beta_r \in [0, 1].$$

Theorem 2 *The strategy y^* and vector β^* are POSS and security level of satisfaction respectively for Player-II, iff the pair (y^*, β^*) is an efficient solution to the multiobjective problem $(EOP - II)$.*

The proof of this theorem follows on the lines of Theorem 1.

2.2 Model in Pessimistic Framework

The decision maker has pessimistic attitude in acceptance amounting to saying that complete rejection of a criterion does not mean its full acceptance. Let p_o^r and q_o^r be the tolerances for Player I, associated with acceptance and rejection of the aspiration level V_o^r in $(IFP - I)$. Let $f_j^r(\lambda, x^T A_j^r)$, $j = 1, 2, .., n$, $r = 1, 2, ..., l$ be their indeterminacy resolving functions. Similarly, let s_o^r and t_o^r be the tolerances for Player II, associated with acceptances and rejection of the aspiration level W_o^r in $(IFP-II)$. The membership and the non-membership functions for Player I in a pessimistic situation are described as follows:

$$\mu_j{}^r(x^TA_j^r)=\begin{cases}1; & x^TA_j^r\geq V_o^r,\\ 1+\frac{x^TA_j^r-V_o^r}{p_o^r}; & V_o^r-p_o^r\leq x^TA_j^r\leq V_o^r,\\ 0; & x^TA_j^r\leq V_o{}^r-p_o^r.\end{cases}$$

and

$$\nu_j{}^r(x^TA_j^r)=\begin{cases}1; & x^TA_j^r\leq V_o^r-p_o^r,\\ 1-\frac{x^TA_j^r-(V_o^r-p_o^r)}{q_o^r}; & V_o^r-p_o^r\leq x^TA_j^r\leq V_o^r-p_o^r+q_o^r,\\ 0; & x^TA_j^r\geq V_o^r-p_o^r+q_o^r.\end{cases}$$

The indeterminacy resolving functions for Player I, for $j=1,2,...,n$ and $r=1,2,...,l$ are as follows:

$$f_j{}^r(\lambda,x^TA_j^r)=\begin{cases}0; & x^TA_j^r\leq V_o^r-p_o^r,\\ f_{1j}=\frac{p_o^r\lambda+(1-\lambda)q_o^r}{q_o^r}\left(1+\frac{x^TA_j^r-V_o^r}{p_o^r}\right); & V_o^r-p_o^r\leq x^TA_j^r\leq V_o^r-p_o^r+q_o^r,\\ f_{2j}=1+(1-\lambda)\frac{x^TA_j^r-V_o^r}{p_0^r}; & V_o^r-p_o^r+q_o^r\leq x^TA_j^r\leq V_o^r,\\ 1; & x^TA_j^r\geq V_o{}^r.\end{cases}$$

Similarly, for Player II, the membership and the non-membership functions are as follows:

$$\mu_i{}^r(A_i^ry)=\begin{cases}1; & A_i^ry\leq W_o^r\\ 1+\frac{W_o^r-A_i^ry}{s_o^r}; & W_o^r\leq A_i^ry\leq W_o^r+s_o^r,\\ 0; & A_i^ry\geq W_o{}^r+s_o^r,\end{cases}$$

and

$$\nu_i{}^r(A_i^ry)=\begin{cases}0; & A_i^ry\leq W_o^r+s_o^r-t_o^r,\\ 1+\frac{A_i^ry-(W_o^r+s_o^r)}{t_o^r}; & W_o^r+s_o^r-t_o^r\leq A_i^ry\leq W_o^r+s_o^r,\\ 1; & A_i^ry\geq W_o^r+s_o^r,\end{cases}$$

And the indeterminacy resolving functions $g_i^r(\lambda,A_i^ry)$, for $i=1,2,..,m$, are as follows:

$$g_i{}^r(\lambda,A_i^ry)=\begin{cases}1; & A_i^ry\leq W_o^r,\\ g_{i1}=1+(1-\lambda)\frac{W_o^r-A_i^ry}{s_o^r}; & W_o^r\leq A_i^ry\leq W_o^r+s_o^r-t_o^r,\\ g_{i2}=\frac{\lambda s_o^r+(1-\lambda)t_o^r}{t_o^r}\left(1+\frac{W_o^r-A_i^ry}{s_o^r}\right); & W_o^r+s_o^r-t_o^r\leq A_i^ry\leq W_o^r+s_o^r,\\ 0; & A_i^ry\geq W_o^r+s_o^r.\end{cases}$$

In the absence of any information about the attitude of the decision maker towards resolving indeterminacy, we continue to take $\lambda=\frac{1}{2}$ only. Hence, for $j=1,2,...,n$ and $i=1,2,...,m$, the indeterminacy resolving functions for Player I and Player II are described as:

$$f_j{}^r(\lambda,x^TA_j^r)=\begin{cases}0; & x^TA_j^r\leq V_o^r-p_o^r,\\ f_{1j}=\frac{p_o^r+q_o^r}{2q_o^r}\left(1+\frac{x^TA_j^r-V_o^r}{p_o^r}\right); & V_o^r-p_o^r\leq x^TA_j^r\leq V_o^r-p_o^r+q_o^r,\\ f_{2j}=1+\frac{x^TA_j^r-V_o^r}{2p_0^r}; & V_o^r-p_o^r+q_o^r\leq x^TA_j^r\leq V_o^r,\\ 1; & x^TA_j^r\geq V_o{}^r.\end{cases}$$

and

$$g_i{}^r(\lambda,A_i^ry)=\begin{cases}1; & A_i^ry\leq W_o^r,\\ g_{i1}=1+\frac{W_o^r-A_i^ry}{2s_o^r}; & W_o^r\leq A_i^ry\leq W_o^r+s_o^r-t_o^r,\\ g_{i2}=\frac{s_o^r+t_o^r}{2t_o^r}\left(1+\frac{W_o^r-A_i^ry}{s_o^r}\right); & W_o^r+s_o^r-t_o^r\leq A_i^ry\leq W_o^r+s_o^r,\\ 0; & A_i^ry\geq W_o^r+s_o^r.\end{cases}$$

respectively. It is important to note that for all $r=1,2,...,l$, $f_j^r(x^TA_j^r)$, $j=1,2,...,n$ and $g_i^r(A_iy)$, $i=1,2,...,m$, are piecewise linear S-shaped functions with concave type break points. Therefore solving $(IFP-I)$ and $(IFP-II)$ for Player I and

Player II are equivalent to solving the following two programs, respectively.

$(EPP-I)$ $\max(\alpha_1, \alpha_2, ..., \alpha_l)$

subject to

$$\frac{p_o^r + q_o^r}{2q_o^r}\left(1 + \frac{x^T A_j^r - V_o^r}{p_o^r}\right) \geq \alpha_r,\ j = 1, 2, ..., n,$$

$$1 + \frac{x^T A_j^r - V_o^r}{2p_0^r} \geq \alpha_r,\ j = 1, 2, ..., n,$$

$$e^T x = 1,$$

$$x \geq 0,\ \alpha_r \in [0, 1],\ r = 1, 2, ...l.$$

and

$(EPP-II)$ $\max(\beta_1, \beta_2, ..., \beta_l)$

subject to

$$1 + \frac{W_o^r - A_i^r y}{2s_o^r} \geq \beta_r,\ i = 1, 2, ..., m,$$

$$\frac{s_o^r + t_o^r}{2t_o^r}\left(1 + \frac{W_o^r - A_i^r y}{s_o^r}\right) \geq \beta_r,\ i = 1, 2, ..., m,$$

$$e^T y = 1,$$

$$y \geq 0,\ \beta_r \in [0, 1],\ r = 1, 2, ...l.$$

Note that $(EPP-I)$ and $(EPP-II)$ are the crisps equivalent of $(IFP-I)$ and $(IFP-II)$, respectively. It is well understood, from section 2.1 that x^* is a POSS and α^* is the security level for Player I in the Pessimistic view, iff (x^*, α^*) is an efficient solution of $(EPP-I)$. The proof is similar to Theorem 1. Similarly, y^* is a POSS and β^* is the security level for Player II in the Pessimistic view iff (y^*, β^*) is an efficient solution of $(EPP-II)$.

2.3 Numerical Illustration

Let us consider the numerical example as taken by [3], but having intuitionistic fuzzy goals. Here, we solve all numerical problems using GAMS [21]. Consider the multi-objective matrix game having payoff matrices

$$A^1 = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{bmatrix},\ A^2 = \begin{bmatrix} -3 & 7 & 2 \\ 0 & -2 & 0 \\ 3 & -1 & 6 \end{bmatrix},\ A^3 = \begin{bmatrix} 8 & 2 & 3 \\ -5 & 6 & 0 \\ -3 & 1 & 6 \end{bmatrix}$$

as the cost matrix, the time matrix and the productivity matrix, respectively.

Solution by the proposed method

We solve this problem with same parameters as in [22,3,8] so that we can compare the results. Thus $V_o^1 = \overline{a}^1 = 6$, $W_o^1 = \underline{a}^1 = -2$, ${p_o}^1 = {s_o}^1 = \overline{a}^1 - \underline{a}^1 = 8$; $V_o^2 = \overline{a}^2 = 7$, $W_o^2 = \underline{a}^2 = -2$, ${p_o}^2 = {s_o}^2 = \overline{a}^2 - \underline{a}^2 = 10$; $V_o^3 = \overline{a}^3 = 8$, $W_o^3 = \underline{a}^3 = -5$, ${p_o}^3 = {s_o}^3 = \overline{a}^1 - \underline{a}^1 = 10$. Also, we take $q_o^1 = 1$, $q_o^2 = 4$, $q_o^3 = 10$, and $t_o^1 = 6$, $t_o^2 = 4$, $t_o^3 = 7$ as in [8]. Assuming $\lambda = \frac{1}{2}$. After resolving indeterminacy, we get the piecewise linear indeterminacy resolving functions, with convex break points. Incorporating Inuiguchi et al. [18] method, we transform the indeterminacy functions with convex break points into concave break points. Thus making the problem ready to solve as a multiobjective programming problem, resolving the ambiguity. The equivalent crisp problem with constraints having concave breakpoints for Player I is given by (in optimistic sense):

(EOP-I) $\max(\alpha_1, \alpha_2, \alpha_3)$

subject to

$$0.6691x_1 - 0.3345x_2 + 1.0036 \geq \alpha_1,$$

$$0.2788x_1 - 0.1394x_2 + 0.7578 \geq \alpha_1,$$

$$0.0995x_1 - 0.4976x_2 + 0.7014 \geq \alpha_1,$$

$$1.674x_1 - 0.6698x_2 + 1.0048x_3 + 1.0044 \geq \alpha_1,$$

$$0.697x_1 - 0.2788x_2 + 0.4182x_3 + 0.7578 \geq \alpha_1,$$

$$0.2488x_1 - 0.0995x_2 + 0.1492x_3 + 0.7014 \geq \alpha_1,$$

$$0.3345x_1 - 2.0072x_2 - 0.3345x_3 + 1.0036 \geq \alpha_1,$$

$$0.1394x_1 + 0.8364x_2 - 0.1394x_3 + 0.7578 \geq \alpha_1,$$
$$0.0498x_1 + 0.2985x_2 - 0.0497x_3 + 0.7014 \geq \alpha_1,$$
$$-0.6458x_1 + 0.6458x_3 + 1.5069 \geq \alpha_2,$$
$$-0.3038x_1 + 0.3038x_3 + 0.9739 \geq \alpha_2,$$
$$-0.1084x_1 + 0.1084x_3 + 0.7470 \geq \alpha_2,$$
$$1.5069x_1 - 0.4305x_2 - 0.2152x_3 + 1.5069 \geq \alpha_2,$$
$$0.7089x_1 - 0.2025x_2 - 0.1012x_3 + 0.8859 \geq \alpha_2,$$
$$0.2529x_1 - 0.0722x_2 - 0.0361x_3 + 0.7470 \geq \alpha_2,$$
$$0.4304x_1 + 1.2912x_3 + 1.5064 \geq \alpha_2,$$
$$0.2024x_1 + 0.6072x_3 + 0.8856 \geq \alpha_2,$$
$$0.0722x_1 + 0.2166x_3 + 0.747 \geq \alpha_2,$$
$$1.0482x_1 - 0.6551x_2 - 0.3931x_3 + 1.9655 \geq \alpha_3,$$
$$0.4977x_1 - 0.31108x_2 - 0.1866x_3 + 1.1089 \geq \alpha_3,$$
$$0.2031x_1 - 0.1269x_2 - 0.0761x_3 + 0.7971 \geq \alpha_3,$$
$$0.2620x_1 - 0.7862x_2 + 0.13103x_3 \geq \alpha_3,$$
$$0.1244x_1 - 0.3732x_2 + 0.0622x_3 + 1.1089 \geq \alpha_3,$$
$$0.0507x_1 + 0.1523x_2 + 0.0253x_3 + 0.797 \geq \alpha_3,$$
$$0.03931x_1 + 0.7862x_3 + 1.9655 \geq \alpha_3,$$
$$0.1866x_1 + 0.3732x_3 + 1.1087 \geq \alpha_3,$$
$$0.0759x_1 + 0.1518x_3 + 0.7962 \geq \alpha_3,$$
$$x_1 + x_2 + x_3 = 1,$$
$$0 \leq \alpha_1,\ \alpha_2,\ \alpha_3 \leq 1,$$
$$x_1, x_2, x_3 \geq 0.$$

The Pareto-optimal security strategies with corresponding security levels for Player I are depicted in Table 1.

Similarly, the equivalent crisp problem for Player II is:

(EOP-II) $\max(\beta_1, \beta_2, \beta_3)$

subject to

$$-0.3410y_1 - 0.8525y_2 - 0.1705y_3 + 2.046 \geq \beta_1,$$
$$-0.1416y_1 - 0.354y_2 - 0.0708y_3 + 1.2484 \geq \beta_1,$$
$$-0.0556y_1 - 0.139y_2 - 0.0278y_3 + 0.94318 \geq \beta_1,$$
$$0.1705y_1 + 0.341y_2 + 1.023y_3 + 2.046 \geq \beta_1,$$
$$0.0708y_1 + 0.1416y_2 - 0.4248y_3 + 1.2484 \geq \beta_1,$$
$$0.0278y_1 + 0.0556y_2 - 0.1668y_3 + 0.9431 \geq \beta_1,$$
$$-0.5115y_2 + 0.1705y_3 + 2.046 \geq \beta_1,$$
$$-0.0834y_2 + 0.0278y_3 + 0.9431 \geq \beta_1,$$
$$-0.2124y_2 + 0.0708y_3 + 1.2484 \geq \beta_1,$$
$$0.5112y_1 - 1.1928y_2 - 0.1704y_3 + 1.8744 \geq \beta_2,$$
$$0.0198y_1 - 0.0462y_2 - 0.0132y_3 + 0.7895 \geq \beta_2,$$
$$0.0507y_1 - 0.1183y_2 - 0.0338y_3 + 0.8806 \geq \beta_2,$$
$$0.0341y_2 + 1.8747 \geq \beta_2,$$
$$0.0132y_2 + 0.7895 \geq \beta_2,$$
$$0.0338y_2 + 0.8806 \geq \beta_2,$$
$$-0.5112y_1 + 0.1704y_2 - 1.0224y_3 + 1.8744 \geq \beta_2,$$
$$-0.0198y_1 + 0.0066y_2 - 0.0396y_3 + 0.7895 \geq \beta_2,$$
$$-0.0507y_1 + 0.0169y_2 - 0.1014y_3 + 0.8806 \geq \beta_2,$$
$$-0.9544y_1 - 0.2386y_2 - 0.3579y_3 + 1.7895 \geq \beta_3,$$
$$-0.3968y_1 - 0.0992y_2 - 0.1488y_3 + 1.1426 \geq \beta_3,$$
$$-0.144y_1 - 0.036y_2 - 0.054y_3 + 0.9092 \geq \beta_3,$$
$$0.5965y_1 - 0.7158y_2 + 1.7895 \geq \beta_3,$$
$$0.2481y_1 - 0.2976y_2 + 1.1426 \geq \beta_3,$$
$$0.09y_1 - 0.108y_2 + 0.9092 \geq \beta_3,$$
$$0.3579y_1 - 0.1193y_2 - 0.7158y_3 + 1.7895 \geq \beta_3,$$

Table 1 POSS and Security levels for Player I in Optimistic Approach

#	x_1^*	x_2^*	x_3^*	α_1^*	α_2^*	α_3^*
1	0.9412	0.0587	0	0.7658	0.6449	0.2928
2	0.8727	0.0587	0.0684	0.7590	0.6598	0.2838
3	0.8042	0.0587	0.1369	0.7521	0.6746	0.2748
4	0.7357	0.0587	0.2054	0.7453	0.6895	0.2658
5	0.6672	0.0587	0.2739	0.7385	0.7043	0.2569

$$0.1488y_1 - 0.0496y_2 - 0.2976y_3 + 1.14267 \geq \beta_3,$$
$$0.054y_1 - 0.018y_2 - 0.108y_3 + 0.9092 \geq \beta_3,$$
$$y_1 + y_2 + y_3 = 1,$$
$$0 \leq \beta_1,\ \beta_2,\ \beta_3 \leq 1,\ y_1, y_2, y_3 \geq 0.$$

The POSS and the corresponding security levels for Player II are depicted in Table 2. Similarly, the Pareto-optimal security strategies with corresponding security levels for Player I and Player II in Pessimistic sense are depicted in Table 3 and Table 4 respectively.

Table 2 POSS and Security levels for Player II in Optimistic Approach

#	y_1^*	y_2^*	y_3^*	β_1^*	β_2^*	β_3^*
1	0.625	0	0.375	0.898	0.7622	0.7989
2	0.6287	0.01869	0.3525	0.8958	0.7632	0.7989
3	0.6366	0.0581	0.3052	0.8912	0.7651	0.7989
4	0.6445	0.0975	0.2579	0.8866	0.7671	0.7989
5	0.6523	0.1369	0.2106	0.8820	0.7691	0.7989

Table 3 POSS and Security levels for Player I in Pessimistic Approach

#	x_1^*	x_2^*	x_3^*	α_1^*	α_2^*	α_3^*
1	0.875	0.125	0	0.7265	0.1312	0.6634
2	0.8214	0.1785	0	0.7165	0.1874	0.6823
3	0.7679	0.2320	0	0.7064	0.2436	0.7013
4	0.7144	0.2855	0	0.6964	0.2998	0.7202
5	0.6609	0.3390	0	0.6864	0.3560	0.7392

Table 4 POSS and Security levels for Player II in Pessimistic Approach

#	y_1^*	y_2^*	y_3^*	β_1^*	β_2^*	β_3^*
1	0.625	0	0.375	0.6380	0.5031	0.2060
2	0.6485	0	0.3514	0.6345	0.5154	0.1931
3	0.7019	0.0097	0.2883	0.6211	0.5554	0.1648
4	0.7079	0.0397	0.2522	0.6027	0.5953	0.1648
5	0.7139	0.0697	0.2162	0.5843	0.6353	0.1648

3 Conclusion

1. A new model is constructed for studying multiobjective two person zero-sum matrix games with I-fuzzy goals via resolving the indeterminacy function. Thereby extending the results of Khan et al. [20] to the multiobjective case. The game is shown equivalent to two fuzzy multiobjective fuzzy linear programming problems involving piecewise linear membership functions. The crisp equivalent programs are formulated using Yang et al. [24] and Inuiguchi et al. [18] approaches.
2. The efficient solutions of the equivalent multiobjective (crisp) problems are POSS and security levels of the I-fuzzy model.
3. Although this problem has not been much discussed in literature so far, but Bashir et al. [8] has examined the same with a different approach, using score function.
4. The security levels for Player I defined by [8] are

$$\alpha_r(x) = \min_{1\leq j\leq n}[{\mu_j}^r(x^T{A_j}^r), {\nu_j}^r(x^T{A_j}^r)]$$
$$=[\min_{1\leq j\leq n}{\mu_j}^r(x^T{A_j}^r), \max_{1\leq j\leq n}{\nu_j}^r(x^T{A_j}^r)],$$

and the security levels for Player II are defined as

$$\beta_r(x) = \min_{1\leq i\leq m}[{\mu_i}^r(A_i^r y), {\nu_i}^r(A_i^r y)]$$
$$=[\min_{1\leq i\leq m}{\mu_i}^r(A_i^r y), \max_{1\leq j\leq m}{\nu_i}^r(A_i^r y)].$$

However, our choice of security levels are motivated by the approach of Yager [23].

There is a great scope to extend the results to multiobjective Bi-matrix games with I-fuzzy goals by resolving indeterminacy Also, it would be interesting and challenging to explore the multiobjective two-person zero sum matrix games with I-fuzzy goals as well as I-fuzzy payoff matrix. Further discussions can be made of the problem by third approach, called mixed approach.

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