

Fields Accelerated Math Summer Program Quiz 2016

Instructions

Complete the following five problems to the best of your abilities and submit your answer and solution to **xiangtaoliu@gmail.com**. Note that the sub-parts of each question are not related to each other.

The purpose of this quiz is to see how well you can use the basic knowledge from school and apply them to more difficult questions. What matters to us are not just your final answers, but your method of obtaining the result. Correct answers on their own will count for very little. Finally, even if you cannot solve a problem, do not hesitate to submit a partial solution. We would also be interested in reading your attempts.

If you need clarification on any problem, please contact us. You may not consult or get help from anyone else. You can use books or the Web to look up definitions, formulas, or standard techniques, but any information obtained in this way must be clearly referenced in your solution. Please do not try to look for the problems themselves: we want to see how well you can do math, not how well you can use Google!

You may hand write or type your solutions.

1. System of equations

- (a) Solve the following system of equations

$$\begin{cases} x + y = 1 \\ y + z = 2 \\ z + x = 3 \end{cases}$$

- (b) Let a and b be two numbers such that $a + b \neq 1$. Solve the following system of equations

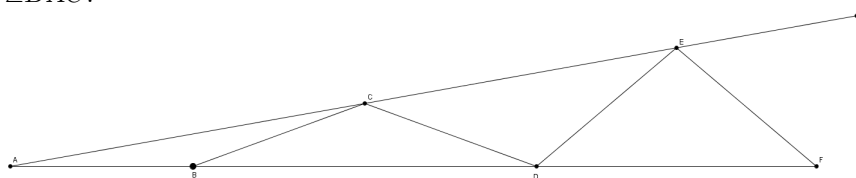
$$\begin{cases} y + \frac{x}{x+y} = a \\ x + \frac{y}{x+y} = b \end{cases}$$

- (c) Solve the following system of equations

$$\begin{cases} xyz = 6 \\ x + y + z = 6 \\ xy + yz + zx = 11 \end{cases}$$

2. Geometry

- (a) Call a triangle simple if each of its sides has length 4, 6, or 8. Find all possible simple triangles.
- (b) In the following figure, $\overline{AB} = \overline{BC} = \overline{CD} = \overline{DE} = \overline{EF} = \overline{EG}$. If $\angle GEF = 50^\circ$, find the value of $\angle BAC$.



- (c) Consider a square PQRS. Let F be the midpoint of PS and G be the midpoint of FS . Determine the value of $\frac{\angle GQR}{\angle PQF}$.

3. Number Theory

- (a) Determine the unit digit of 2^{2016} .
- (b) List all prime numbers p such that $p + 2$ and $p + 4$ are also prime. (Be sure to explain why your list is complete)
- (c) List all prime numbers p such that $p^{2016} + p^{2017}$ is a perfect square. (Be sure to explain why your list is complete)

4. Factoring

- (a) Find the solutions to the following equations.

- i. $x^2 + 2x + 1 = 0$
- ii. $x^2 + 3x + 2 = 0$
- iii. $x^3 + x^2 + x + 1 = 0$
- iv. $x^4 + x^2 + 1 = 0$

- (b) Compute the value of n such that

$$(2 + 1)(2^2 + 1)(2^4 + 1)(2^8 + 1) \cdots (2^{2^{10}} + 1) + 1 = 2^n$$

- (c) Let a and b be two numbers such that

$$a^2 + 6a + 25 + 8b + b^2 = 0$$

Find all possible pairs that satisfy the above equation.

5. Miscellaneous

- (a) Let $\diamond$ be an operator such that for any two numbers x and y ,

$$x \diamond y = 5x - y$$

Let a be a fixed number. Find the value of x that satisfy the equation

$$(a \diamond x) \diamond a = a$$

- (b) For any number a , we define its **absolute value**, denoted by $|a|$, as follows:

$$|a| = \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{if } a = 0 \\ -a, & \text{if } a < 0 \end{cases}$$

If a and b are non-zero real number, find all possible values of the expression

$$\frac{a}{|a|} + \frac{b}{|b|}$$

- (c) Given that the following equation is true,

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Compute the value of

$$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \frac{1}{(4)(5)} + \cdots + \frac{1}{(99)(100)}$$