Math 3331 ODEs 1st Order Review

1. Identify the type of first order ODE: Separable, linear, Bernoulli, homogeneous or exact.

(1) $\dfrac{dy}{dx} = \dfrac{x-y}{x}$

Soln. The ODE is linear so putting it in standard form gives

$$\frac{dy}{dx} + \frac{y}{x} = 1$$

The integrating factor is $\mu = exp(\int \frac{dx}{x}) = x$ and so the standard form ODE becomes

$$\frac{d}{dx}(x \cdot y) = x$$

which integrates to give

$$xy = \frac{x^2}{2} + c.$$

Solving for y gives

$$y = \frac{x}{2} + \frac{c}{x}.$$

(2) $(x+1)\dfrac{dy}{dx} = 10 - y$

Soln. The ODE is separable so putting it in standard form gives

$$\frac{dy}{10-y} = \frac{dx}{x+1}.$$

Integrating gives

$$-\ln|10-y| = \ln|x+1| + c.$$

(3) $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{y}{x} + 1$

Soln. The ODE is homogeneous so letting $y = xu$ gives

$$x\frac{du}{dx} + u = \frac{1}{u} + u + 1.$$

Canceling u and separating gives

$$\frac{u}{u+1}\,du = \frac{dx}{x}$$

which integrates to give

$$u - \ln|u+1| = \ln|x| + c.$$

Back substituting gives

$$\frac{y}{x} - \ln|\frac{y}{x}+1| = \ln|x| + c.$$

(4) $2xy\dfrac{dy}{dx} + y^2 = x^2$

Soln. The ODE is Bernoulli so putting it in standard form gives

$$\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{2y}$$

Mulltiplying by y gives

$$y\frac{dy}{dx} + \frac{y^2}{2x} = \frac{x}{2}.$$

Setting $u = y^2$ so $\dfrac{du}{dx} = 2y\dfrac{dy}{dx}$ giving the ODE as

$$\frac{1}{2}\frac{du}{dx} + \frac{u}{2x} = \frac{x}{2}$$

or, in standard form linear

$$\frac{du}{dx} + \frac{u}{x} = x.$$

With the integrating factor x, the ODE integrates giving

$$xu = \frac{x^3}{3} + c$$

and back substituting gives the final solution as

$$xy^2 = \frac{x^3}{3} + c.$$

(5) $\dfrac{dy}{dx} = -\dfrac{3y^2 + 2x}{6xy + 3y^2}$

Soln. The ODE is exact so putting it in standard form gives

$$\left(3y^2 + 2x\right)dx + \left(6xy + 3y^2\right)dy = 0$$

and identifying that $M = 3y^2 + 2x$ and $N = 6xy + 3y^2$ it is easy to check that $M_y = 6y = N_x$. Thus z exists such that

$$\begin{aligned} z_x &= M = 3y^2 + 2x \quad \Rightarrow \quad z = 3xy^2 + x^2 + A(y) \\ z_y &= N = 6xy + 3y^2 \quad \Rightarrow \quad z = 3xy^2 + y^3 + B(x) \end{aligned}$$

and choosing $A = y^3$ and $B = x^2$ gives the solution of the ODE as

$$3xy^2 + x^2 + y^3 = c.$$

(6) $\dfrac{dy}{dx} = y - xy^2$

Soln. The ODE is Bernoulli so putting it in standard form gives

$$\frac{dy}{dx} - y = -xy^2$$

Dividing by y^2 gives

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = -x.$$

Setting $u = \dfrac{1}{y}$ so $\dfrac{du}{dx} = -\dfrac{1}{y^2}\dfrac{dy}{dx}$ giving the ODE as

$$-\frac{du}{dx} - u = -x$$

or, in standard form linear

$$\frac{du}{dx} + u = x.$$

With the integrating factor e^x, the ODE integrates giving

$$e^x u = xe^x - e^x + c$$

and back substituting gives the final solution as

$$\frac{e^x}{y} = xe^x - e^x + c.$$

(7) $\dfrac{dy}{dx} = \dfrac{2x}{y} + \dfrac{1}{y} + 2x + 1$

Soln. The ODE is separable as we can see by factoring

$$\frac{dy}{dx} = \left(\frac{1}{y} + 1\right)(2x+1)$$

so putting it in standard form gives

$$\frac{y}{y+1}\,dy = (2x+1)\,dx.$$

Integrating gives

$$y - \ln|y+1| = x^2 + x + c.$$

(8) $x\frac{dy}{dx}+2y = x\sin x$

The ODE is linear so putting it in standard form gives

$$\frac{dy}{dx}+\frac{2y}{x} = \sin x$$

The integrating factor is $\mu = exp(\int \frac{2dx}{x}) = x^2$ and so the standard form ODE becomes

$$\frac{d}{dx}\left(x^2\cdot y\right) = x^2\sin x$$

which integrates to give

$$x^2y = 2\cos x + 2x\sin x - x^2\cos x + c.$$

Solving for y leads to

$$y = \frac{2\cos x + 2x\sin x - x^2\cos x + c}{x^2}.$$

(9) $\frac{dy}{dx} = -\frac{x^2+y}{x+y^2}$

Soln. The ODE is exact so putting it in standard form gives

$$\left(x^2+y\right)dx + \left(x+y^2\right)dy = 0$$

and identifying that $M = x^2+y$ and $N = x+y^2$ it is easy to check that $M_y = 1 = N_x$. Thus z exists such that

$$\begin{aligned} z_x &= M = x^2+y \;\Rightarrow\; z = \frac{x^3}{3}+xy+A(y) \\ z_y &= N = x+y^2 \;\Rightarrow\; z = xy+\frac{y^2}{3}+B(x) \end{aligned}$$

and choosing $A = \frac{y^3}{3}$ and $B = \frac{x^3}{3}$ gives the solution of the ODE as (we multiplied by 3 and absorbed the 3 into c)

$$x^3+3xy+y^3 = c.$$

(10) $\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$

The ODE is homogeneous so letting $y = xu$ gives

$$x\frac{du}{dx}+u = \frac{1+u^2}{1+u}.$$

Isolating $\dfrac{du}{dx}$ and separating gives

$$\frac{1+u}{1-u}\,du = \frac{dx}{x}$$

which integrates to give

$$-u - 2\ln|1-u| = \ln|x| + c.$$

Back substituting gives

$$-\frac{y}{x} - 2\ln\left|1-\frac{y}{x}\right| = \ln|x| + c.$$

(11) $\dfrac{dy}{dx} = x^2 + y$

The ODE is linear so putting it in standard form gives

$$\frac{dy}{dx} - y = x^2$$

The integrating factor is $\mu = exp(\int -dx) = e^{-x}$ and so the standard form ODE becomes

$$\frac{d}{dx}\left(e^{-x}\cdot y\right) = x^2 e^{-x}$$

which integrates to give

$$e^{-x}y = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c.$$

Solving for y leads to

$$y = -x^2 - 2x - 2 + ce^x.$$

(12) $\dfrac{dy}{dx} = \dfrac{1 - y\cos x}{2y + \sin x}$

Soln. The ODE is exact so putting it in standard form gives

$$(y\cos x - 1)\,dx + (2y + \sin x)\,dy = 0$$

and identifying that $M = y\cos x - 1$ and $N = 2y + \sin x$ it is easy to check that $M_y = \cos x = N_x$. Thus z exists such that

$$\begin{aligned} z_x &= M = y\cos x - 1 \;\Rightarrow\; z = y\sin x - x + A(y) \\ z_y &= N = 2y + \sin x \;\Rightarrow\; z = y^2 + y\sin x + B(x) \end{aligned}$$

and choosing $A = y^2$ and $B = -x$ gives the solution of the ODE as

$$y\sin x - x + y^2 = c.$$

(13) $\dfrac{dy}{dx} + \dfrac{y}{2x} = \dfrac{x}{y^3}$

Soln. The ODE is Bernoulli already in standard form so multiply by y^3 gives

$$y^3 \frac{dy}{dx} + \frac{y^4}{2x} = x.$$

Setting $u = y^4$ so $\dfrac{du}{dx} = 4y^3 \dfrac{dy}{dx}$ giving the ODE as

$$\frac{1}{4}\frac{du}{dx} + \frac{u}{2x} = x$$

or, in standard form linear

$$\frac{du}{dx} + \frac{2u}{x} = 4x.$$

With the integrating factor x^2, the ODE integrates giving

$$x^2 u = x^4 + c$$

and back substituting gives the final solution as

$$x^2 y^4 = x^4 + c.$$

(14) $\dfrac{dy}{dx} = \dfrac{y\,(x-y)}{x\,(x+y)}$

Soln. The ODE is homogeneous so letting $y = xu$ gives

$$x\frac{du}{dx} + u = \frac{u(1-u)}{1+u}.$$

Isolating u' gives

$$x\frac{du}{dx} = \frac{-2u^2}{1+u}$$

and separating gives

$$-\frac{1+u}{u^2}\,du = 2\frac{dx}{x}$$

which integrates to give

$$\frac{1}{u} - \ln|u| = 2\ln|x| + c.$$

Back substituting gives

$$\frac{x}{y} - \ln\left|\frac{y}{x}\right| = 2\ln|x| + c.$$

(15) $\dfrac{dy}{dx} = x^2y^2 + x^2$

Soln. The ODE is separable so putting it in standard form gives

$$\frac{dy}{y^2+1} = x^2\,dx.$$

Integrating gives

$$\tan^{-1} y = \frac{x^3}{3} + c$$

or

$$y = \tan\left(\frac{x^3}{3} + c\right)$$