Math 1497 Calc 2

Partial Fractions

Consider $\int \frac{x\,dx}{x^2-1}$ & $\int \frac{dx}{x^2-1}$

How do we integrate these

1st one u-sub

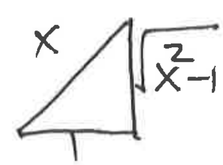
let $u = x^2 - 1$ so $\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c$

$du = 2x\,dx$

$$= \frac{1}{2}\ln|x^2-1| + c$$

Next one - u sub won't work but a trig sub will $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$, $x^2-1 = \sec^2\theta - 1 = \tan^2\theta$

$$\int \frac{\sec\theta\tan\theta}{\tan^2\theta}\,d\theta = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\,d\theta$$



$$= -\ln|\csc\theta + \cot\theta| + c = -\ln\frac{x+1}{\sqrt{x^2-1}} + c$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$$ (after some simplification)

However notice that

$$\frac{x}{x^2-1} = \frac{1}{2}\left\{\frac{1}{x-1} + \frac{1}{x+1}\right\}$$

&

$$\frac{1}{x^2-1} = \frac{1}{2}\left\{\frac{1}{x-1} - \frac{1}{x+1}\right\}$$

So why is this of use. Each of the terms in the { } are easy to integrate

$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\left\{\int \frac{dx}{x-1} + \int \frac{dx}{x+1}\right\} + c$$

$$= \frac{1}{2}\left\{\ln|x-1| + \ln|x+1|\right\} + c$$

$$= \frac{1}{2}\ln|(x-1)(x+1)| + c$$

$$= \frac{1}{2}\ln|x^2-1| + c$$

So now we want to integrate expressions like

$$\int \frac{p(x)}{q(x)}\,dx \qquad p, q \text{ are polynomials}$$

For example,

$$\int \frac{x+7}{x^2-x-2}\,dx$$

well first notice that $x^2-x-2=(x+1)(x-2)$

so can we break up

$$\frac{x+7}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \ ?$$

A & B are to be found

so $$x+7 = \left(\frac{A}{x+1} + \frac{B}{x-2}\right)(x+1)(x-2)$$

$$= \frac{A\cancel{(x+1)}(x-2)}{\cancel{x+1}} + \frac{B(x+1)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= A(x-2) + B(x+1)$$

$$x+7 = (A+B)x + (-2A+B)$$

compare left & right sides

$$\Rightarrow \left.\begin{aligned} A+B&=1 \\ -2A+B&=7 \end{aligned}\right\} \quad A=-2,\ B=3$$

so $$\int \frac{x+7}{(x+1)(x-2)}\,dx = \int\left(\frac{-2}{x+1} + \frac{3}{x-2}\right)dx$$

$$= -2\ln|x+1| + 3\ln|x-2| + C$$

Ex 2

$$\int \frac{2x^2+6x+6}{(x+1)(x+2)(x+3)}\,dx$$

$$\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} = \frac{2x^2+6x+6}{(x+1)(x+2)(x+3)}$$

$$A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2) = 2x^2+6x+6$$

$$A(x^2+5x+6) + B(x^2+4x+3) + C(x^2+3x+2) = \text{" "}$$

$$\left.\begin{array}{ll} x^2) & A+B+C = 2 \\ x) & 5A+4B+3C = 6 \\ 1) & 6A+3B+2C = 6 \end{array}\right\}$$

$A = 1$, $B = -2$, $C = 3$

Hint: solve 1st for C

sub into 2nd & 3rd

2 eq^n for 2 unknowns.

so $\int \frac{2x^2+6x+6}{(x+1)(x+2)(x+3)}\,dx = \int\left(\frac{1}{x+1} - \frac{2}{x+2} + \frac{3}{x+3}\right)dx$

$= \ln|x+1| - 2\ln|x+2| + 3\ln|x+3| + c$

Repeated Factors

$\int \frac{2x^2+3x+2}{x^2(x+1)}\,dx$

Note downstair cubic so need 3 constants

Does

$\frac{A}{x^2} + \frac{B}{x+1}$ work?

~~Not~~ No! Not enough const. (need 1 more!)

Consider $\int \frac{2x+1}{x^2}\,dx = \int\left(\frac{2}{x} + \frac{1}{x^2}\right)dx$

2 terms

so I think

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{2x^2+3x+2}{x^2(x+1)}$$

Note: 3 constants ✓

$$Ax(x+1) + B(x+1) + Cx^2 = 2x^2+3x+2$$

x^2) $A + C = 2$ so $B = 2$

x) $A + B = 3$ $A = 3 - B = 3 - 2 = 1$

1) $B = 2$ $C = 2 - A = 2 - 1 = 1$

$$\int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+1} \right) dx$$

$$= + \ln|x| - \frac{2}{x} + \ln|x+1| + C$$

Quadratic Factors

$$\int \frac{3x^2+x+1}{x(x^2+1)}\,dx$$

downstairs
cubic so 3 const

dos it $\frac{A}{x} + \frac{B}{x^2+1}$ note enough constant !

Note from previous problem

$$\frac{A}{x} + \frac{B}{x^2} = \frac{Ax+B}{x^2}$$ ← linear ← quadratic

so guess $\frac{A}{x} + \frac{Bx+C}{x^2+1}$ ← lin ← quad

also 3 const.

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{3x^2+x+1}{x(x^2+1)}$$

$$A(x^2+1) + (Bx+C)x = 3x^2+x+1$$

$x^2)\quad A+B = 3 \Rightarrow B=2$

$x)\quad C = 1$

$1)\quad A = 1$

so $\displaystyle\int \left(\frac{1}{x} + \frac{2x+1}{x^2+1}\right) dx$

$$\int \left(\frac{1}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right) dx$$

let $u = x^2+1$ to $\int$

$$\ln|x| + \ln|x^2+1| + \tan^{-1} x + C$$

Two Quad factors

$$\int \frac{2x^3+x^2+2x+4}{(x^2+1)(x^2+4)} dx$$

denominator 4th order so 4 unk's

$$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} = \frac{2x^3+x^2+2x+4}{(x^2+1)(x^2+4)}$$

$$(Ax+B)(x^2+4) + (Cx+D)(x^2+1) = 2x^3+x^2+2x+4$$

x^3) $A+C = 2$

x^2) $B+D = 1$

x) $4A+C = 2$

1) $4B+D = 4$

$A=0,\ C=2$

$B=1,\ D=0$

so $\displaystyle\int \left(\frac{1}{x^2+1} + \frac{2x}{x^2+4}\right) dx$ $\leftarrow u = x^2+4$

$= \tan^{-1}x + \ln|x^2+4| + c$

Note: $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

if needed.

Repeated Quadratic Factors

$$\int \frac{2x^3+6x-1}{(x^2+1)^2}\,dx$$

$$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{2x^3+6x-1}{(x^2+1)^2}$$

$$(Ax+B)(x^2+1) + (Cx+D) = 2x^3+6x-1$$

$x^3)\quad A = 2$ $\qquad A=2,\ B=0$

$x^2)\quad B = 0$ $\qquad C=4,\ D=-1$

$x)\quad A+C = 6$

$1)\quad B+D = -1$

$u = x^2+1$

$x = \tan\theta$

$$\int \frac{2x}{x^2+1} + \frac{4x}{(x^2+1)^2} - \frac{1}{(x^2+1)^2}\,dx$$

$$= \ln|x^2+1| - \frac{2}{x^2+1} - \left\{ \frac{1}{2}\cdot\frac{x}{x^2+1} + \frac{\tan^{-1}x}{2} \right\} + C$$