Math 1497 - Calc 2

Chapter 9 (Briggs) Power Series

Last chapter we introduced infinite series

$$\sum_{n=1}^{\infty} a_n$$

where a_n #s. Now we extend this to include powers of x. These power series provide a way to represent familiar functions.

What are they

$$\sum_{n=0}^{\infty} c_n x^n$$ where c_n depends only on n.

Ex $$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad c_n = \frac{1}{n!}$$

Ex $$\sum_{n=0}^{\infty} 2^n x^n \qquad c_n = 2^n$$

In general $$\sum_{n=0}^{\infty} c_n (x-a)^n$$ $x=a$ called the center of the series

Briggs Section 9.2

we ask – for what values does this series converge? This set of values is called the "interval of convergence."

What are the possibilities

(i) converges at only $x = a$

(ii) converges for all x ↙ R called radius of convergence

(iii) converges for $|x-a| < R$

We 1st use the ratio test. If it converges at only $x=a$ or for all x – we're done!
If it converges ~~at~~ on some interval, we also need to check the end points

The following examples illustrate

Ex1 $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ this series is famous.

1st use ratio test

required for conv.

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{x \cdot \cancel{x^n}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right| < 1$$

$$\lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \quad \text{for all } x$$

so $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x

Ex 2 $\sum_{n=0}^{\infty} n!(x-1)^n$ center at $x=1$

required

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x-1)^{n+1}}{n!\,(x-1)^n} \right| < 1$$

series converges only at $x=1$

$$\lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}\,(x-1)\cancel{(x-1)^n}}{\cancel{n!}\,\cancel{(x-1)^n}} \right| < 1$$

$\therefore \lim_{n\to\infty} n+1 = \infty$

$$\lim_{n\to\infty} (n+1)|x-1| < 1$$

this only occurs if $x=1$

Ex 3 $\quad \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\,2^n}$

$$\lim_{n\to\infty} \left| \frac{(x)^{n+1}}{(n+1)\,2^{n+1}} \Big/ \frac{x^n}{n\,2^n} \right| < 1$$

$$\lim_{n\to\infty} \frac{|x|\; n2^n}{(n+1)\,2^{n+1}} = \lim_{n\to\infty} |x| \frac{n}{2(n+1)} = \frac{|x|}{2} < 1$$

so $|x| < 2 \Rightarrow -2 < x < 2$

Now we check the end pts

$x = -2 \qquad \displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \leftarrow$ we saw this Monday converges by AST

$x = 2 \qquad \displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \leftarrow$ div harmonic

so this series converges for $-2 \le x < 2$

$\uparrow$

include this ~~end~~ endpt.

Ex 4 $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \Big/ \frac{(x-2)^n}{n^2+1} \right| < 1$$

$$\lim_{n\to\infty} |x-2| \frac{n^2+1}{n^2+2n+2} = |x-2| < 1$$

so $-1 < x-2 < 1 \Rightarrow 1 < x < 3$

End pts

$x=+1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ alt.

$x = 3$ $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ doesn't alt check 1ST

LCT w/ $\sum \frac{1}{n^2}$ $\lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = 1$ so
$(p=2)$

$\sum \frac{1}{n^2+1}$ conv so $\sum \frac{(-1)^n}{n^2+1}$ conv. abs

and series conv. for $1 \le x \le 3$

Ex 5 $\displaystyle\sum_{n=1}^{\infty} \frac{n(-1)^n (x+3)^n}{3^n}$

$$\lim_{n\to\infty} \left| \frac{(n+1)(x+3)^{n+1}}{3^{n+1}} \bigg/ \frac{n(x+3)^n}{3^n} \right| < 1$$

$$\lim_{n\to\infty} \frac{n+1}{3n} |x+3| = \frac{|x+3|}{3} < 1$$

so $|x+3| < 3 \Rightarrow -3 < x+3 < 3 \Rightarrow -6 < x < 0$

Endpts

$x = -6$ $\displaystyle\sum_{n=1}^{\infty} \frac{n(-1)^n(-3)^n}{3^n} = \sum_{n=1}^{\infty} \frac{n\,3^n}{3^n} = \sum_{n=1} n$

$x = 0$ $\displaystyle\sum_{n=1}^{\infty} \frac{n(-1)^n 3^n}{3^n} = \sum_{n=1}^{\infty} n(-1)^n$

both div
so neither
endpt is included

So our series converges $-6 < x < 0$