Last class we considered

$$\frac{dy}{dx} = \frac{x+y+y^2}{xy} \tag{1}$$

which was shown to be invariant under

$$\bar{x} = \frac{x}{1+\varepsilon x}, \quad \bar{y} = \frac{y}{1+\varepsilon x} \tag{2}$$

we also show that under

$$x = \frac{1}{s}, \quad y = \frac{r}{s} \tag{3}$$

that we obtain the separable ODE

$$\frac{ds}{dr} = \frac{-r}{1+r} \tag{4}$$

which is invariant under

$$\bar{r} = r, \quad \bar{s} = s + \varepsilon \tag{5}$$

Note: All separable eq^{n}'s $\frac{ds}{dr} = G(r)$ are.

Further more ~~the~~ we showed the transformation (3) is invariant under (2) & (5).

However we need to be able to come up with (1) the Lie Group leaving the original ODE invariant

(2) the transformation $x = A(r,s)$, $y = B(r,s)$

we tried to derive that latter with no avail

Infinitesimal Lie Group:

Consider the LG

$$\bar{x} = e^{\varepsilon} x, \quad \bar{y} = e^{-\varepsilon} y$$

If we assume ε small Then a Taylor expansion about $\varepsilon = 0$ gives

$$\bar{x} = (1 + \varepsilon + O(\varepsilon^2))\, x$$

$$= x + \varepsilon x + O(\varepsilon^2)$$

$$\bar{y} = y - \varepsilon y + O(\varepsilon^2)$$

Recall Taylor Series in general

$$f(x) = f(a) + f'(a)(x-a) + \cdots$$

so if $\bar{x} = f(x,y,\varepsilon)$, $\bar{y} = g(x,y,\varepsilon)$

$$\bar{x} = f(x,y,0) + f_\varepsilon(x,y,0)\,\varepsilon + O(\varepsilon^2)$$

similarly

$$\bar{y} = g(x,y,0) + g_\varepsilon(x,y,0)\,\varepsilon + O(\varepsilon^2)$$

Now $f(x,y,0) = x$, $g(x,y,0) = y$

we define

$$\left.\frac{\partial \bar{x}}{\partial \varepsilon}\right|_{\varepsilon=0} = X(x,y) \qquad \left.\frac{\partial \bar{y}}{\partial \varepsilon}\right|_{\varepsilon=0} = Y(x,y)$$

so
$$\bar{x} = x + X(x,y)\,\varepsilon + O(\varepsilon^2)$$
$$\bar{y} = y + Y(x,y)\,\varepsilon + O(\varepsilon^2)$$

Infinitesimal Lie Group (or transformation)

Examples

(1) $\bar{x} = \cos\varepsilon\, x - \sin\varepsilon\, y \qquad \bar{x}|_{\varepsilon=0} = x$ ✓

$\bar{y} = \cos\varepsilon\, y + \sin\varepsilon\, x \qquad \bar{y}|_{\varepsilon=0} = y$

$$\frac{\partial \bar{x}}{\partial \varepsilon} = -\sin\varepsilon\, x - \cos\varepsilon\, y$$

$$\frac{\partial \bar{y}}{\partial \varepsilon} = -\sin\varepsilon\, y + \cos\varepsilon\, x$$

$$X(x,y) = \left.\frac{\partial \bar{x}}{\partial \varepsilon}\right|_{\varepsilon=0} = -y, \quad Y(x,y) = \left.\frac{\partial \bar{y}}{\partial \varepsilon}\right|_{\varepsilon=0} = x$$

(2) $\bar{x} = \dfrac{x}{1+\varepsilon x}, \quad \bar{y} = \dfrac{y}{1+\varepsilon x}$

$$\frac{\partial \bar{x}}{\partial \varepsilon} = -\frac{x^2}{(1+\varepsilon x)^2} \qquad \frac{\partial \bar{y}}{\partial \varepsilon} = \frac{-xy}{(1+\varepsilon x)^2}$$

$$X(x,y) = -x^2, \quad Y(x,y) = -xy$$

Finding the Transformation

So we seek

$$x = A(r,s) \qquad y = B(r,s)$$

we will assume that it is invertible so

$$r = R(x,y) \qquad s = S(x,y)$$

and it is invariant under some Lie Group

LG1 $\bar{x} = f(x,y,\varepsilon), \quad \bar{y} = g(x,y,\varepsilon)$
LG2 $\bar{r} = r, \quad \bar{s} = s + \varepsilon$

so $$R(f(x,y,\varepsilon), g(x,y,\varepsilon)) = r$$
$$S(f(x,y,\varepsilon), g(x,y,\varepsilon)) = s + \varepsilon$$

we want these to be independent of ε

so $$\frac{\partial}{\partial \varepsilon} R = 0 \qquad \frac{\partial S}{\partial \varepsilon} = 1$$

or expanding gives

$$R_f(f,g)\frac{\partial f}{\partial \varepsilon} + R_g(f,g)\frac{\partial g}{\partial \varepsilon} = 0$$

$$S_f(f,g)\frac{\partial f}{\partial \varepsilon} + S_g(f,g)\frac{\partial g}{\partial \varepsilon} = 1$$

now set $\varepsilon = 0$ $\quad f = x,\ g = y \quad f_\varepsilon = X,\ g_\varepsilon = Y$

so $\quad XR_x + YR_y = 0 \quad XS_x + YS_y = 1$

Now $r = R$ & $s = S$ so

$$\boxed{Xr_x + Yr_y = 0, \quad Xs_x + Ys_y = 1}$$

<u>examples</u>

we return to the first example where

$$LG = \left\{ \bar{x} = \frac{x}{1+\varepsilon x},\ \bar{y} = \frac{y}{1+\varepsilon x} \right\}$$

$$T = \left\{ x = \frac{1}{s},\ y = \frac{r}{s} \right\}$$

Now the infinitesimals are

$$X = -x^2, \quad Y = -xy$$

Now we solve

$$-x^2 r_x - xy\, r_y = 0, \quad -x^2 s_x - y^2 s_y = 1$$

More

(1) $$\frac{dx}{-x^2} = \frac{dy}{-xy}\,; \quad dr = 0$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow c_1 = y/x \ \ \& \ \ c_2 = r$$

sol^n $\quad r = R(y/x)$

(2) $$\frac{dx}{-x^2} = -\frac{dy}{xy} = \frac{ds}{1}$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow c_1 = y/x \quad \text{(like above)}$$

$$\frac{dx}{-x^2} = ds \Rightarrow s - \frac{1}{x} = c_2$$

$$S = \frac{1}{x} + S(y/x)$$

choose $r = y/x, \ s = \frac{1}{x}$

or $x = \frac{1}{s}, \ y = \frac{r}{s}$ ✓

ex2 Consider

$$\frac{dy}{dx} = 2y^2 + xy^3$$

It is easy to show this is invariant under

$$\bar{x} = e^{\varepsilon} x, \quad \bar{y} = e^{-\varepsilon} y$$

so $\quad X = x, \quad Y = -y$

solve $\quad x r_x - y r_y = 0 \qquad x s_x - y s_y = 1$

$$\frac{dx}{x} = \frac{dy}{-y}; \; dr = 0 \qquad \frac{dx}{x} = -\frac{dy}{y} = \frac{ds}{1}$$

$$r = R(xy) \qquad s = \ln x + S'(xy)$$

choose $\; r = xy, \; s = \ln x \;$ or $\; x = e^{s}, \; y = r e^{-s}$

$$\frac{dy}{dx} = \frac{e^{-s} - r e^{-s} s'}{e^{s} s'} = e^{-2s}\left(\frac{1 - r s'}{s'}\right)$$

sub $\quad \cancel{e^{-2s}}\left(\frac{1 - r s'}{s'}\right) = 2r^2 \cancel{e^{-2s}} + r^3 \cancel{e^{-2s}}$

$$\Rightarrow \quad s' = \frac{1}{r + 2r^2 + r^3} \qquad \text{separable}$$